# 1

# Multilevel Extension Techniques in Domain Decomposition Preconditioners

GUNDOLF HAASE

One component in Additive Schwarz Method (ASM) Domain Decomposition (DD) preconditioners [BPS89, SBG96] using inexact subdomain solvers [Boe89, HLM91] consists in an operator extending the boundary data into the interior of each subdomain, i.e. a homogeneous extension with respect to the differential operator given in that subdomain. This paper is concerned with the construction of cheap extension operators using multilevel nodal bases [Yse86, Xu89, BPX90, Osw94] from an implementation viewpoint. Additional smoothing sweeps in the extension operators further improve the condition number of the preconditioned system. The paper summarizes and improves results given in [HLMN94, Nep95, Haa97].

## 1.1 The ASM-DD-Preconditioner

Consider the symmetric, $\mathbb{V}_0 = \overset{\circ}{\mathbb{H}}{}^1(\Omega)$–elliptic and $\mathbb{V}_0$–bounded variational problem

$$\text{find } u \in \mathbb{V}_0 : \int_\Omega \lambda(x)\, \nabla^T u(x)\, \nabla v(x)\, dx = \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in \mathbb{V}_0\,, \quad (1.1)$$

arising from the weak formulation of a scalar second–order, symmetric and uniformly bounded elliptic boundary value problem given in a plane bounded domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. The material coefficients $\lambda(x) \geq \lambda_0 > 0 \quad \forall x \in \overline{\Omega}$ have to be restricted for certain extension techniques.

The domain $\Omega$ will be decomposed into $p$ non-overlapping subdomains $\Omega_i$ $(i = 1, \ldots, p)$ such that $\overline{\Omega} = \bigcup_{i=1}^p \overline{\Omega}_i$. The discretization process using Courant's linear triangular finite elements in each subdomain $\Omega_i$ results in a conforming triangulation of $\Omega$. In the following, the indices "$C$" and "$I$" correspond to the nodes belonging to the coupling boundaries $\Gamma_C = \bigcup_{i=1}^p \partial\Omega_i \setminus \Gamma_D$ and to the interior $\Omega_I = \bigcup_{i=1}^p \Omega_i$ of the subdomains, respectively. $\Gamma_D$ is that part of $\partial\Omega$ where Dirichlet–type boundary conditions are given whereas Neumann boundary conditions will be handled as coupling boundaries.

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Define the usual finite elements (FE) nodal basis

$$\Phi = [\Phi_C, \Phi_I] = \left[\psi_1, \cdots, \psi_{N_C}, \psi_{N_C+1}, \cdots, \psi_{N_C+N_{I,1}}, \cdots, \psi_{N=N_C+N_I}\right] , \tag{1.2}$$

where the first $N_C$ basis functions belong to $\Gamma_C$, the next $N_{I,1}$ to $\Omega_1$, the next $N_{I,2}$ to $\Omega_2$ and so on such that $N_I = \sum_{i=1}^{p} N_{I,i}$ . Then the FE isomorphism leads to the symmetric and positive definite system of equations

$$K\,\underline{u} \;:=\; \begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} \;=\; \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix} \;=:\; \underline{f} \;, \tag{1.3}$$

where $K_I = \text{blockdiag}\,(K_{I,i})_{i=1,\dots,p}$ is block diagonal and symmetric, positive definite.

Solving system (1.3) with some parallelized iterative method, e.g., CG-method, we use the ASM–DD preconditioner

$$C = \begin{pmatrix} I_C & -B_{IC}^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ -B_{IC} & I_I \end{pmatrix} . \tag{1.4}$$

This preconditioner contains the three components $C_C$, $C_I = \text{diag}\,(C_{I,i})_{i=1,\dots,p}$ and $B_{IC} = \text{blockmatrix}\,(B_{IC,i})_{i=1,\dots,p}$ , which can freely be chosen in order to adapt the preconditioner to the particulars of the problem under consideration. For the choice $B_{IC,i} = -B_{I,i}K_{IC,i}$ see [HLM91]. As preconditioner $C_C$ for the Schur complement $S_C = K_C - K_{CI}K_I^{-1}K_{IC}$ the BPS [BPS89] and the S(chur)-BPX [TCK92] are used.

