MAT.405 / WS 2015/2016
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Advanced Functional Analysis

Problem Sheet 2

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Remark: From the Alaoglu theorem shown in the basic course on functional analysis (or by a direct proof) one can deduce (by equivalence of weak and weak* convergence) that every bounded sequence in a reflexive normed space has a weakly convergent subsequence. We assume this to be given.

Problem 2.1. Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$. Consider

$$\text{(P)} \quad :\Leftrightarrow \quad \forall (x_n)_n \text{ in } X : x_n \rightharpoonup x \text{ weakly } \Rightarrow Tx_n \to Tx \text{ strongly.}$$

Show

i) If $T \in \mathcal{K}(X, Y)$, then (P) holds.

ii) If X is reflexive and (P) holds, then $T \in \mathcal{K}(X, Y)$.

iii) Provide an example where (P) holds but T is not a compact operator. (Hint: You might need that $(\ell^1)^* \cong \ell^\infty$)

Problem 2.2. [The approximation problem]. Let X, Y be Banach spaces. Show

- For any linear continuous projection P on Y it holds that either $P = 0$ or $\|P\| \geq 1$.
- If Y is a Hilbert space and P is a continuous, linear, orthogonal projection on Y (i.e. $\text{rg}(P)$ is orthogonal to $\ker(P)$), then $\|P\| = 1$ (or $P = 0$).
- Show that, if Y is a Hilbert space, then for every $T \in \mathcal{K}(X, Y)$ there exists a sequence of finite rank operators from X to Y converging to T with respect to the operator norm.

Problem 2.3. Show

i) If H is a Hilbert space and $T \in \mathcal{L}(H)$ is such that $T^*T = TT^*$ (i.e., is a normal operator), then $r(T) = \|T\|$.

ii) Provide an example (e.g. in $\mathbb{R}^2$) of an operator such that $r(T) = 0$ but $\|T\| = 1$.

iii) Provide an example (e.g. in $\mathbb{R}^3$) of an operator such that $\sigma(T) = \{0\}$ but $r(T) = 1$.

Problem 2.4. On the $\mathbb{R}$-Hilbert space $\ell^2 = \{(x_i)_{i\in\mathbb{N}} \,|\, x_i \in \mathbb{R},\ \|x\|_{\ell^2}^2 := \sum_i x_i^2 < \infty\}$, define the operators

$$S_r x = (0, x_1, x_2, \ldots)$$

and

$$S_l x = (x_2, x_3, x_4, \ldots)$$

mapping again to ℓ^2 for $x = (x_1, x_2, \ldots)$.

i) Determine $\|S_r\|$, $\|S_l\|$, S_r^*, S_l^*. Does S_r or S_l belong to $\mathcal{K}(\ell^2)$?

ii) Determine the spectrum and the resolvent set of S_r and S_l.

iii) Determine the decomposition of $\sigma(S_r)$ and $\sigma(S_l)$ into the point, continuous and residual spectrum and determine the Eigenspaces corresponding to all Eigenvalues.