THE AMERICAN MATHEMATICAL MONTHLY

Volume 111, Number 3 March 2004

Robert E. Megginson	Yueh-Gin Gung and Dr. Charles Y. Hu Award to T. Christine Stevens for Distinguished Service to Mathematics	185
Alan Edelman, Gilbert Strang	Pascal Matrices	189
William T. Ingram, William S. Mahavier	Interesting Dynamics and Inverse Limits in a Family of One-Dimensional Maps	198
Jaume Paradís, Josep Pla, Pelegrí Viader	Fermat and the Quadrature of the Folium of Descartes	216
Detlef Gronau	The Spiral of Theodorus	230
Leslie Blackwell Galen	Putting Fractions in Their Place	238
NOTES		
Tsuneo Ishikawa, Nobuhiko Ishida, Yoshito Yukimoto	On Prime Factors of $A^n - 1$	243
Lou M. Pretorius, Konrad J. Swanepoel	An Algorithmic Proof of the Motzkin-Rabin Theorem on Monochrome Lines	245
Martin Engman	A Note on Isometric Embeddings of Surfaces of Revolution	251
T. Y. Lam	On Subgroups of Prime Index	256
PROBLEMS AND SOLUTIONS		259
REVIEWS		
John B. Little	*A **Singular** Introduction to Commutative Algebra*. By Gert-Martin Greuel and Gerhard Pfister	267
Stephen A. Fulling	*Asymptotology: Ideas, Methods, and Applications*. By I. V. Andrianov and L. I. Manevitch	271
TELEGRAPHIC REVIEWS		276

AN OFFICIAL PUBLICATION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The Spiral of Theodorus

Detlef Gronau

1. INTRODUCTION. Philip J. Davis's lively book on spirals [**2**] considers these curves generally but has its main focus on the beautiful *spiral of Theodorus* (named after Theodorus of Cyrene, circa 470–399 B.C.). The discrete points of this spiral can be represented in the Gaussian plane by $z_0 = 1$ and $z_{n+1} = (1 + i/\sqrt{n+1}) \cdot z_n$ for $n = 0, 1, 2, \ldots$, where $i = \sqrt{-1}$ (see [**2**, p. 33]). There are infinitely many possibilities for connecting these points by a continuous curve. This is the old problem of "interpolating" a discrete function, defined say on the nonnegative integers, by a function that is defined on the nonnegative reals. It was, for example, Euler's motivation for introducing the gamma function as the interpolating function of the factorials (see [**4**]).

Davis [**2**] raised the question: Is there a "natural" way to draw a nice curve through the "Theodorus points" that looks spiral-like. But first we ask, as Davis did: How can one characterize such a spiral-like curve?

2. SPIRALS. Spirals are very popular and therefore there is a vast literature treating them. In addition to the Davis book, we mention three references on this topic. The book of Lawrence [**10**] gives a concise overview on special curves, including spirals. Heitzer [**5**] is full of pictures of spirals in mathematics and nature. Gray [**3**] describes curves and surfaces, including spirals, and provides formulas that are additionally formulated in the language of MATHEMATICA.

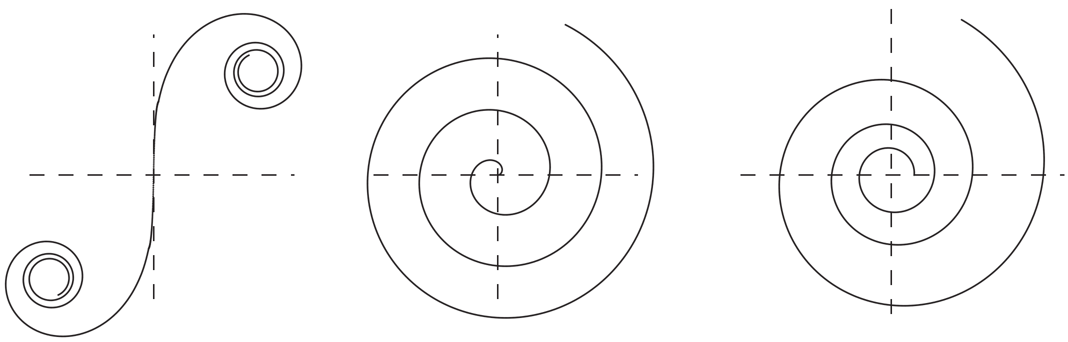

Figure 1. The spiral of Cornu, Archimedes' spiral, and the logarithmic spiral.