The preconditioning step $\underline{w} = C^{-1}\underline{r}$ can be rewritten in the form

**Algorithm 1** : The ASM-DD Preconditioner [HLM91]

$$\begin{aligned} \underline{\mathbf{w}}_C &= C_C^{-1} \sum_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} + B_{IC,i}^T \underline{r}_{I,i}\right) \\ \underline{\mathbf{w}}_{I,i} &= C_{I,i}^{-1}\underline{r}_{I,i} + B_{IC,i}\underline{\mathbf{w}}_{C,i} \qquad ;\; i = 1, 2, \dots, p \end{aligned}$$

where $A_i = \left(\begin{smallmatrix} A_{C,i} & A_{CI,i} \\ A_{IC,i} & A_{I,i} \end{smallmatrix}\right)$ denotes the subdomain connectivity matrix which is used for a convenient notation only. The subdomain FE assembly process which is connected with nearest neighbour communication stands behind this notation. Other DD-preconditioners and modifications of Algorithm 1 can be found in [HL92].

Assume positive, $h$-independent spectral equivalence constants $\underline{\gamma}_C, \overline{\gamma}_C, \underline{\gamma}_I, \overline{\gamma}_I$ fulfilling the spectral equivalence inequalities

$$\underline{\gamma}_C\, C_C \;\le\; S_C \;\le\; \overline{\gamma}_C\, C_C \qquad \text{and} \qquad \underline{\gamma}_I\, C_I \;\le\; K_I \;\le\; \overline{\gamma}_I\, C_I \quad . \tag{1.5}$$

If we have a constant $c_E$ so that

$$\left\| \begin{pmatrix} \underline{v}_C \\ B_{IC}\underline{v}_C \end{pmatrix} \right\|_K \;\le\; c_E \;\| \underline{v}_C \|_{S_C} \qquad \forall \underline{v}_C \in \mathbb{R}^{N_c} \tag{1.6}$$

holds then the upper and lower bounds of the condition number $\kappa(C^{-1}K)$ [HLM91, Che93] can be estimated as

$$\mathcal{O}(c_E^2) \;\le\; \kappa(C^{-1}K) \;\le\; \mathcal{O}(c_E^4) \quad . \tag{1.7}$$

In the remaining chapter we construct extension techniques defining $B_{IC,i}$ which are cheap to implement and should result in a constant $c_E$ independent of or slightly dependent on the discretization parameter $h$ and the number of levels $\ell$.

In the following the subscript $i$ denoting the subdomain number will be omitted.

## 1.2 Multilevel Extension Operators

Let space $\mathbb{W}_k$ consist of real-valued functions which are continuous on $\Omega$ and linear on the triangles in $\Omega_k^h$. The space $\mathbb{V}_k$ is the space of traces on $\Gamma$ of functions from $\mathbb{W}_k$.

The multilevel extension of a FE function $\varphi^h \in \mathbb{V}_\ell$ on the boundary $\partial\Omega$ into a function $u^h \in \mathbb{W}_\ell$ in the interior of the domain $\Omega$ via the extension operator $\mathcal{E}_{\overline{\Omega}\Gamma}$ consists of three steps:

1. Choose the projection $\mathcal{Q}_k$ from $\mathbb{V}_\ell$ into $\mathbb{V}_k$ $(k = 0, \ldots, \ell)$.
2. Split the function $\varphi^h$ into a multilevel nodal basis according to the projections $\mathcal{Q}_k$

$$\chi_0^h = \mathcal{Q}_0\varphi^h, \tag{1.8a}$$
$$\chi_k^h = (\mathcal{Q}_k - \mathcal{Q}_{k-1})\varphi^h \qquad k = 1, \ldots, \ell\,. \tag{1.8b}$$

3. Define the extensions $u_k^h \in \mathbb{W}_k \setminus \mathbb{W}_{k-1}$ of the function $\chi_k^h$

$$u_0^h(x_i^{(0)}) = \begin{cases} \chi_0^h(x_i^{(0)}) & , x_i^{(0)} \in \Gamma, \\ \overline{\chi} & , x_i^{(0)} \notin \Gamma, \end{cases} \tag{1.9a}$$
$$u_k^h(x_i^{(k)}) = \begin{cases} \chi_k^h(x_i^{(k)}) & , x_i^{(k)} \in \Gamma, \\ 0 & , x_i^{(k)} \notin \Gamma. \end{cases} \quad k = 1, \ldots, \ell \tag{1.9b}$$

For $\overline{\chi}$ we choose either the mean value of the boundary function $\chi_0^h$ or the solution of the proper PDE on the coarsest grid with Dirichlet boundary conditions $\chi_0^h$.

