The Non-overlapping Domain Decomposition Multiplicative Schwarz Method

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Abstract

Once a decomposition of the finite element space $\mathbf{V}$ into two or more subspaces is given, e.g. , via domain decomposition, a specific Multiplicative Schwarz Method (MSM) and Additve Schwarz Method (ASM) is defined. In this paper, we analyse the MSM for the decomposition induced by the approximate discrete harmonic finite element basis which was introduced in a joint paper of the authors with A. Meyer (1990). The main theorem of the present paper states that a special symmetric version of the MSM with approximate orthoprojections is equivalent to some ASM with specially chosen basic transformation and block preconditioners. From this observation we can benefit twice. Indeed, the MSM-DD-preconditioner can be analysed in the MSM framework and implemented as specific ASM-DD-preconditioner in the parallel PCG method studied previously. Emphazis that we look at the ASM and MSM as techniques for defining and analysing parallel DD preconditioners used then in a parallelized version of the PCG-method which is well suited for computations on MIMD computers with local memory and message passing principle.

Key words: elliptic problems, finite elements, substructuring, domain decomposition, Schwarz' methods, preconditioners, parallel algorithms

AMS (MOS) subject classification: 65N55, 65N22, 65F10, 65Y05, 65Y10, 65F35

1 Introduction

In 1869, H. Schwarz proposed an iterative method called later alternating Schwarz method [23]. He used this method in order to show the existence of harmonic functions on regions Ω with non-smooth boundaries $\partial\Omega$ which can be decomposed into two simpler, overlapping domains Ω_1 und Ω_2 for which the existence is known. The basic idea of the method consists in an alternating iterative solution of the boundary value problem (b.v.p.) first in Ω_1 and then in Ω_2 taking the boundary values from the foregoing iterate. Schwarz analysed the convergence of his method by means of the maximum principle. More than 60 years later, S.L. Sobolev gave the variational formulation of the alternating Schwarz method [24]. Let us define the energy orthoprojections $P_i : \mathbf{V}_0 \longrightarrow \mathbf{V}_{0i}$ of the energy space $\mathbf{V}_0 = \overset{\circ}{H}{}^1(\Omega)$ into the subspace $\mathbf{V}_{0i} = \left\{ v \in \overset{\circ}{H}{}^1(\Omega_i) : v = 0 \quad in \quad \overline{\Omega} \setminus \Omega_i \right\}$ with respect to the energy inner product $a(\cdot,\cdot)$

induced by the PDE considered. Then the variational formulation of the alternating Schwarz method can be rewritten in the form

$$u^{k+1} = (I - P)\, u^k + Pu \quad , \qquad k = 0, 1, ..., \tag{1}$$

where $P = P_1 + P_2 - P_2 P_1$, $u^0 \in \mathbf{V}_0$ is some given initial guess and u denotes the solution of the b.v.p. . Therefore, the error $z^k = u - u^k$ satisfies the iteration scheme

$$z^{k+1} = (I - P_2)\,(I - P_1)\, z^k, \tag{2}$$

i.e. one performs alternating orthoprojections of the error into $V_1^{\perp}$ and $V_2^{\perp}$. The iteration operator is a product of orthoprojectors, namely, of the co-projectors to P_1 and P_2. That is the reason why the alternating Schwarz method is also called Multiplicative Schwarz Method (MSM). The MSM can be easily generalized to a decomposition of $\mathbf{V}$ into more than two subspaces. An excellent analysis of the MSM on the "PDE level" is given in a series of three papers by P.L. Lions [17]. Replacing the energy orthoprojectors P_i by the Ritz projectors $P_{ih} : \mathbf{V}_h \longrightarrow \mathbf{V}_{hi}$, we naturally arrive at some MSM for solving the finite element (f.e.) scheme generated by the f.e. subspace $\mathbf{V}_h \subset \mathbf{V}_0$, where $\mathbf{V}_{hi} = \mathbf{V}_h \cap \mathbf{V}_{0i}$. Many papers have been devoted to the construction and analysis of various versions of the MSM for solving finite element equations via overlapping domain decomposition (see, e.g. [2] [21] [10] [15] [18], and the references given therein). Further papers dealing with various aspects of the Domain Decomposition (DD) techniques for f.e. equations can be found in the proceedings [4] [5] [11] [14]. The products of orthoprojectors arising in the operator P, especially, in the case of more than two subspaces can affect the parallelizability of the MSM in practical applications considerably. M. Dryja and O.B. Widlund proposed to cancel all these products and arrived at the so-called Additive Schwarz Method (ASM)

$$u^{k+1} = (I - \tau P)\, u^k + \tau Pu \quad , \; k = 0, 1, \dots \quad , \tag{3}$$

with $P = P_1 + \cdots + P_p$ in the case of a decomposition into p subspaces, where τ denotes some appropriately chosen iteration parmeter [7] [9], (see also [8] [2] and the references therein).

Once a decomposition of the finite element space $\mathbf{V}$ into two or more subspaces is given, e.g., via domain decomposition, a specific MSM and ASM is defined. In this paper, we analyse the MSM for the decomposition induced by the approximate discrete harmonic basis which was introduced in a joint paper of the authors with A. Meyer [12]. In Section 2, we consider a non-overlapping domain decomposition on the basis of the classical f.e. substructuring technique and connect the latter one with the discrete harmonic basis. In Section 3, we introduce the approximate discrete harmonic basis resulting in the representation

$$\mathbf{V} \;=\; \widetilde{\mathbf{V}}_C + \widetilde{\mathbf{V}}_I \tag{4}$$

of the f.e. subspace $\mathbf{V}$ as the direct sum of the subspaces $\widetilde{\mathbf{V}}_C$ of the approximate discrete harmonic f.e. functions and $\widetilde{\mathbf{V}}_I$ of the subdomain f.e. functions, and we determine $\gamma = \cos \angle(\widetilde{\mathbf{V}}_C, \widetilde{\mathbf{V}}_I)$. Section 4 is devoted to the ASM and MSM based on the decomposition (4). The main theorem of this paper is formulated in Section 5 and states that a special symmetric version of the MSM with approximate orthoprojections is equivalent to some ASM with specially chosen basis transformation and block preconditioners. From this observation we can benefit twice: The MSM-DD-preconditioner can be analysed in the MSM framework and implemented as specific ASM-DD-preconditioner in a parallelized version of the PCG-method which is given in the Appendix for completeness. Moreover, the special form of the preconditioner can be exploited in the implementation in order to save arithmetical operations without changing the convergence properties.

2 The Finite Element Substructuring Technique and the Discrete Harmonic Basis

Let us consider the symmetric, $\mathbf{V}_0$-elliptic and $\mathbf{V}_0$- bounded variational problem,

$$Find \quad u \in \mathbf{V}_0 : \quad a(u,v) \ = < F, v > \quad \forall v \in \mathbf{V}_0 \quad , \tag{5}$$

arising from the week formulation of a symmetric, (2n)-th-order (n $\geq$ 1), uniformly elliptic b.v.p. given in a d-dimensional bounded domain Ω with a piecewise smooth boundary $\partial\Omega$. We divide Ω into p non-overlapping subdomains $\Omega_i \quad (i = 1, 2, \ldots, p)$ called also substructures or superelements such that $\overline{\Omega} = \bigcup\limits_{i=1}^{p} \overline{\Omega}_i$, and each subdomain Ω_i into finite elements δ_r such that this discretization process results in a conform triangulation of Ω. In the following, the indices "C" and "I" correspond to the nodes belonging to the coupling boundary $\Gamma_C = \bigcup\limits_{i=1}^{p} \partial\Omega_i \setminus \Gamma_D$ (interfaces) and to the interior $\Omega_I = \bigcup\limits_{i=1}^{p} \Omega_i$ of the subdomains, respectively, where Γ_D is that part of $\partial\Omega$ where pure Dirichlet-type boundary conditions are given.

