Institute for Mathematics
and Scientific Computing

Advanced Functional Analysis

Problem Sheet 3

Due date: November 6, 2015

Problem 3.1. For H a Hilbert space and $T \in \mathcal{L}(H)$ a self-adjoint compact operator, let $(\mu_i)_{i\in N}$ for $N \subset \mathbb{N}$ be the sequence of non-zero eigenvalues and E_i the orthogonal projections on the corresponding eigenspaces. With $\mu_0 = 0$ and E_0 an orthogonal projection on $\ker(T)$, decompose T according to the lecture as

$$T = \sum_{i\in N\cup\{0\}} \mu_i E_i.$$

Show that

$$\Phi : C(\sigma(T)) \to \mathcal{L}(H) \tag{1}$$

$$f \mapsto f(T) := \sum_{i\in N\cup\{0\}} f(\mu_i)E_i \tag{2}$$

defines the continuous functional calculus of T.

Problem 3.2. For H a Hilbert space and $T \in \mathcal{L}(H)$, we say that $\lambda \in \sigma(T)$ is an approximate Eigenvalue if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H such that $\|x_n\| = 1$ for all n and $\lambda x_n - Tx_n \to 0$ as $n \to \infty$. Show that, if T is normal, every $\lambda \in \sigma(T)$ is an approximate eigenvalue.

Problem 3.3. For H a Hilbert space, $(\lambda_n)_{n\in\mathbb{N}}$ a bounded sequence of numbers in $\mathbb{K}$ and $(e_n)_{n\in\mathbb{N}}$ an orthonormal system, define

$$Tx = \sum_{n\in\mathbb{N}} \lambda_n (x, e_n)e_n.$$

i) Show that $T \in \mathcal{L}(H)$ and T is normal

ii) Show that T is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$

iii) Determine the eigenvalues and eigenvectors of T

Problem 3.4. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \lambda_n = \infty$. Define V to be the set of all (real) sequences $(u_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\in\mathbb{N}} \lambda_n u_n^2 < \infty$$

and the scalar product

$$(u, v)_V = \sum_{n\in\mathbb{N}} \lambda_n u_n v_n.$$

Show that V equipped with $(\cdot,\cdot)_V$ is a Hilbert space and that canonical injection (i.e. the identity mapping) from V to ℓ^2 (as real vector space) is compact.

Problem 3.5. Let X be a Banach space over $\mathbb{R}$ and $T \in \mathcal{L}(X)$. For a polynomial $Q(t) = \sum_{k=0}^{p} a_k t^k$ with $a_k \in \mathbb{R}$, let $Q(T) = \sum_{k=0}^{p} a_k T^k$. Show

i) $Q(\sigma(T)) \subset \sigma(Q(T))$. Give an example where the inclusion is strict.

ii) If X is a Hilbert space, T is self-adjoint and Q has no real root, then $Q(T)$ is bijective.

iii) If X is a Hilbert space, T is self-adjoint, then $Q(\sigma(T)) = \sigma(Q(T))$.

Hint: For the second point, start with $Q(t) = t^2 + 1$ and apply Lax-Milgram (see, e.g., [Brezis, Corollary 5.8]). This exercise deals with the validity of Proposition 1.23 of the lecture for real-Hilbert spaces. You might use some techniques of its proof.