Now, the extension $u^h$ is simply

$$\mathcal{E}_{\overline{\Omega}\Gamma}\varphi^h := u^h = \sum_{k=0}^{\ell} u_k^h \qquad . \tag{1.10}$$

In Algorithm 1 the discrete representation $B_{IC}^T$ of the transposed operator $\mathcal{E}_{\overline{\Omega}\Gamma}^T$ is needed so we prefer the recursive version of definition (1.10) :

$$v_0 := u_0^h \tag{1.11a}$$
$$v_k := v_{k-1} + u_k^h \qquad k = 1, \ldots, \ell \tag{1.11b}$$

and set $\mathcal{E}_{\overline{\Omega}\Gamma}\varphi^h := v_\ell$. Note, that $v_k \in \mathbb{W}_k$ is the extension of $\mathcal{Q}_k\varphi^h$ on level $k = 0, \ldots, \ell$.

## 1.3 Matrix representation of Multilevel Extension Operators

Using the FE isomorphism we change from the operator description of the extension $\mathcal{E}_{\overline{\Omega}\Gamma}$ to the matrix representation. The bilinear FE basis $\Psi^{(k)}$ will be defined similarly to (1.2) and the matrices $I_{C,k}$ denote the proper identities on level $k$.

The multilevel extension of a function $\varphi^h = \sum_i^{N_{C,\ell}} \varphi_i \psi_i^{(\ell)} \in \mathbb{V}_\ell$ represented by the vector $\underline{\varphi} \in \mathbb{R}^{N_{C,\ell}}$ into a function $u^h = \sum_i^{N_\ell} u_i \psi_i^{(\ell)} \in \mathbb{W}_\ell$ represented by the vector $\underline{u} \in \mathbb{R}^{N_\ell}$ consists of three steps :

1. Determine the rectangular $N_{C,k} \times N_{C,\ell}$ projection matrix $Q_k$ and define the coefficients of the projection $\mathcal{Q}_k \varphi^h$ in the FE nodal basis of level $k$

$$\underline{\beta}_k := Q_k \underline{\varphi} \qquad k = 0, \ldots, \ell \qquad . \tag{1.12}$$

2. According to (1.8) split the vectors $\underline{\beta}_k$ into the coefficients of the multilevel nodal basis presentation of $\varphi^h = \sum_{k=0}^{\ell} \sum_i^{N_{C,k}} \alpha_i^{(k)} \psi_i^{(k)}$. Denoting by $P_{C,k}^{k+1}$, $P_{I,k}^{k+1}$, $P_{IC,k}^{k+1}$ the usual linear FE interpolation matrices on the proper subsets of nodes we can determine the coefficient vectors $\underline{\alpha}_k$

$$\underline{\alpha}_0 := \underline{\beta}_0 \tag{1.13a}$$

$$\underline{\alpha}_k := \left(-P_{C,k-1}^k \quad I_{C,k}\right) \begin{pmatrix} \underline{\beta}_{k-1} \\ \underline{\beta}_k \end{pmatrix} \qquad k = 1, \ldots, \ell. \tag{1.13b}$$

3. The coefficients $\underline{v}_k$ of the extensions $v_k = \sum_{i=1}^{N_k} v_i^{(k)} \psi_i^{(k)}$ are determined by

$$\underline{v}_0 = \begin{pmatrix} \underline{v}_{C,0} \\ \underline{v}_{I,0} \end{pmatrix} := \begin{pmatrix} I_{C,0} \\ B_{IC,0} \end{pmatrix} \underline{\alpha}_0 \tag{1.14a}$$

$$\underline{v}_k = \begin{pmatrix} \underline{v}_{C,k} \\ \underline{v}_{I,k} \end{pmatrix} := \begin{pmatrix} I_{C,k} & P_{C,k-1}^k & 0 \\ 0 & P_{IC,k-1}^k & P_{I,k-1}^k \end{pmatrix} \begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} \tag{1.14b}$$

Set $E_{IC}\underline{\varphi} := \underline{v}_\ell$.
The matrix $B_{IC,0}$ can be chosen as $\frac{1}{N_{C,0}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{N_{I,0} \times N_{C,0}}$, mapping the mean value of the boundary data into the interior. Another approach is the discrete harmonic extension on the coarsest grid with respect to the PDE, i.e. $B_{IC,0} = -K_{I,0}^{-1} K_{IC,0}$.