Define the usual nodal basis

$$\Phi = \left\{ \varphi_1, \cdots, \varphi_{N_C}, \varphi_{N_C+1}, \cdots, \varphi_{N_C+N_{I,1}}, \cdots, \varphi_{N=N_C+N_I} \right\} \quad ,$$

where the first N_C basis functions belong to Γ_C, the following $N_{I,1}$ basis functions to Ω_1 and so on, with $N_I = N_{I,1} + \cdots + N_{I,p}$. The corresponding finite element subspace

$$\mathbf{V} = \mathbf{V}_h = span(\Phi) = span(\Phi V) \subset \mathbf{V}_0$$

can be represented in the form

$$\mathbf{V} = \mathbf{V}_C + \mathbf{V}_I \tag{6}$$

of the direct sum of the subspaces $\mathbf{V}_C = span(\Phi V_C)$ and $\mathbf{V}_I = span(\Phi V_I)$, where

$$V = (V_C \; V_I) = I = \begin{pmatrix} I_C & O \\ O & I_I \end{pmatrix}_{N\times N} \quad , V_C = \begin{pmatrix} I_C \\ O \end{pmatrix}_{N\times N_C} \quad \text{and} \quad V_I = \begin{pmatrix} O \\ I_I \end{pmatrix}_{N\times N_I} \quad .$$

The f.e. isomorphism is given by the relation

$$\mathbf{V} \ \ni u = \Phi V \underline{u} = \Phi \underline{u} \quad \stackrel{\Phi}{\longleftrightarrow} \quad \underline{u} \in \mathbf{R}^N \quad ,$$

where $\underline{u} = \left(\underline{u}_C^T, \underline{u}_I^T \right)^T$ is the vector of the nodal parameters.

The f.e. approximation to (5) is defined as the solution of the following f.e. scheme:

$$\text{Find} \quad u = \Phi V \underline{u} \in \mathbf{V} \ : \ a(\Phi V \underline{u}, \Phi V \underline{v}) =< F, \Phi V \underline{u} > \ \forall v = \Phi V \underline{v} \in \mathbf{V} \ . \tag{7}$$

Once the basis Φ for $\mathbf{V}$ is chosen, (7) immediately leads to the system

$$K\,\underline{u} \;=\; \underline{f} \tag{8}$$

of the f.e. equations for determining the unknown nodal parameter vector $\underline{u} \in \mathbf{R}^N$, where the stiffness matrix K and the right-hand side $\underline{f}$ are defined by the relations

$$(K\underline{u}, \underline{v}) = (V^T K V \underline{u}, \underline{u}) = a(\Phi V \underline{u}, \Phi V \underline{v}) \quad \forall \underline{u}, \underline{v} \in \mathbf{R}^N \tag{9}$$

and

$$(\underline{f}, \underline{v}) = (\underline{f}, V\underline{u}) = < F, \Phi V \underline{v} > \qquad \forall \underline{v} \in \mathbf{R}^N \;, \tag{10}$$

respectively. Here $(\cdot,\cdot)$ denotes the usual Euclidian inner product. Because of the arrangment of the unknowns defined above, system (8) can be rewritten in the block form

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix} \quad ,$$

where the matrix K_I has the block diagonal form

$$K_I = diag\,(K_{I,i})_{i=1,2,\ldots,p} \quad . \tag{11}$$

The block diagonal entries $K_{I,i}$ are of the dimension $N_{I,i} \times N_{I,i}$ and arise from the f.e. approximation to the PDE considered in Ω_i under homogenous Dirichlet boundary conditions on $\partial\Omega$. Indeed, we have

$$(K_{I,i}\underline{u}_i, \underline{v}_i) = a(\Phi V_{I,i}\underline{u}_i, \Phi V_{I,i}\underline{v}_i) \quad \forall \underline{u}_i, \underline{v}_i \in \mathbf{R}^{N_{I,i}}$$

and $\mathbf{V}_{I,i} = span(\Phi V_{I,i}) \;\subset \overset{\circ}{\mathbf{W}}{}_2^n(\Omega_i)$, where $V_{I,i}^T = (O\ I_{I,i}\ O)_{N_{I,i}\times N}$. Furthermore,

$$\mathbf{V}_I = \mathbf{V}_{I,1} \oplus \cdots \oplus \mathbf{V}_{I,p}$$

can be decomposed into the orthogonal sum of the subspaces $\mathbf{V}_{I,i}$ with respect to the energy inner product $a(\cdot,\cdot)$. The properties of the bilinear form $a(\cdot,\cdot)$ mentioned above and identity (9) imply that the stiffness matrix K is symmetric and positive definite.

Beside the nodal basis Φ, we now introduce the so-called discrete harmonic basis

$$\overset{*}{\Psi} = \left\{ \overset{*}{\psi}_1, \cdots, \overset{*}{\psi}_{N_C}, \overset{*}{\psi}_{N_C+1}, \cdots, \overset{*}{\psi}_N \right\} = \Phi\, \overset{*}{V}$$

of $\mathbf{V} = span(\Phi) = span(\overset{*}{\Psi})$, where the basis transformation matrix $\overset{*}{V} = \begin{pmatrix} \overset{*}{V}_C & \overset{*}{V}_I \end{pmatrix}$ is given by the submatrices

$$\overset{*}{V}_C = \begin{pmatrix} I_C \\ -K_I^{-1} K_{IC} \end{pmatrix}_{N\times N_C} \quad \text{and} \quad \overset{*}{V}_I = \begin{pmatrix} O \\ I_I \end{pmatrix}_{N\times N_I} \quad .$$

Therefore, the basis functions belonging to the interior of the subdomains remain unchanged, i.e. $\overset{*}{\psi}_j = \phi_j$ for all $j = N_C+1, \ldots, N$. Analogously to (6), the f.e. subspace $\mathbf{V}$ can be represented in the form

$$\mathbf{V} = \overset{*}{\mathbf{V}}_C \oplus \overset{*}{\mathbf{V}}_I$$

where $\overset{*}{\mathbf{V}}_C = span(\Phi \overset{*}{V}_C)$ and $\overset{*}{\mathbf{V}}_I = span(\Phi \overset{*}{V}_I) = \mathbf{V}_I$. Emphasize that this decomposition is orthogonal in the energy inner product $a(\cdot,\cdot)$. The f.e. variational problem (7) can now be rewritten in the the matrix form

$$\begin{pmatrix} S_C & O \\ O & K_I \end{pmatrix} \begin{pmatrix} \overset{*}{\underline{u}}_C \\ \overset{*}{\underline{u}}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C - K_{CI}K_I^{-1}\underline{f}_I \\ \underline{f}_I \end{pmatrix} \tag{12}$$

where $S_C = K_C - K_{CI}K_I^{-1}K_{IC}$ denotes the Schur complement. The solution vector $\overset{*}{\underline{u}} = \left(\overset{*}{\underline{u}}{}_C^T, \overset{*}{\underline{u}}{}_I^T\right)^T$ is nothing but the coefficients vector with respect to the discrete harmonic basis $\overset{*}{\Psi}$, i.e., $u = \overset{*}{\Psi}\, \overset{*}{\underline{u}} = \Phi\, \overset{*}{V}\, \overset{*}{\underline{u}}$ is the solution of the f.e. scheme (7). In order to recover the coefficient vector $\underline{u} = \left(\underline{u}_C^T, \underline{u}_I^T\right)^T$ in the nodal basis, we simply have to apply the basis transformation matrix $\overset{*}{V}$ to $\overset{*}{\underline{u}}$, i.e.

$$\underline{u} = \overset{*}{V}\, \overset{*}{\underline{u}} = \begin{pmatrix} I_C & O \\ -K_I^{-1}K_{IC} & I_I \end{pmatrix} \begin{pmatrix} \overset{*}{\underline{u}}_C \\ \overset{*}{\underline{u}}_I \end{pmatrix} = \begin{pmatrix} \overset{*}{\underline{u}}_C \\ \overset{*}{\underline{u}}_I - K_I^{-1}K_{IC}\, \overset{*}{\underline{u}}_C \end{pmatrix} \tag{13}$$

The relations (12) - (13) are obviously equivalent to the well-known formulas

$$\begin{aligned} \overset{*}{\underline{u}}_I &= K_I^{-1}\underline{f}_I\,, \\ \underline{g}_C &= \underline{f}_C - K_{CI}\, \overset{*}{\underline{u}}_I\,, \quad S_C \;=\; K_C - K_{CI}K_I^{-1}K_{IC}\,, \\ \underline{u}_C &= \overset{*}{\underline{u}}_C \;=\; S_C^{-1}\underline{g}_C\,, \\ \underline{u}_I &= \overset{*}{\underline{u}}_I - K_I^{-1}K_{IC}\, \overset{*}{\underline{u}}_C \end{aligned}$$

of the classical f.e. substructuring technique. Because of the block diagonal form (11) of K_I, the application of K_I^{-1} means the (parallel) solution of p independent systems with the system matrices $K_{I,1}, \ldots, K_{I,p}$, respectively. Although the solution of these subsystems can be performed completely in parallel, in practice the solution of one subsystem is usually too expensive. Thus, we prefer to apply the parallel conjugate gradient (cg) method (cf. Appendix) to (2) with an appropriate DD-preconditioner [12] instead of the classical f.e. substructuring technique. Replacing the actions of the operators K_I^{-1} and S_C^{-1} by appropriate, possibly non-linear mappings induced usually by iterative methods, O. Axelsson and P.S. Vassilevski construct variable-step preconditioners for K on the basis of the factorization $K^{-1} = \overset{*}{V} diag\left(S_C^{-1}, K_I^{-1}\right) \overset{*}{V}{}^T$, [1].