There are several different "definitions" of spirals in the plane. Roughly speaking, a spiral is a curve in the plane around a center point that, with a proper choice of coordinates, can be placed at the origin. In [**5**, p. 12] we find among others the characterization: a spiral is a curve in the plane for which the distance between a point of the curve and its center is a monotonic function of the argument of the rotating ray connecting the point with the center. This means that, if we introduce polar coordinates (r, φ) in the plane, then a spiral is given by an equation $r = f(\varphi)$, where f is a monotonic real function. This definition, in general, proved rather unsatisfactory. Thus the *clothoid* or *spiral of Cornu* (Figure 1; for the parametric representation see [**10**, p. 190]) does not fit this definition. Other spirals, for example the *spiral of Archimedes* (its defining equation is $r = a\varphi$ for a real constant a), the *logarithmic spiral* ($r = k\exp(a\varphi)$, where a and k are real constants), and also our spiral of Theodorus are covered by this definition. Another characterization of spirals appears in [**6**] (see Remark 4).

In analogy with Euler's infinite product for the gamma function, Davis [**2**, pp. 37–38] proposes the product

$$T(x) = \prod_{k=1}^{\infty} \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{x+k}} \qquad (-1 < x < \infty) \tag{1}$$

as the parametric representation for an appropriate interpolating curve for the discrete spiral of Theodorus, i.e., $T(n) = z_n$ for $n = 0, 1, 2, \ldots$ (see Figure 2). The infinite product (1) is convergent (very slowly), for each term has an asymptotic representation

$$\frac{1 + i/\sqrt{k}}{1 + i/\sqrt{x+k}} = 1 + O(k^{-3/2})$$

(see [**2**, p. 38]).

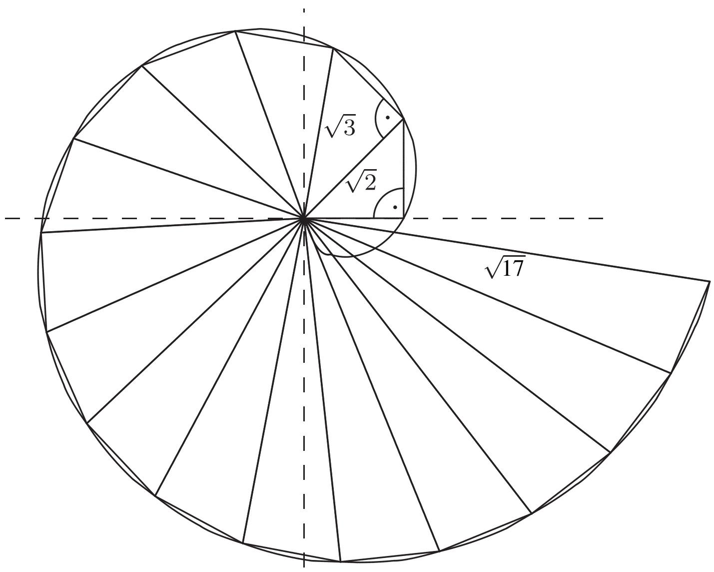

Figure 2. The discrete spiral and its interpolating curve defined by (1).

3. THE DIFFERENCE EQUATION. The function T, which Davis calls the *Theodorus function*, is a particular solution of the difference equation

$$f(x+1) = \left(1 + \frac{i}{\sqrt{x+1}}\right) \cdot f(x) \quad (-1 < x < \infty),\ f(0) = 1. \tag{2}$$

Equation (2) is a linear first order difference equation for the real- or complex-valued unknown function f. Equations of this type and their nondiscrete solutions arise very often in the literature (see for example [**7**], [**8**], [**9**] [**11**]). In particular the functional equation of the gamma function is of this type: $\Gamma(x+1) = x \cdot \Gamma(x)$. The gamma function is a kind of *normal solution* to its equation (Nörlund [**11**] calls it the "Hauptlösung"), characterized in the famous Bohr-Mollerup theorem by a logarithmic convexity condition (see, for instance, the nice exposition of Rudin [**12**]). In this connection F. John [**7**] considered monotonic solutions of linear first order difference equations to be "normal" solutions, but his methods were different from those used in what follows. One should also mention W. Krull's papers [**8**] and [**9**], which Krull himself calls "marginal notes" on the gamma function.