## 1.4 Various Multilevel Extension Techniques

Although having in mind the operator representation from Section 1.2 we give the following extension techniques mostly in the matrix representation from Section 1.3. Note, that the theoretical results given in subsections 1.4.1 and 1.4.2 require material coefficients $\lambda(x) = \lambda_i > 0 \quad \forall x \in \overline{\Omega}_i \quad (i = 1, \ldots, p)$ whereas the technique described in 1.4.3 is also available for more general coefficients.

### *1.4.1 The Hierarchical extension technique*

Using just the injection from level $\ell$ to level $k$ for defining the projections $Q_k$ we get exactly the hierarchical extension technique $\widehat{E}_{IC}$ which was first proposed in [HLMN94]. The proofs of the following statements can be found therein.
Setting $B_{IC} := \widehat{E}_{IC}$ the constant $c_E$ in (1.6) behaves like

$$\text{in 2D}: \quad c_E(\widehat{E}_{IC}) \;=\; \mathcal{O}(\ln h^{-1}) \;=\; \mathcal{O}(\ell) \qquad \text{in 3D}: \quad c_E(\widehat{E}_{IC}) \;=\; \mathcal{O}(h^{-1})\,. \tag{1.15}$$

This hierarchical extension can also be used as initial guess in an iteration approximating the extension, i.e. $B_{IC} := M_I^s \widehat{E}_{IC} + (I_I - M_I^s) K_I^{-1}(-K_{IC})$ with some iteration operator $M_I$. If there exists an $h$-independent positive constant such that $\| M_I \|_{K_I} \leq \eta < 1$ holds, e.g. using multigrid to define $M_I$, then in the 2D case $s = \mathcal{O}(\ln(\ln h^{-1}))$ iterations are sufficient to achieve an $h$-independent constant $c_E$.

### *1.4.2 The BPX-like extension technique*

When approaching a BPX splitting with the $L_2$ ortho-projections of the boundary data $\varphi^h$ the projection matrix will be expressed by $Q_k = M_{k,k}^{-1} M_{k,\ell}$, where the entries in the mass matrix $M_{k,p}$ are defined via the $L_2$ inner product $m_{i,j}^{(k,p)} = (\psi_i^{(k)}, \psi_j^{(p)})_{L_2}$. Using bilinear FE functions in the 2D case the matrix $M_{k,k}$ possesses exactly 3 non-zero entries per row. Whereas this matrix is rather easy to invert in the 3D case this will require too much arithmetic work.
Therefore a mass lumping or the proposal in [Nep95] for defining the projection operator $\mathcal{Q}_k$ is used. This results in an easily invertible diagonal matrix $\overline{M}_{k,k}$ defining $Q_k := \overline{M}_{k,k}^{-1} M_{k,\ell}$ and leads to the BPX-like extension technique $\overline{E}_{IC}$ for which

$$c_E(\overline{E}_{IC}) \;=\; \mathcal{O}(1) \tag{1.16}$$

was proven in [Nep95] for the 2D as well as for the 3D case.

### *1.4.3 Multilevel extension techniques plus smoothing*

The recursive definition of the extension in (1.14) leads to the idea of an additional improvement of the extensions given above via some linear smoothing procedure $S_{I,k} : \mathbb{R}^{N_{I,k}} \to \mathbb{R}^{N_{I,k}}$, $k = 1, \ldots, \ell$ with the properties

$$\begin{array}{lll} \text{High frequencies}: & \| S_{I,k}\, \underline{v}^h \|_{K_I} \leq \varrho_k \| \underline{v}^h \|_{K_I} & \forall \Phi_I \underline{v}^h \in \mathbb{W}_k \setminus \mathbb{W}_{k-1} \\ \text{Low frequencies}: & \| S_{I,k}\, \underline{v}^h \|_{K_I} \leq \sigma_k \| \underline{v}^h \|_{K_I} & \forall \Phi_I \underline{v}^h \in \mathbb{W}_k \quad , \end{array} \tag{1.17}$$