3 The Approximate Discrete Harmonic Basis and the Corresponding Splitting of the F.E. Space

Replacing the matrix K_I^{-1} in the basis transformation matrix $\overset{*}{V}$ by some non-singular matrix B_I^{-1}, we introduce an Approximate Discrete Harmonic Basis (ADHB)

$$\widetilde{\Psi} = \left\{\widetilde{\psi}_1, \cdots, \widetilde{\psi}_{N_C}, \widetilde{\psi}_{N_C+1}, \cdots, \widetilde{\psi}_N\right\} = \Phi\widetilde{V}$$

of $\mathbf{V} = span(\Phi) = span(\widetilde{\Psi})$, where the transformation matrix $\widetilde{V} = \left(\widetilde{V}_C\ \widetilde{V}_I\right)$ is given by the submatrices

$$\widetilde{V}_C = \begin{pmatrix} I_C \\ -B_I^{-1}K_{IC} \end{pmatrix}_{N\times N_C} \quad \text{and} \quad \widetilde{V}_I = \overset{*}{V}_I = V_I = \begin{pmatrix} O \\ I_I \end{pmatrix}_{N\times N_I} .$$

We suppose that the matrix $B_I = diag\,(B_{I,i})_{i=1,2,\dots,p}$ has the same block diagonal structure as K_I and that systems with the system matrix $B_{I,i}$ can be solved much faster than systems with the original matrix $K_{I,i}$ (see [12] and Section 5 for examples). The symmetry and the positive definiteness of B_I are not required. Again, the f.e. subspace $\mathbf{V}$ can be represented in the form

$$\mathbf{V} = \widetilde{\mathbf{V}}_C + \widetilde{\mathbf{V}}_I$$

of the direct sum of the subspaces $\widetilde{\mathbf{V}}_I = span(\Phi\widetilde{V}_I) = \mathbf{V}_I$ and $\widetilde{\mathbf{V}}_C = span(\Phi\widetilde{V}_C)$. We lose now the orthogonality, but we can expect that the angle $\angle(\widetilde{V}_C, \widetilde{V}_I)$ between $\widetilde{V}_C$ and $\widetilde{V}_I$ is near $90°$ if B_I approximates K_I in some sense. Indeed, the following Lemma was proved in [12].

Lemma 3.1 *The angle between $\widetilde{V}_C$ and $\widetilde{V}_I$ is given by the relation*

$$\gamma = \cos\angle(\widetilde{V}_C, \widetilde{V}_I) = \sqrt{\frac{\mu}{1+\mu}}\,,$$

where $T_C = K_{CI}(K_I^{-1} - B_I^{-T})K_I(K_I^{-1} - B_I^{-1})K_{IC}$, $\mu = \rho(S_C^{-1}T_C)$ is the spectral radius of $S_C^{-1}T_C$ and $S_C = K_C - K_{CI}K_I^{-1}K_{IC}$ denotes the Schur complement.

Therefore, the strengthened Cauchy inequality

$$|a(u,v)| \le \gamma \, ||| \, u \, ||| \; ||| \, v \, ||| \quad \forall u \in \widetilde{\mathbf{V}}_C \text{ and } \forall v \in \widetilde{\mathbf{V}}_I$$

holds with $\gamma \in [\,0,1)$ from Lemma 3.1. Here $|||\cdot||| = (a(\cdot,\cdot))^{0.5}$ denotes the energy norm.

For completeness, we now rewrite the f.e. equations in the approximate discrete harmonic basis $\widetilde{\Psi}$. From (7) with $u = \Phi\widetilde{V}\underline{\widetilde{u}}$ and $v = \Phi\widetilde{V}\underline{\widetilde{v}}$, we obtain the system

$$\begin{pmatrix} S_C + T_C & K_{CI}(K_I^{-1} - B_I^{-T})K_I \\ K_I(K_I^{-1} - B_I^{-1})K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}}_C \\ \underline{\widetilde{u}}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C - K_{CI}B_I^{-T}\underline{f}_I \\ \underline{f}_I \end{pmatrix}$$

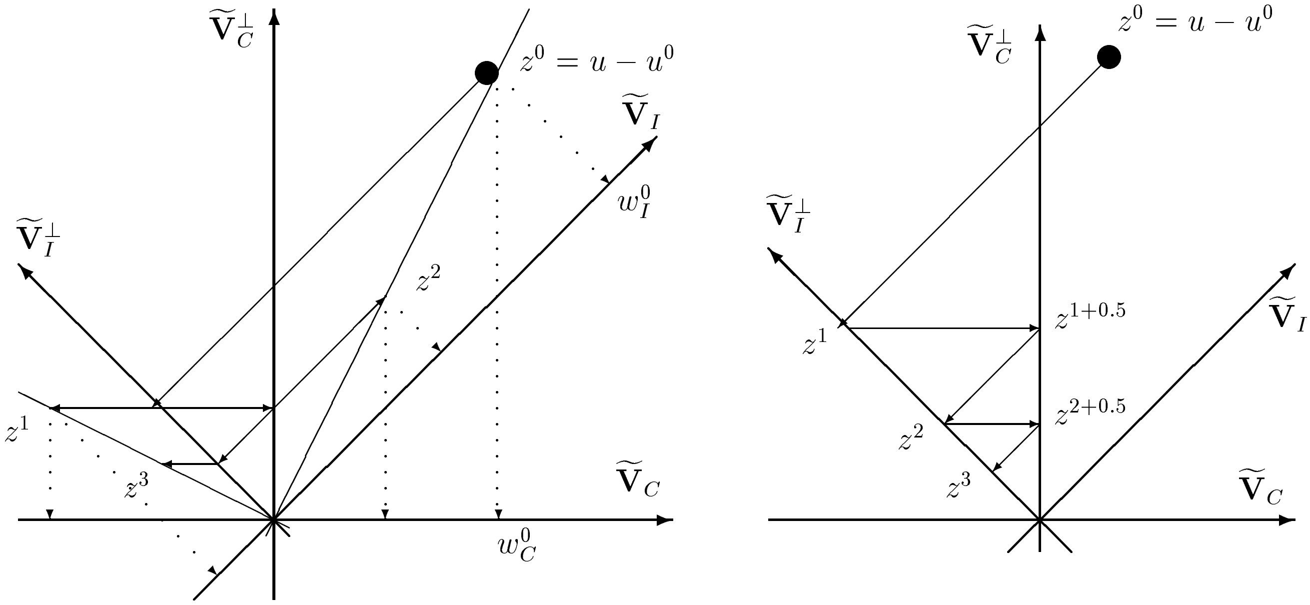
for determining the coefficient vector $\underline{\widetilde{u}} = \left(\underline{\widetilde{u}}_C^T, \underline{\widetilde{u}}_I^T\right)^T \in \mathbf{R}^N$ with respect to the approximate discrete harmonic basis $\widetilde{\Psi}$. As in the foregoing case, the coefficient vector $\underline{u} = \left(\underline{u}_C^T, \underline{u}_I^T\right)^T$ in the nodal basis can be obtained from the relation

$$\underline{u} = \widetilde{V}\underline{\widetilde{u}} = \begin{pmatrix} I_C & O \\ -B_I^{-1}K_{IC} & I_I \end{pmatrix} \begin{pmatrix} \underline{\widetilde{u}}_C \\ \underline{\widetilde{u}}_I \end{pmatrix} = \begin{pmatrix} \underline{\widetilde{u}}_C \\ \underline{\widetilde{u}}_I - B_I^{-1}K_{IC}\underline{\widetilde{u}}_C \end{pmatrix}.$$

4 ASM and MSM Based on the ADHB-Splitting

The decomposition $\mathbf{V} = \widetilde{\mathbf{V}}_C + \widetilde{\mathbf{V}}_I$ of the f.e. space $\mathbf{V}$ into the direct sum of the subspaces $\widetilde{\mathbf{V}}_C$ of the approximate discrete harmonic functions and $\widetilde{\mathbf{V}}_I$ of the subdomain functions uniquely defines a specific Additive Schwarz Method (ASM) as well as a specific Multiplicative Schwarz Method (MSM) which are illustrated in Fig. 1:

ASM MSM



$$\gamma = \cos \angle(\widetilde{\mathbf{V}}_C, \widetilde{\mathbf{V}}_I) = \sqrt{\frac{\mu}{1+\mu}}$$

$|||\, u - u^{k+1} \,||| \leq \gamma \,|||\, u - u^k \,|||$ $|||\, u - u^{k+1} \,||| \leq \gamma^2 \,|||\, u - u^k \,|||$

Figure 1: ASM and MSM corresponding to the ADHB-Splitting

We observe from Fig. 1, that the convergence rate of the MSM is much better than the rate of the ASM, namely γ^2 instead of γ.