Davis asked for a characterization of the Theodorus function T among the various solutions of (2), similar to the characterization of the gamma function. We provide such a characterization here.

4. CHARACTERIZATIONS. An elementary characterization of the function T is the following.

Lemma. *The unique solution f of* (2) *satisfying the additional condition that*

$$\lim_{n\to\infty} \frac{f(x+n)}{f(n)} = 1 \tag{3}$$

for x in $(0, 1)$ *is the Theodorus function.*

Proof. For any solution of (2) and for arbitrary x in $(-1, \infty)$ and n in $\mathbb{N}$ we have

$$f(x+n) = f(x) \cdot \prod_{k=1}^{n} \left(1 + i/\sqrt{x+k}\right). \tag{4}$$

Let f be a solution of (2) satisfying condition (3). Note that (3) holds automatically for $x = 0$ and $x = 1$. First we show that, if (3) holds for x in $(0, 1)$, then this condition is also valid for all x in $(-1, \infty)$.

If $x > 1$, we can write $x = m + y$ with y in $[0, 1)$ and m in $\mathbb{N}$ and we infer from (4) that

$$\frac{f(x+n)}{f(n)} = \frac{f(y+n)}{f(n)} \cdot \prod_{k=1}^{m} \left(1 + i/\sqrt{y+n+k}\right).$$

Each term of this product converges to 1 as $n \to \infty$, hence $f(x+n)/f(n)$ does as well.

If $-1 < x < 0$, we have

$$\frac{f(x+n)}{f(n)} = \frac{f(x+1+n)}{f(n)} \cdot \left(1 + i/\sqrt{x+1+n}\right)^{-1},$$

which also converges to 1 as $n \to \infty$ since $x + 1$ belongs to $(0, 1)$. Thus (3) holds for all x in $(-1, \infty)$.

Finally we have

$$\frac{f(x+n)}{f(n)} = \frac{f(x)}{f(0)} \cdot \prod_{k=1}^{n} \frac{1 + i/\sqrt{x+k}}{1 + i/\sqrt{k}}$$

for each x in $(-1, \infty)$. Because the limits of both sides of this equality exist as $n \to \infty$ and because $f(0) = 1$, we arrive at the representation (1) for all x in $(-1, \infty)$. ■

The main result in this article gives a characterization of the Theodorus function via monotonicity criteria. We call a function f increasing or decreasing if $x < y$ implies $f(x) \le f(y)$ or $f(x) \ge f(y)$, respectively. The function f is said to be strictly increasing or strictly decreasing, respectively, if the related inequalities are strict. We call a function monotonic, if it is either increasing or decreasing on its domain of definition (but not necessarily strictly increasing or strictly decreasing).

Theorem 1. *If f is a solution of* (2) *on* $(-1, \infty)$ *such that* $|f(x)|$ *is monotonic and* $arg(f(x))$ *is monotonic and continuous, then f is the Theodorus function.*

Proof. We write equation (2) in polar coordinates. First we have

$$1 + \frac{i}{\sqrt{x+1}} = a(x) \cdot e^{ib(x)}$$

with

$$a(x) = \left|1 + \frac{i}{\sqrt{x+1}}\right| = \sqrt{\frac{x+2}{x+1}}, \qquad b(x) = \pi/2 - \arctan\sqrt{x+1}. \tag{5}$$