where the number of smoothing sweeps is denoted by $\nu_k$. We require smoothing factors $\varrho_k \leq \sigma_k \leq 1$ independent of $h = 2^{-l}$. So, definition (1.14b) changes into

$$\begin{pmatrix} \underline{v}_{C,k} \\ \underline{v}_{I,k} \end{pmatrix} := \begin{pmatrix} I_{C,k} & 0 \\ -(I_{I,k} - S_{I,k}^{\nu_k}) K_{I,k}^{-1} K_{IC,k} & S_{I,k}^{\nu_k} \end{pmatrix} \begin{pmatrix} I_{C,k} & P_{C,k-1}^k & 0 \\ 0 & P_{IC,k-1}^k & P_{I,k-1}^k \end{pmatrix} \begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} \tag{1.18}$$

Using the hierarchical extension together with the smoothing, i.e. setting $B_{IC} = \widehat{E}_{IC}(\nu)$, it was shown in [Haa97] that in the 2D case

$$c_E(\widehat{E}_{IC}(\nu)) \;\leq\; c\left(1 + \sqrt{\ell}\cdot\sqrt{\sum_{k=1}^{\ell}\varrho_k^{2\nu_k}\prod_{j=k+1}^{\ell}\sigma_j^{2\nu_j}}\right) \;=\; \mathcal{O}(\ell) \tag{1.19}$$

holds with a positive and $h$-independent constant $c$. In comparison to the hierarchical extension the order remains the same but the constants hidden in that order statement are partially controlled by the smoothing. Now, $\nu_k = \mathcal{O}(\ln(\ln h^{-1}))$ iterations of the smoothing procedure $S_{I,k}$ are sufficient to achieve an $h$-independent constant $c_E$ so that no longer an additional iteration as in subsection 1.4.1 is needed.

In the BPX-like extension additional smoothing sweeps are also feasible but the theory for this is still open. In that case the parameter $\sigma_k$ will have more influence on the behavior of the constant $c_E$.

## 1.5 An Algorithmic Improvement of the Preconditioner

The improvement of Algorithm 1 is based upon three observations :

A) In Algorithm 1 the matrices $B_{IC}^T$ and $C_I^{-1}$ are applied to the same vector $\underline{r}_I$.

B) In the transposed operation to (1.14b) used in $B_{IC}^T$

$$\begin{pmatrix}\underline{\alpha}_k\\ \underline{v}_{C,k-1}\\ \underline{v}_{I,k-1}\end{pmatrix} \;:=\; \begin{pmatrix} I_{C,k} & 0\\ \left(P_{C,k-1}^k\right)^T & \left(P_{IC,k-1}^k\right)^T\\ 0 & \left(P_{I,k-1}^k\right)^T\end{pmatrix}\begin{pmatrix}\underline{v}_{C,k}\\ \underline{v}_{I,k}\end{pmatrix}$$

the last row is simply the usual linear restriction from the finer to the coarser grid. A similar observation for the extension technique plus smoothing in (1.18) leads to

$$\underline{v}_{I,k-1} \;:=\; \left(P_{I,k-1}^k\right)^T \left(S_{I,k}^T\right)^{\nu_k}\underline{v}_{I,k} \quad . \tag{1.20}$$

C) Perform in a multigrid algorithm at level $k$ $\nu_k$ smoothing sweeps with the iteration matrix $\overset{*}{S}_{I,k}$, ($\overset{*}{S}_{I,k}$ means the adjoint matrix to $S_{I,k}$ in the $K_{I,k}$-energy inner product) together with calculation and restriction of the defect. Then that brief algorithm

$$\underline{w}_{I,k} := (I_{I,k} - \left(\overset{*}{S}_{I,k}\right)^{\nu_k})K_{I,k}^{-1}\underline{f}_{I,k} \quad \text{and} \quad \underline{d}_{I,k-1} := \left(P_{I,k-1}^k\right)^T(\underline{f}_{I,k} - K_{I,k}\underline{w}_{I,k})$$

can be simplified into

$$\underline{d}_{I,k-1} \;:=\; \left(P_{I,k-1}^k\right)^T \left(S_{I,k}^T\right)^{\nu_k}\underline{f}_{I,k} \quad . \tag{1.21}$$

Now, choosing $\overset{*}{S}_{I,k}$ as pre-smoother and $S_{I,k}$ as post-smoother in a multigrid algorithm defining $C_I$ we preserve the required symmetry and may use restriction and pre-smoothing from the implementation of the transposed extension $B_{IC}^T$. When

additionally in that extension a coarse grid solver is used more than half of the algorithmic work can be saved when applying $C_I^{-1}$. This algorithmic improvement does not depend on the choice of the projections $Q_k$ !