However, let us first turn our attention to the ASM. Introducing the orthoprojectors

$$P_I : \mathbf{V} \longrightarrow \widetilde{\mathbf{V}}_I \quad \text{and} \quad P_C : \mathbf{V} \longrightarrow \widetilde{\mathbf{V}}_C$$

defined by the relations

$$a(P_I u, v) = a(u,v) \quad \forall v \in \mathbf{V}_I \quad \text{and} \quad a(P_C u, v) = a(u,v) \quad \forall v \in \mathbf{V}_C\,,\ \forall u \in \mathbf{V}\ ,$$

respectively, we can rewrite the ASM in the form

$$u^{k+1} = u^k + P_C z^k + P_I z^k \quad , \tag{14}$$

where $u^0 \in \mathbf{V}$ is some initial guess, $z^k = u - u^k$ denotes the k-th iteration error and $k = 0, 1, \ldots$. It was shown in [12] that the ASM (14) is nothing but the method of simple iterations for (8) with the preconditioner $D = \widetilde{V}^{-T} \widetilde{D} \widetilde{V}^{-1}$ and the iteration parameter $\tau = 1$, where $\widetilde{D} = diag\,(S_C + T_C, K_I)$. In other words, the ASM (14) is equivalent to the iteration process

$$\tau^{-1} D\,(\underline{u}^{k+1} - \underline{u}^k) + K\,\underline{u}^k = \underline{f}\ . \tag{15}$$

Lemma 3.1 implies the spectral equivalence inequalities

$$(1-\gamma)\,D \ \leq\ K \ \leq\ (1+\gamma)\,D\,, \tag{16}$$

where the spectral equivalence constants are sharp. Therefore, the relative spectral condition number $\kappa(D^{-1}K) = \lambda_{max}(D^{-1}K)\ /\ \lambda_{min}(D^{-1}K)$ is equal to $(1+\gamma)\,/\,(1-\gamma) = \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2$. Here and in the following, inequalities of the form (16) are to be understood in the sense of the Euclidian inner product. From (16) we obtain the optimal iteration parameter $\tau = \tau_{opt} = 2\,/\,(1+\gamma+1-\gamma) = 1$ and the corresponding optimal rate $q_{opt} = (1+\gamma-(1-\gamma))\,/\,(1+\gamma+1-\gamma) = \gamma$. Replacing the diagonal block entries $S_C + T_C$ and K_I of $\widetilde{D}$ by some appropriate symmetric and positive definite block preconditioners C_C and C_I, we arrive at the ASM-DD-preconditioner

$$C = \begin{pmatrix} I_C & K_{CI}B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1}K_{IC} & I_I \end{pmatrix} \tag{17}$$

studied in [12]. If the symmetric and positive definite matrices C_C and C_I satisfy the spectral equivalence inequalities

$$\underline{\gamma}_C C_C \leq S_C + T_C \leq \overline{\gamma}_C C_C \qquad \text{and} \qquad \underline{\gamma}_I C_I \leq K_I \leq \overline{\gamma}_I C_I$$

with positive constants $\underline{\gamma}_C$, $\overline{\gamma}_C$, $\underline{\gamma}_I$, $\overline{\gamma}_I$, then the ASM-DD-preconditioner C is spectral equivalent to K, i.e.

$$\underline{\gamma}\,C \ \leq\ K \ \leq\ \overline{\gamma}\,C\,,$$

with the spectral equivalence constants

$$\underline{\gamma} \,=\, min\{\,\underline{\gamma}_C\,,\,\underline{\gamma}_I\,\}\,\left(1 - \sqrt{\tfrac{\mu}{1+\mu}}\right) \ \text{ and } \ \overline{\gamma} \,=\, max\{\,\overline{\gamma}_C\,,\,\overline{\gamma}_I\,\}\,\left(1 + \sqrt{\tfrac{\mu}{1+\mu}}\right).$$

Furthermore, we have the estimates

$$(\,\underline{\gamma}_1\,/\,\overline{\gamma}_1)\left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \ \leq\ \kappa(C^{-1}K) \ \leq\ (\,\overline{\gamma}_1\,/\,\underline{\gamma}_1)\left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2$$

of relative spectral condition number from above and below, where $\underline{\gamma}_1 = min\{\,\underline{\gamma}_C\,,\,\underline{\gamma}_I\,\}$ and $\overline{\gamma}_1 = max\{\,\overline{\gamma}_C\,,\,\overline{\gamma}_I\,\}$ (see [12] for more details). In practice, the conjugate gradient method with the ASM-DD-preconditioner C is used instead of the method of simple iterations. In this context, the ASM should be considered as a method for determining preconditioners rather than a solution method. In [12], we propose a parallel PCG algorithm with the ASM-DD-preconditioner C which is suited for the implementation on a multiprocessor computer system with distributed memory and message passing principle. For completeness, we include this algorithm as Appendix.

Now we return to the MSM belonging to the decomposition $\mathbf{V} = \widetilde{\mathbf{V}}_C + \widetilde{\mathbf{V}}_I$. Using again the orthoprojections defined above, we can write the MSM in the form

$$\begin{array}{rll} & u^1 := u^0 + P_I z^0 \ , & u^{k+1} = (I-P)\, u^k + Pu \\ \text{and} \qquad & z^1 := (I-P_I)\, z^0 \ , & z^{k+1} = (I-P_I)\,(I-P_C)\, z^k \end{array}$$

for the iterates and the errors, respectively, where $u^0 \in \mathbf{V}$ is some initial guess, $P = P_I + P_C - P_I P_C$, $z^k = u - u^k$ and $k = 1, 2, \dots$. The orthoprojections into $\widetilde{\mathbf{V}}_C$ and $\widetilde{\mathbf{V}}_I$ require the solutions of algebraic system with the system matrices $S_C + T_C$ and K_I, respectively. The exact solution of these systems is, in general, quite expensive. In practice, it is necessary to replace the exact solution of these systems by a preconditioned solution or by an approximate solution via some efficient iterative method.

5 Symmetric MSM and its ASM Interpretation

Let us consider a symmetric modification of the MSM-algorithm with an approximate performance of the orthoprojections:

<u>Initialization :</u> Choose some initial guess

$$u^0 \in \mathbf{V} = \widetilde{\mathbf{V}}_C + \widetilde{\mathbf{V}}_I \tag{18}$$

<u>Iteration :</u> <u>for</u> k=0 <u>step</u> 1 <u>until</u> convergence <u>do</u>

- Determine

$$u^{k,1} = u^k + w_I^k \ , \tag{19}$$

  where $w_I^k \in \widetilde{\mathbf{V}}_I$ is an approximation to $v_I^k = P_I z^k$ such that

$$|||\, v_I^k - w_I^k \,||| \ \le\ \rho_I \,|||\, v_I^k \,||| \ , \tag{20}$$

  with $z^k = u - u^k$ and some fixed accuracy $\rho_I \in [\,0, 1)$.

- Determine

$$u^{k,2} = u^{k,1} + w_C^{k,1} \ , \tag{21}$$

  where $w_C^{k,1} \in \widetilde{\mathbf{V}}_C$ is an approximation to $v_C^{k,1} = P_C z^{k,1}$ such that

$$|||\, v_C^{k,1} - w_C^{k,1} \,||| \ \le\ \rho_C \,|||\, v_C^{k,1} \,||| \ , \tag{22}$$

  with $z^{k,1} = u - u^{k,1}$ and some fixed accuracy $\rho_C \in [\,0, 1)$.

- Determine

$$u^{k+1} = u^{k,2} + w_I^{k,2} \ , \tag{23}$$

  where $w_I^{k,2} \in \widetilde{\mathbf{V}}_I$ is an approximation to $v_I^{k,2} = P_I z^{k,2}$ such that

$$|||\, v_I^{k,2} - w_I^{k,2} \,||| \ \le\ \rho_I \,|||\, v_I^{k,2} \,||| \ , \tag{24}$$

  with $z^{k,2} = u - u^{k,2}$.