Thus (2) becomes

$$f(x+1) = a(x) \cdot e^{ib(x)} \cdot f(x) \qquad (-1 < x < \infty).$$

Set $f(x) = \psi(x) \cdot e^{i\varphi(x)}$ with real-valued functions ψ and φ on $(-1, \infty)$. This gives

$$\psi(x+1) \cdot e^{i\varphi(x+1)} = a(x) \cdot \psi(x) \cdot e^{i(\varphi(x)+b(x))},$$

hence we obtain the two equations

$$\psi(x+1) = a(x) \cdot \psi(x) \tag{6}$$

and

$$e^{i\varphi(x+1)} = e^{i(\varphi(x)+b(x))}.$$

The latter is equivalent to

$$\varphi(x+1) = \varphi(x) + b(x) + 2\pi\kappa(x), \tag{7}$$

where $\kappa : (-1, \infty) \to \mathbb{Z}$.

Equation (6). The initial condition $f(0) = 1$ implies that $\psi(0) = 1$. We show that $\psi(x) = \sqrt{x+1}$, which is obviously a strictly increasing solution of (6), is the unique monotonic solution of (6) with $\psi(0) = 1$. For this we consider an arbitrary monotonic solution ψ of (6) on $(-1, \infty)$. For given x in $(-1, \infty)$ and n in $\mathbb{N}$ we have

$$\psi(x+n) = \sqrt{\frac{x+n+1}{x+1}} \cdot \psi(x),$$

so $\psi(n) = \sqrt{n+1}$. Thus ψ has to be increasing. Let m be the smallest integer greater than or equal to x. Then by the monotonicity of ψ we have

$$1 = \frac{\psi(n)}{\psi(n)} \leq \frac{\psi(n+x)}{\psi(n)} \leq \frac{\psi(n+m)}{\psi(n)} = \sqrt{\frac{n+m+1}{n+1}}$$

in the case $x \geq 0$. Since the right-hand term tends to 1 as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \frac{\psi(n+x)}{\psi(n)} = 1. \tag{8}$$

An argument similar to the one in the proof of the lemma demonstrates that (8) holds as well when x lies in $(-1, 0)$.

Now

$$\frac{\psi(x+n)}{\psi(n)} = \sqrt{\frac{x+n+1}{n+1}} \cdot \frac{1}{\sqrt{x+1}} \cdot \psi(x).$$

Taking the limit of both sides of this equation as $n \to \infty$, we obtain $\psi(x) = \sqrt{x+1}$, this for each x in $(-1, \infty)$.

Equation (7). We are interested in the monotonic, continuous solutions φ of (7). Note that $b(x) = \pi/2 - \arctan\left(\sqrt{x+1}\right)$ takes values in $(0, \pi/2)$, $b(-1) = \pi/2$, and $b(x)$ decreases to 0 as $x \to \infty$. Equation (7) and the continuity of b and φ imply that the integer-valued function κ is likewise continuous, hence it must be constant.

The initial condition $f(0) = 1$ implies that $\varphi(0)$ is a multiple of 2π. Since κ is constant and the solutions of (7) are determined up to an additive constant, we may assume without loss of generality that $\kappa(x) = 0$ for all x in $(-1, \infty)$ and $\varphi(0) = 0$.

In other words, φ is a solution of the equation

$$\varphi(x+1) = \varphi(x) + b(x) \quad (-1 < x < \infty), \tag{9}$$

with $\varphi(0) = 0$. We proceed in a manner similar to what we did earlier. It is clear from (9) and the hypotheses of Theorem 1 that φ has to be increasing. For x in $(0, 1)$ and n in $\mathbb{N}$ we infer from the monotonicity of φ that

$$0 \leq \varphi(n+x) - \varphi(n) \leq \varphi(n+1) - \varphi(n) = b(n).$$

Since $b(n) \to 0$ as $n \to \infty$, we find as before that

$$\lim_{n\to\infty} \left[\varphi(n+x) - \varphi(n)\right] = 0$$

for all x in $(-1, \infty)$. We make use of the formula

$$\varphi(x+n) - \varphi(n) = \varphi(x) - \varphi(0) + \sum_{k=0}^{n-1} \left[b(x+k) - b(k)\right]. \tag{10}$$