Numerical tests using this improved algorithm together with the hierarchical extension technique can be found in [Haa97].

## 1.6 A test example

For checking the theoretical results concerning the behavior of the constant $c_E$ it is necessary that the spectral equivalence constants in (1.5) are independent of $h$. Therefore, the test example consists of the PDE

$$-\Delta\, u(x) \;=\; 1 \quad \text{in } \Omega = (0,1) \times (0,0.5) \qquad \text{and} \qquad u(x) = 0 \quad \text{on } \partial\Omega \;\;,$$

where the domain $\overline{\Omega}$ was decomposed into two squares. Using in the preconditioner $C$ (1.4) Dryja's approach [Dry82] as Schur complement preconditioner $C_C$ and exact solvers for $C_{I,i}^{-1}$ we achieve $h$-independent spectral equivalence constants in (1.5) so that the condition number of the preconditioned system $\kappa(C^{-1}K)$ will be influenced only by the extension $B_{IC}$.

The discrete system (1.3) was solved with a preconditioned parallelized CG until a relative accuracy of $10^{-6}$ measured in the $\|\cdot\|_{K C^{-1} K}$-norm of the error was reached. Due to estimates (1.7) the number of CG iterations behaves like $\mathcal{O}(c_E^2)$ .
On level 0 with the discretization parameter $h = 0.25$, i.e. just one node on the interface between the two subdomains, the automatic mesh generator produces a triangular mesh with 4 inner nodes per subdomain. All finer meshes were produced by simply subdividing each triangle into 4 congruent ones. The Gauß-Seidel smoother $S_{I,k}$ applied $\nu = \nu_k$-times $(k = 1, \ldots, \ell)$ and a coarse grid solver were used in the extension $B_{IC}$.

| Splitting | $\nu$ | $\ell = 0$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ | $\ell = 4$ | $\ell = 5$ | $\ell = 6$ |
|---|---|---|---|---|---|---|---|---|
| hier. | 0 | 2 | 7 | 11 | 17 | 24 | 30 | 36 |
| hier. | 1 | 2 | 6 | 8 | 11 | 13 | 16 | 19 |
| hier. | 2 | 2 | 5 | 7 | 8 | 10 | 13 | 14 |
| BPX-like | 0 | 2 | 7 | 8 | 11 | 12 | 13 | 13 |

**Table 1.1** # CG-iterations for the test example using 2 processors

Whereas in Table 1.1 the iteration numbers connected with the hierarchical splitting grow linearly with the number of levels $\ell$ the iteration numbers of the BPX-like splitting tend to an upper bound. This confirms the theoretical results from (1.15), (1.16) and (1.19). Obviously the additional smoothing sweeps decrease the number of iterations and also the condition number $\kappa(C^{-1}K)$ rapidly.

## 1.7 Final Remarks

Although the development was done on rather simple operators the results carry over to systems of symmetric elliptic second order PDE s resulting in symmetric $\mathbb{V}_0$-elliptic and $\mathbb{V}_0$-bounded bilinear forms. The extension techniques used in the test example work also very well in more challenging examples and result in a faster solver when the algorithmic improvement from Paragraph 1.4.3 is used; see also [Haa97].

In the latter paper, the full proof for estimate (1.19) can be found and it was shown that this estimate holds for the whole range from extension without smoothing to the exact harmonic extension. Especially smoothing controls similar to the multigrid slash cycle and generalized slash cycle were investigated, so that $O(\ln \ell)$ smoothing sweeps are sufficient to achieve a condition number $\kappa(C^{-1}K) = O(1)$. The algorithmic improvement from Paragraph 1.5 was derived in a more heuristic way. The electrical machine with jumping coefficients and a complicated geometry proved that the new extension method is also successfully applicable to practical problems.

The implementation and theoretical analysis of the BPX-like extension together with smoothing sweeps will be done in a forthcoming paper. Also, a comparison to other extension techniques proposed in [Nep91, Che93, BPV96] should be done on a more challenging example with respect to the CPU-time needed to solve (1.1).

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