<u>end for</u>

THEOREM 5.1 *The symmetric MSM-algorithm (18)-(24) with approximate orthoprojections converges in the energy norm* $||| \cdot |||$ *such that the error estimate*

$$||| \, u - u^n \, ||| \;\leq\; q^n \, ||| \, u - u^0 \, ||| \tag{25}$$

holds for the n-th iterate, where

$$q = \rho_C + (1 - \rho_C) \left(\rho_I + (1 - \rho_I) \sqrt{\mu \,/\, (1 + \mu)} \right)^2 \;\in [\,0, 1) \tag{26}$$

for ρ_C *and* $\rho_I \in [\,0, 1)$.

Proof. The proof of this theorem was given by N. Schieweck in connection with projection-type multigrid methods (see Theorem 2.2 in [22] p.51). ∎

REMARK 5.1 *In the symmetric MSM-algorithm (18)-(24) the arrangement of the approximate orthoprojections can be changed. In this case, we must also change* ρ_C *and* ρ_I *in the rate formula (26).*

The orthoprojections $v_I^k = P_I z^k$, $v_C^{k,1} = P_C z^{k,1}$ and $v_I^{k,2} = P_I z^{k,2}$ are related to the solution of algebraic systems with the system matrices K_I , $S_C + T_C$ and K_I, respectively. More precisely, we have the relations

$$\begin{aligned}
v_I^k \;&\equiv\; \Phi \widetilde{V}_I \underline{\widetilde{v}}_I^k = P_I z^k \;\; \longleftrightarrow \;\; \underline{\widetilde{v}}_I^k \in \mathbf{R}^{N_I} : \; K_I \underline{\widetilde{v}}_I^k = \widetilde{V}_I^T \underline{r}^k \, , && (27)\\
v_C^{k,1} \;&\equiv\; \Phi \widetilde{V}_C \underline{\widetilde{v}}_C^{k,1} = P_C z^{k,1} \;\; \longleftrightarrow \;\; \underline{\widetilde{v}}_C^{k,1} \in \mathbf{R}^{N_C} : \; (S_C + T_C) \underline{\widetilde{v}}_C^{k,1} = \widetilde{V}_C^T \underline{r}^{k,1} \, , && (28)\\
v_I^{k,2} \;&\equiv\; \Phi \widetilde{V}_I \underline{\widetilde{v}}_I^{k,2} = P_I z^{k,2} \;\; \longleftrightarrow \;\; \underline{\widetilde{v}}_I^{k,2} \in \mathbf{R}^{N_I} : \; K_I \underline{\widetilde{v}}_I^{k,2} = \widetilde{V}_I^T \underline{r}^{k,2} \, , && (29)
\end{aligned}$$

with $\underline{r}^k = K \underline{z}^k = \underline{f} - K \underline{u}^k$ and $\underline{r}^{k,j} = K \underline{z}^{k,j} = \underline{f} - K \underline{u}^{k,j}$, $j = 1, 2$. Therefore, $\underline{\widetilde{w}}_I^k \longleftrightarrow w_I^k = \Phi \widetilde{V}_I \underline{\widetilde{w}}_I^k$, $\underline{\widetilde{w}}_C^{k,1} \longleftrightarrow w_C^{k,1} = \Phi \widetilde{V}_C \underline{\widetilde{w}}_C^{k,1}$ and $\underline{\widetilde{w}}_I^{k,2} \longleftrightarrow w_I^{k,2} = \Phi \widetilde{V}_I \underline{\widetilde{w}}_I^{k,2}$ can be considered as approximate solutions of the systems (27), (28) and (29), respectively. Such approximate solutions can be obtained by iterative methods, including variational-type iterative methods such as the PCG methods, with the zero initial guess. Note that the matrix

$$S_C + T_C = K_C - K_{CI} B_I^{-1} K_{IC} - K_{CI} B_I^{-T} K_{IC} + K_{CI} B_I^{-T} K_I B_I^{-1} K_{IC} \tag{30}$$

does not involve the matrix K_I^{-1} as the Schur complement S_C does. Thus, both products $K_I * \underline{v}_I$ and $(S_C + T_C) * \underline{v}_C$ are easily available provided, in the latter case, that the actions $B_I^{-1} \underline{v}_I$ and $B_I^{-T} \underline{v}_I$ are cheap as was assumed above.

Let us now suppose that

$$\underline{\widetilde{w}}_I^k = C_I^{-1} \widetilde{V}_I \, \underline{r}^k \, , \; \underline{\widetilde{w}}_C^{k,1} = C_C^{-1} \widetilde{V}_C \, \underline{r}^{k,1} \quad \text{and} \quad \underline{\widetilde{w}}_I^{k,2} = C_I^{-1} \widetilde{V}_I \, \underline{r}^{k,2} \, , \tag{31}$$

where C_I and C_C are symmetric and positive definite preconditioners for K_I and $S_C + T_C$, respectively. Then the symmetric MSM-algorithm (18) - (24), (31) can be rewritten in the form

$$\underline{u}^{k+1} = (I - C^{-1} K) \, \underline{u}^k + C^{-1} \underline{f} \tag{32}$$

of the simple iteration method with repect to the nodal basis (i.e. $u^j = \Phi \underline{u}^j$), where

$$C^{-1} \;=\; U \begin{pmatrix} C_C^{-1} & O \\ O & 2C_I^{-1} - C_I^{-1} K_I C_I^{-1} \end{pmatrix} U^T \,, \tag{33}$$

with

$$U \;=\; \tilde{V} \begin{pmatrix} I_C & O \\ -C_I^{-1} \tilde{V}_I^T K \tilde{V}_C & I_I \end{pmatrix} \;=\; \begin{pmatrix} I_C & O \\ -(B_I^{-1} + C_I^{-1} - C_I^{-1} K_I B_I^{-1}) K_{IC} & I_I \end{pmatrix} .$$

Indeed, setting $\underline{f} = O$ or considering the iteration scheme for the error $\underline{z}^k = \underline{u} - \underline{u}^k$, we obtain

$$\begin{aligned} \underline{z}^{k+1} &= \underline{z}^{k,2} - \tilde{V}_I \underline{w}^{k,2} \;=\; \underline{z}^{k,2} - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \underline{z}^{k,2} \\ &= (I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K)\, \underline{z}^{k,2} \\ &= (I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K)\,(I - \tilde{V}_C C_C^{-1} \tilde{V}_C^T K)\, \underline{z}^{k,1} \\ &= (I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K)\,(I - \tilde{V}_C C_C^{-1} \tilde{V}_C^T K)\,(I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K)\, \underline{z}^k \,. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I - C^{-1} K &= (I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K)\,(I - \tilde{V}_C C_C^{-1} \tilde{V}_C^T K)\,(I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K) \\ &= I - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K - \tilde{V}_C C_C^{-1} \tilde{V}_C^T K \\ &\quad + \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \tilde{V}_C C_C^{-1} \tilde{V}_C^T K - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \\ &\quad + \tilde{V}_I C_I^{-1} K_I C_I^{-1} \tilde{V}_I^T K + \tilde{V}_C C_C^{-1} \tilde{V}_C^T K \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \\ &\quad - \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \tilde{V}_C C_C^{-1} \tilde{V}_C^T K \tilde{V}_I C_I^{-1} \tilde{V}_I^T K \,. \end{aligned}$$

Writing

$$\tilde{C}_I^{-1} \;=\; 2C_I^{-1} - C_I^{-1} K_I C_I^{-1} + C_I^{-1} \tilde{V}_I^T K \tilde{V}_C C_C^{-1} \tilde{V}_C^T K \tilde{V}_I C_I^{-1} \quad ,$$

this relation immediately implies that

$$\begin{aligned} C^{-1} &= \tilde{V} \begin{pmatrix} C_C^{-1} & -C_C^{-1} \tilde{V}_C^T K \tilde{V}_I C_I^{-1} \\ -C_I^{-1} \tilde{V}_I^T K \tilde{V}_C C_C^{-1} & \tilde{C}_I^{-1} \end{pmatrix} \tilde{V}^T \\ &= \tilde{V} \begin{pmatrix} I_C & O \\ -C_I^{-1} \tilde{V}_I^T K \tilde{V}_C & I_I \end{pmatrix} \begin{pmatrix} C_C^{-1} & O \\ O & 2C_I^{-1} - C_I^{-1} K_I C_I^{-1} \end{pmatrix} \begin{pmatrix} I_C & -\tilde{V}_C^T K \tilde{V}_I C_I^{-1} \\ O & I_I \end{pmatrix} \tilde{V}^T , \end{aligned}$$

which proves relation (33). Note that C^{-1} is positive definite if and only if the eigenvalues $\lambda(C_I^{-1} K_I)$ of $C_I^{-1} K_I$ belong to the interval $[\,0, 2\,)$. The latter condition is equivalent to the fact that the spectral radius $\rho(I_I - C_I^{-1} K_I)$ of $I_I - C_I^{-1} K_I$ is less than 1. Block preconditioners satisfying one of the two equivalent conditions are called scaled preconditioners. We suppose in our studies that both block preconditioners C_I and C_C are scaled, i.e. $\rho(I_I - C_I^{-1} K_I) \;<\; 1$ and $\rho(I_C - C_C^{-1}(S_C + T_C)) \;<\; 1$.