From the expansion

$$\arctan z = \pi/2 - 1/z + 1/3z^3 - 1/5z^5 + \cdots$$

for $z > 1$ it follows that

$$\begin{aligned} b(x+k) - b(k) &= \arctan\sqrt{k} - \arctan\sqrt{x+k} \\ &= \frac{1}{\sqrt{x+k}} - \frac{1}{\sqrt{k}} + O(k^{-3/2}) \\ &= O(k^{-3/2}). \end{aligned}$$

Thus the series $\sum_{k=0}^{\infty}[b(x+k)-b(k)]$ converges, and we get

$$\varphi(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left[b(k) - b(x+k)\right] = \sum_{k=1}^{\infty} \left[\arctan\sqrt{x+k} - \arctan\sqrt{k}\right].$$

The upshot of this discussion is that

$$f(x) = \sqrt{x+1} \cdot e^{i\sum_{k=1}^{\infty}\left[\arctan\sqrt{x+k}-\arctan\sqrt{k}\right]} \tag{11}$$

furnishes the unique solution of (2) on $(-1,\infty)$ that satisfies the monotonicity hypotheses of Theorem 1. Since this f also satisfies (3), it follows from the lemma that $f = T$, i.e., (11) and (1) represent the same function. ■

The continuity assumptions in Theorem 1 can be weakened in the following way:

Theorem 2. *If f is a solution of* (2) *on* $(-1,\infty)$ *such that* $|f(x)|$ *and* $\arg(f(x))$ *are monotonic and* $\arg(f(n+1)) = \arg(f(n)) + \arg(1 + i/\sqrt{n+1})$ *for* $n = 0, 1, 2, \ldots,$ *then f is the Theodorus function.*

Proof. We have again to treat equation (7) with $\varphi(0) = 0$. By the monotonicity of $x \mapsto \arg(f(x))$, we conclude that the arguments of the discrete points add up, i.e., $\varphi(n+1) = \varphi(n) + b(n)$ for $n = 0, 1, 2, \ldots$. This means that $\kappa(n) = 0$ for $n = 0, 1, 2, \ldots$.

Now for $n = 0, 1, 2, \ldots$ and for x in $(0, 1)$ the monotonicity of φ ensures that

$$\begin{aligned} 0 \le \varphi(n+x+1) - \varphi(n+x) &= b(n+x) + 2\pi\kappa(n+x) \\ &\le \varphi(n+2) - \varphi(n) = b(n+1) + b(n) < \pi. \end{aligned}$$

Hence $\kappa(n+x) = 0$, i.e., $\kappa(y) = 0$ for all nonnegative y. The same inequality with $n = 0$ implies that $\kappa(x) = 0$ for x in $(-1, 0)$. Accordingly $\kappa(x) = 0$ for all x in $(-1,\infty)$. The rest of the proof is completely the same as that of Theorem 1. ■

Remark 1. Although the series in (11) converges very slowly (each term of this sum is $O(k^{-3/2})$), numerical experiments show that the Theodorus function T can be calculated with the aid of (11) on a personal computer in a reasonable length of time and with reasonable precision. All the data for the drawings in this paper have been calculated using the mathematical program DERIVE on the author's desktop computer.

Remark 2. Note that any solution of (2) is a product of the function T in (1) or (11) and a function of period 1. Thus the general solution of (2) takes the form

$$f(x) = \sqrt{x+1} \cdot \Psi(x) \cdot e^{i\Phi(x)+\sum_{k=1}^{\infty} i\left[\arctan\sqrt{x+k}-\arctan\sqrt{k}\right]}, \tag{12}$$

where Ψ and Φ are arbitrary real-valued periodic functions on $(-1,\infty)$ of period 1.

One sees easily that the monotonicity of $|f(x)|$ alone is not sufficient to characterize the Theodorus function, since (12) with $\Psi(x) \equiv 1$ and arbitrary Φ yields a solution of (2) with monotonic $|f(x)|$. Similarly, one can conclude that the monotonicity of $\arg(f(x))$ alone does not characterize (1).