Let us now define the basis transformation B_I and the block preconditioners C_I and C_C via iterative methods for the corresponding subproblems (see [12] for more details), i.e.

$$B_I = K_I \left(I_I - \overline{M_I}^r \right)^{-1} \quad , \quad B_I^T = K_I \left(I_I - \overset{*}{\overline{M_I}^r} \right)^{-1} , \tag{34}$$

$$C_I = K_I \; (I_I - M_I^s\,)^{-1} \quad , \quad C_C = (S_C + T_C) \left(I_C - Q_C^t \right)^{-1} , \tag{35}$$

where $\overline{M_I}, M_I, Q_I$ and r, s, t denote, respectively, the iteration operators and the iteration numbers of the iteration processes applied. In all cases, the starting guess is the zero vector. $\overset{*}{\overline{M_I}}$ denotes that operator which is adjoint to $\overline{M_I}$ in the K_I-energy inner product $(\cdot,\cdot)_{K_I} = (K_I\cdot,\cdot)$, i.e.

$$(\overline{M_I}\underline{u}_I, \underline{v}_I)_{K_I} = (\underline{u}_I, \overset{*}{\overline{M_I}}\underline{v}_I)_{K_I} \quad \forall \underline{u}_I, \underline{v}_I \in \mathbf{R}^{N_I} . \tag{36}$$

The iteration operators $M_I = \overset{*}{M_I}$ and $Q_C = \overset{*}{Q_C}$ are supposed to be self-adjoint in the K_I- and (S_C+T_C)- energy inner products, respectively. This property ensures the symmetry of the block preconditioners C_I and C_C. In [12], several examples for B_I, C_I and C_C of the type (34)-(35) were given. In particular, multigrid techniques are well suited in order to define the basis transformation B_I and the block preconditioner C_I. Note that the polygonal preconditioner $C_C = (S_C+T_C)\,(I_C - Q_C^t)^{-1}$ involves the matrix-by-vector multiplication $(S_C + T_C) * \underline{v}_C$ which is cheap as was already mentioned above. Indeed, inserting (34) into (30), we obtain

$$\begin{aligned}(S_C+T_C)\,\underline{v}_C &= \Big(K_C - K_{CI}K_I^{-1}K_{IC} + K_{CI}\,(K_I^{-1} - B_I^{-T})\,K_I\,(K_I^{-1} - B_I^{-1})\,K_{IC}\Big)\,\underline{v}_C \\ &= \Big(K_C - K_{CI}\overline{B_I}^{-1}K_{IC}\Big)\,\underline{v}_C \quad ,\end{aligned}$$

with

$$\overline{B_I} = K_I\,\left(I_I - \overset{*}{\overline{M_I}}{}^r\,\overline{M_I}^r\right)^{-1} .$$

Therefore, the operation $(S_C+T_C) * \underline{v}_C$ involves the application of r iteration steps with the iteration operator $\overline{M_I}$ followed by r steps with the adjoint iteration operator $\overset{*}{\overline{M_I}}$ to the system

$$K_I\underline{w}_I = K_{IC}\underline{v}_C$$

with the initial guess $\underline{w}_I^0 = O$, obtaining $\underline{w}_I^{2r} = \overline{B_I}^{-1}K_{IC}\underline{v}_C$.

Now we are in a position to formulate our main theorem.

THEOREM 5.2 *Suppose that the operator B_I involved in the basis transformation matrix $\widetilde{V}$ is defined by (34) and the preconditioners C_I and C_C are given by (35), where the iteration operators $\overline{M_I}$, M_I and Q_C satisfy the following conditions:*

1. *M_I and Q_C are self-adjoint in the K_I- and (S_C+T_C)-energy inner product, respectively.*
2. $\| \overline{M_I} \|_{K_I} \le \overline{\eta} < 1$.
3. $\varrho(M_I) = \| M_I \|_{K_I} \le \eta < 1$.
4. $\varrho(Q_C) = \| Q_C \|_{S_C+T_C} \le \nu < 1$.

Then the symmetric MSM-algorithm (18)-(24) with (31) is nothing but the method of simple iteration (32) with

$$\begin{aligned} C &= \begin{pmatrix} I_C & K_{CI}(I_I - \overset{*}{\overline{M_I}}{}^r M_I^s)K_I^{-1} \\ O & I_I \end{pmatrix} * \begin{pmatrix} C_C & O \\ O & K_I(I_I - M_I^{2s})^{-1} \end{pmatrix} \\ &\quad * \begin{pmatrix} I_C & O \\ (I_I - M_I^s\overline{M_I}^r)K_I^{-1}K_{IC} & I_I \end{pmatrix} . \end{aligned} \tag{37}$$

Furthermore, the matrix C is symmetric, positive definit and spectrally equivalent to K, i.e.

$$(1-q)\,C \;\leq\; K \;\leq\; (1+q)\,C\,, \tag{38}$$

where q is defined by (26) with $\rho_C = \nu^t$ and $\rho_I = \eta^s$ and

$$\mu \;=\; \rho(S_C^{-1}T_C) \;=\; \sup_{\underline{v}\in \overset{*}{V}_C \mathrm{R}^{N_C}\setminus\{O\}} \frac{\left\| \begin{pmatrix} O & O \\ O & \overline{M_I}^r \end{pmatrix} \underline{v} \right\|_K^2}{\|\,\underline{v}\,\|_K^2} \;=\; \left\| \begin{pmatrix} O & O \\ O & \overline{M_I} \end{pmatrix}^r \Big|_{\overset{*}{V}_C \mathrm{R}^{N_C}} \right\|_K^2 .$$

Proof. It follows from the representation (33) of C^{-1} in the general case, that we have to analyse the operators $B^{-1} + C^{-1} - C^{-1}K_I B^{-1}$ and $2C^{-1} - C^{-1}K_I C^{-1}$ for our special choice (34)-(35). Using (34)-(35), we obtain

$$\begin{array}{rl}
\text{a)} & 2C_I^{-1} - C_I^{-1}K_I C_I^{-1} = \\
= & 2(I_I - M_I^s)K_I^{-1} - (I_I - M_I^s)K_I^{-1}K_I(I_I - M_I^s)K_I^{-1} \\
= & 2(I_I - M_I^s)K_I^{-1} - (I_I - 2M_I^s + M_I^{2s})K_I^{-1} \\
= & (I_I - M_I^{2s})K_I^{-1} \qquad\qquad\qquad \text{and} \\
\text{b)} & B_I^{-1} + C_I^{-1} - C_I^{-1}K_I B_I^{-1} = \\
= & (I_I - \overline{M_I}^r)K_I^{-1} + (I_I - M_I^s)K_I^{-1} - (I_I - M_I^s)K_I^{-1}K_I(I_I - \overline{M_I}^r)K_I^{-1} \\
= & (I_I - \overline{M_I}^r + I_I - M_I^s - (I_I - M_I^s - \overline{M_I}^r + M_I^s\overline{M_I}^r))K_I^{-1} \\
= & (I_I - M_I^s\overline{M_I}^r)K_I^{-1} \qquad .
\end{array}$$

Taking into account these relations, the representation (33) and the formula $K_I^{-1}\overline{M}^T = \overset{*}{\overline{M}}K_I^{-1}$ which is equivalent to (36), we immediately arrive at the representation (37) of C.

Now we proof that $\rho_C = \nu^t$ and $\rho_I = \eta^s$. Indeed, using (27) and (31), we obtain the estimates

$$\begin{array}{rcl}
|||\, v_I^k - w_I^k \,||| & = & \left(a\,(\,\Phi\tilde{V}_I(\underline{\tilde{v}}_I^k - \underline{\tilde{w}}_I^k)\,,\; \Phi\tilde{V}_I(\underline{\tilde{v}}_I^k - \underline{\tilde{w}}_I^k)\,)\right)^{0.5} \\
& = & \|\, \underline{\tilde{v}}_I^k - \underline{\tilde{w}}_I^k \,\|_{K_I} = \|\, K_I^{-1}\tilde{V}_I^T\underline{r}^k - C_I^{-1}\tilde{V}_I^T\underline{r}^k \,\|_{K_I} \\
& = & \|\, K_I^{-1}\tilde{V}_I^T\underline{r}^k - (I_I - M_I^s)\,K_I^{-1}\tilde{V}_I^T\underline{r}^k \,\|_{K_I} \\
& = & \|\, M_I^s\underline{\tilde{v}}_I^k \,\|_{K_I} \leq \|\, M_I \,\|_{K_I}^s \,\|\, \underline{\tilde{v}}_I^k \,\|_{K_I} \\
& \leq & \eta^s \;\|\, \underline{\tilde{v}}_I^k \,\|_{K_I} = \eta^s \,|||\, v_I^k \,||| \quad \text{, i.e. } \rho_I = \eta^s
\end{array}$$

Analogously,

$$\begin{array}{rcl}
|||\, v_C^{k,1} - w_C^{k,1} \,||| & = & \|\, \underline{\tilde{v}}_C^{k,1} - \underline{\tilde{w}}_C^{k,1} \,\|_{S_C+T_C} \leq \|\, Q_C \,\|_{S_C+T_C}^t \,\|\, \underline{\tilde{v}}_C^{k,1} \,\|_{S_C+T_C} \\
& \leq & \nu^t \;\|\, \underline{\tilde{v}}_C^{k,1} \,\|_{S_C+T_C} \;=\; \nu^t \,|||\, v_C^{k,1} \,||| \quad \text{, i.e. } \rho_C = \nu^t.
\end{array}$$

Now let us show the spectral equivalence inequalities (38). From Theorem (5.1) and from well-known convergence rate estimates for the method (32) of simple iterations (cf., e.g., [20] with $\tau = 1$), we conclude that

$$max\{|1 - \lambda_{min}(C^{-1}K)\,|\;,\;|1 - \lambda_{max}(C^{-1}K)\,|\} \;\leq\; q \quad , \tag{39}$$

where $\lambda_{min}(C^{-1}K)$ and $\lambda_{max}(C^{-1}K)$ denote the minimal and maximal eigenvalue of $C^{-1}K$, respectively. Inequality (39) is obviously equivalent to the inequalities

$$1-q \;\leq\; \lambda_{min}(C^{-1}K) \;\leq\; \lambda_{max}(C^{-1}K) \;\leq\; 1+q$$

which prove the spectral equivalence inequalities (38). ∎

REMARK 5.2 *For plane, 2-nd order elliptic b.v.p., one can establish the rough estimate*

$$\mu \;\leq\; \overline{\eta}^{2r}(\overline{c}_2 \underline{c}_1^{-1} h^{-1} - 1)$$

implying $r = O(\ln h^{-1})$ *iterations in order to obtain an h-independent bound on* μ *provided that* $\overline{\eta}$ *is h-independent as in the case of multigrid methods [3][12] [13]. The constants* $\underline{c}_1$ *and* $\overline{c}_2$ *were taken from the well-known inequalities [6]*

$$\underline{c}_1 h I_C \leq S_C \leq \overline{c}_1 I_C \qquad \textit{and} \qquad \underline{c}_2 I_C \leq K_C \leq \overline{c}_2 I_C \;.$$

However, in practice one or two multigrid iterations (i.e. $r=1$ *or* 2*) are sufficient to obtain an h-independent* μ *(see numerical results in [12] and conclusions 5 in Section 5 of [13]).*

COROLLARY 5.1 *The symmetric MSM-algorithm (18)-(24) with (31) defined by (34) and (35) is equivalent to the ASM (cf. Section 4)*

$$\tau^{-1} C\,(\underline{u}^{k+1} - \underline{u}^k) + K\underline{u}^k \;=\; \underline{f} \tag{40}$$

with $\tau = 1$ *and the ASM-DD-preconditioner (17) the components of which are defined by the relations*

$$\left.\begin{array}{rcll} B_I &=& K_I\,(I_I - M_I^s \overline{M_I}^r)^{-1} & , \\ C_I &=& K_I\,(I_I - M_I^{2s})^{-1} & , \\ C_C &=& (S_C + T_C)\,(I_C - Q_C^t)^{-1} & . \end{array}\right\} \tag{41}$$

Therefore, in this case the symmetric MSM is nothing but the ASM with specially chosen basis transformation matrix B_I and block preconditioners C_I and C_C. From this observation we can benefit twice. Firstly, we have the spectral equivalence inequalities (38) based on the MSM rate estimate (26) from Theorem (5.1). Secondly, we can exploite the special form (37) of the MSM-DD-preconditioner C when solving the preconditioning system $C\underline{w}^k = \underline{r}^k$ in the DD-Preconditioned Parallel CG-method given in the Appendix. For the general ASM-DD preconditioner (17), the preconditioning equation $\underline{w}^k = C^{-1}\underline{r}^k$ can be rewritten in the form

$$\underline{w}_C^k \;=\; C_C^{-1}\,(\underline{r}_C^k - K_{CI} B_I^{-T} \underline{r}_I^k) \tag{42}$$

$$\underline{w}_I^k \;=\; C_I^{-1} \underline{r}_I^k - B_I^{-1} K_{IC} \underline{w}_C^k \tag{43}$$

In the case $B_I = C_I$ studied in [13], [19], [14] the second equation (43) simplifies in the sense that only one action of C_I^{-1} is necessary, i.e.

$$\underline{w}_I^k = C_I^{-1}\,(\underline{r}_I^k - K_{IC}\underline{w}_C^k)\;.$$

Similary, we can take advantage of the special form (41) of B_I, C_I and C_C in case of the MSM-DD preconditioner (37). Indeed, using (41), we can rewrite the first preconditioning equation (42) in the form

$$\left.\begin{array}{lllll} 1) & \underline{w}_I^{k,s} & = & (I_I - M_I^s)\, K_I^{-1} \underline{r}_I^k & , \\ 2) & \underline{w}_I^{k,s+r} & \equiv & B_I^{-T} \underline{r}_I^k \, = \overset{*}{\overline{M_I}}{}^r \underline{w}_I^{k,s} + (I_I - \overset{*}{\overline{M_I}}{}^r)\, K_I^{-1} \underline{r}_I^k & , \\ 3) & \underline{w}_C^k & = & C_C^{-1}\, (\underline{r}_C^k - K_{CI} \underline{w}_I^{k,s+r}) & , \end{array}\right\} \tag{44}$$

and the second preconditioning equation (43) in the form

$$\left.\begin{array}{lllll} 4) & \underline{w}_I^{k,r} & = & (I_I - \overline{M_I}^r)\, K_I^{-1}\, (-K_{IC} \underline{w}_C^k) & , \\ 5) & \underline{\hat{r}}_I^k & = & \underline{r}_I^k - K_{IC} \underline{w}_C^k & , \\ 6) & \underline{\hat{w}}_I^k & = & \underline{w}_I^{k,s} + \underline{w}_I^{k,r} & , \\ 7) & \underline{w}_I^k & = & M_I^s \underline{\hat{w}}_I^k + (I_I - M_I^s)\, K_I^{-1} \underline{\hat{r}}_I^k & . \end{array}\right\} \tag{45}$$

Let us now verify that equations 4)-7) from (45) are equivalent to (43) with (41). Inserting 1) and 4)-6) into 7), we obtain

$$\begin{aligned} \underline{w}_I^k &= M_I^s \left[\underline{w}_I^{k,s} + \underline{w}_I^{k,r} \right] + (I_I - M_I^s)\, K_I^{-1} \left[\underline{r}_I^k - K_{IC} \underline{w}_C^k \right] \\ &= M_I^s \left[(I_I - M_I^s)\, K_I^{-1} \underline{r}_I^k + (I_I - \overline{M_I}^r)\, K_I^{-1}\, (-K_{IC} \underline{w}_C^k) \right] \\ &\quad + (I_I - M_I^s)\, K_I^{-1} \left[\underline{r}_I^k - K_{IC} \underline{w}_C^k \right] \\ &= \left[M_I^s (I_I - M_I^s) + (I_I - M_I^s) \right] K_I^{-1} \underline{r}_I^k \\ &\quad + \left[M_I^s (I_I - \overline{M_I}^r) + (I_I - M_I^s) \right] K_I^{-1} (-K_{IC} \underline{w}_C^k) \\ &= \left[I_I - M_I^{2s} \right] K_I^{-1} \underline{r}_I^k + \left[I_I - M_I^s \overline{M_I}^r \right] K_I^{-1} (-K_{IC} \underline{w}_C^k) \\ &= C_I^{-1} \underline{r}_I^k - B_I^{-1} K_{IC} \underline{w}_C^k \,. \end{aligned}$$

The equivalence of equation 1)-3) from (44) to (42) with (41) is obvious.

Equation 1) means the application of s M_I-iteration steps (s iteration steps with the iteration operator M_I) to

$$K_I \, \underline{w}_I \; = \; \underline{r}_I^k \tag{46}$$

using the initial guess $\underline{w}_I^{k,0} = O$, whereas equation 2) corresponds to r $\overset{*}{\overline{M_I}}$-iteration-steps applied to (46) with the initial guess $\underline{w}_I^{k,s}$. Analogeously, equations 4) and 7) are tantamount to the application of r $\overline{M_I}$-iteration-steps to

$$K_I \, \underline{w}_I \; = \; -K_{IC} \underline{w}_C^k$$

with the initial guess $\underline{w}_I^{k,0} = O$ and to the appliocation of s M_I-iteration-steps to

$$K_I \, \underline{\hat{w}}_I \; = \; \underline{\hat{r}}_I^k$$

with the initial guess $\underline{\hat{w}}_I^k$, respectively. Therefore, we economize $2s$ iteration steps defined by the selfadjoint iteration operator M_I. In the Appendix, we have to replace the general preconditioning equations by the MSM preconditioning equations 1)-7) using the data distribution introduced there. The numerical experiments performed show the efficiency of the MSM-DD-preconditioner studied in this paper.

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APPENDIX

A The DD-Preconditioned Parallel Conjugate Gradient Method

Let us now describe a parallel version of the PCG-algorithm with the DD-preconditioner (17) for solving (2) well suited for parallel computation on MIMD computers with local memory and message passing (see also [12],[13],[16],[19]). In correspondence to the p subdomains $\overline{\Omega}_i$, we distribute all vectors and matrices involved in the DD-PCG-algorithm to the p processors P_i indicated by the index $i = 1, 2, \ldots, p$. For the vectors, we use two types of distribution called overlapping (type I) and adding (type II) :

type I : $\underline{\mathbf{u}}, \underline{\mathbf{w}}, \underline{\mathbf{s}}$ are stored in P_i ($\hat{=}\overline{\Omega}_i$) as $\underline{\mathbf{u}}_i = A_i\underline{\mathbf{u}}$, $\underline{\mathbf{w}}_i = A_i\underline{\mathbf{w}}$ and $\underline{\mathbf{s}}_i = A_i\underline{\mathbf{s}}$;
type II : $\underline{r}, \underline{v}, \underline{f}$ are stored inP_i as $\underline{r}_i, \underline{v}_i, \underline{f}_i$ such that $\underline{r} = \sum\limits_{i=1}^{p} A_i^T \underline{r}_i$ etc.,

where A_i^T is a $(N_i \times N)$ Boolean matrix called also superelement (subdomain) connectivity matrix which maps some overall vector of nodal variables into the superelement vector of variables associated with the subdomain $\overline{\Omega}_i$ only ($i = 1, 2, \ldots, p$). Introducing the notations $K = \sum_{i=1}^{p} A_i^T K_i A_i$, $K_{IC,i} = (K_i)_{IC}$, $A_{C,i} = (A_i)_C$ etc., we can now formulate the parallelized DD-PCG algorithm as follows:

For all $i = 1, 2, \ldots, p$ do in parallel

0. Start step

Define $\underline{\mathbf{u}}_i$, e.g., $\underline{\mathbf{u}}_i = O$

$$
\begin{array}{lcl}
\underline{r}_i & = & \underline{f}_i - K_i \underline{\mathbf{u}}_i \\
\underline{\mathbf{w}}_C & = & C_C^{-1} \sum A_{C,i}^T (\underline{r}_{C,i} - K_{CI,i} B_{I,i}^{-T} \underline{r}_{I,i}) \\
\underline{\mathbf{w}}_{I,i} & = & C_{I,i}^{-1} \underline{r}_{I,i} - B_{I,i}^{-1} K_{IC,i} \underline{\mathbf{w}}_{C,i} \\
\underline{\mathbf{s}}_i & = & \underline{\mathbf{w}}_i \\
\sigma_i & = & \underline{\mathbf{w}}_i^T \underline{r}_i \\
\sigma^0 & = & \sum \sigma_i
\end{array}
$$

Iteration

$$
\begin{array}{llcl}
1. & \underline{v}_i & = & K_i \underline{\mathbf{s}}_i \\
 & \delta_i & = & \underline{v}_i^T \underline{\mathbf{s}}_i \\
 & \delta & = & \sum \delta_i \\
 & \alpha & = & \sigma \,/\, \delta \\
2. & \hat{\underline{\mathbf{u}}}_i & = & \underline{\mathbf{u}}_i + \alpha \underline{\mathbf{s}}_i \\
 & \hat{\underline{r}}_i & = & \underline{r}_i - \alpha \underline{v}_i \\
3. & \hat{\underline{\mathbf{w}}}_C & = & C_C^{-1} \sum A_{C,i}^T (\hat{\underline{r}}_{C,i} - K_{CI,i} B_{I,i}^{-T} \hat{\underline{r}}_{I,i}) \\
 & \hat{\underline{\mathbf{w}}}_{I,i} & = & C_{I,i}^{-1} \hat{\underline{r}}_{I,i} - B_{I,i}^{-1} K_{IC,i} \hat{\underline{\mathbf{w}}}_{C,i} \\
4. & \sigma_i & = & \hat{\underline{\mathbf{w}}}_i^T \hat{\underline{r}}_i \\
 & \hat{\sigma} & = & \sum \sigma_i \\
 & \beta & = & \hat{\sigma} / \sigma \\
5. & \hat{\underline{\mathbf{s}}}_i & = & \hat{\underline{\mathbf{w}}}_i + \beta \underline{\mathbf{s}}_i \\
6. & \hat{\sigma} & \leq & \varepsilon^2 * \sigma^0 \quad ? \quad \xrightarrow{\text{no}} \quad \text{goto 1.}
\end{array}
$$

$\downarrow$ yes

stop

In the above algorithm the symbol " ˆ " marks the new iterates, and the sum sign $\sum = \sum_{i=1}^{p}$ means both summing up and transfer operations. From the point of view of parallelization, the most crucial step in the above algorithm is the performance of the block preconditioning equation $\hat{\underline{\mathbf{w}}}_C = C_C^{-1} \sum A_{C,i}^T(\ldots)$. Some preconditioners C_C suited for parallel computers with MIMD architecture are presented in [8], [12], [6], [13], [19] for 2nd-order elliptic b.v.p.

In step 6, the accuracy test is given with respect to the $KC^{-1}K$-energy norm. The defect (K^2-energy) norm test is also feasible. However, theoretical estimates of the relative error of the CG-iterations are made in the K-energy norm (see, e.g., [20]). Indeed, if the spectral equivalence inequalities

$$\underline{\gamma} C \ \leq \ K \ \leq \ \overline{\gamma} C$$

are valid, then the error estimate

$$\| \underline{\mathbf{u}} - \underline{\mathbf{u}}^k \|_K \ \leq \ q_k \ \| \underline{\mathbf{u}} - \underline{\mathbf{u}}^0 \|_K$$

holds, where $\underline{\mathbf{u}} = \left(\underline{\mathbf{u}}_C^T, \underline{\mathbf{u}}_I^T\right)^T$ is the exact solution of system (2), $\underline{\mathbf{u}}^k$ denotes the k-th iterate of the DD-PCG method $\|\cdot\|_K = (K\,\cdot, \cdot)^{0.5}$ is the K-energy norm, $q_k = 2\rho^k \,/\, (1+\rho^{2k})$, $\rho = (1-\xi^{0.5}) \,/\, (1+\xi^{0.5})$, and $\xi = \underline{\gamma} \,/\, \overline{\gamma}$. Thus, in order to reduce the initial error $\underline{\mathbf{u}} - \underline{\mathbf{u}}^0$ by the factor ε $(0 < \varepsilon < 1)$ in the K-energy norm, no more than $k = \sharp \ln(\varepsilon^{-1} + (\varepsilon^{-2}+1)^{0.5}) \,/\, \ln(\rho^{-1}) \sharp$ iterations are required. Here $\sharp x \sharp$ denotes the smallest integer greater or equal to x.