Remark 3. It is well known that any solution of (2) is uniquely determined by prescribing its values on an *initial curve* defined on some half-open interval of length 1, for example, on [0, 1). For $y = n + x$ with x in [0, 1) and n in $\mathbb{N}$, the value $f(y)$ is then given by (4); for x in $(-1, 0)$ the formula

$$f(x) = (1 + i/\sqrt{x+1})^{-1} \cdot f(x+1)$$

determines the value of $f(x)$. These formulas are very useful for calculating the values of the solutions of (2).

5. EXAMPLES.

Example 1. Following the prescription from Remark 3, we take as initial curve the straight line segment from 1 to $1 + i$ (marked in Figure 3 by two bullets), i.e., $f(x) = 1 + ix$ $(0 \leq x < 1)$. This line is transformed with increasing x to a succession of bumps that gradually increase in size. Figure 3 shows the normal solution of (2) together with the solution defined by the line segment.

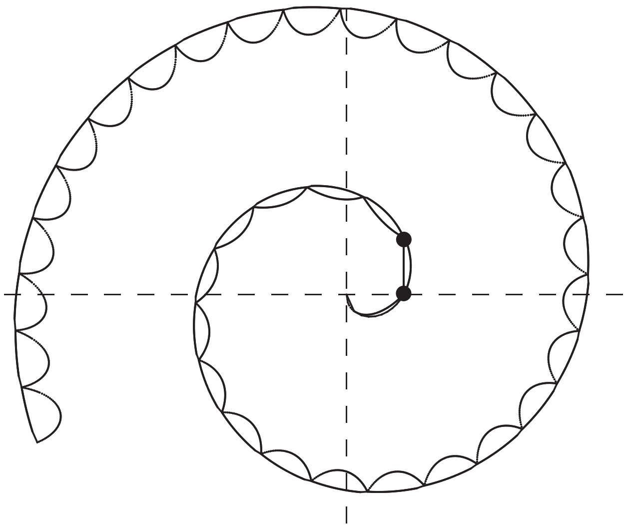

Figure 3. The initial curve is the straight line segment from 1 to $1 + i$.

Example 2. As initial curve we take a loop from 1 to $1 + i$, the precise description of which we suppress (see Figure 4). This loop has the property that its derivative has a (locally) increasing argument, making it *locally curve-convex* (see [**1**]). The picture suggests that the whole curve is also locally curve-convex.

This example gives rise to the following:

Conjecture. *There exist locally curve-convex solutions of* (2) *that are different from* (1).

Remark 4. Konrad Heuvers brought to the attention of the author the paper [**6**], which also deals with the spiral of Theodorus. Heuvers et al. considered the functional equation for the argument of a point of the spiral as a function of its distance from the origin. The solution is the same as (1).

ACKNOWLEDGMENT. The author thanks Phil Davis, who in a personal communication called his attention to the problem of characterizing his spiral-interpolating function.

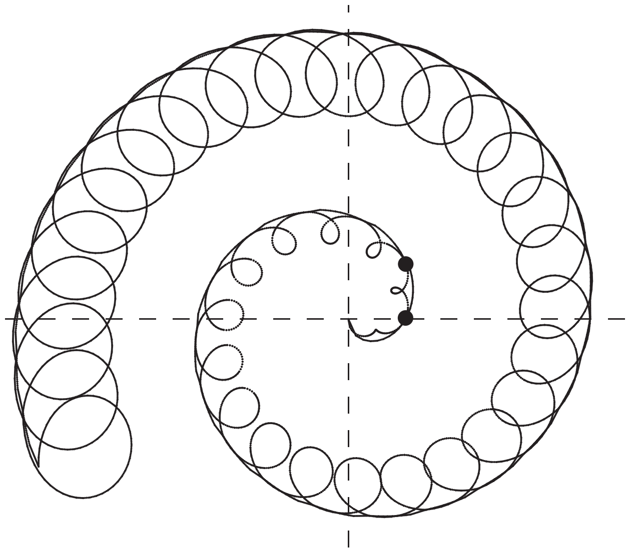

Figure 4. The initial curve is a the looped curve from 1 to $1+i$.

REFERENCES

1. R. D. Bourgin and P. L. Renz, Shortest paths in simply connected regions in R^2, *Advances in Math*. **76** (1989) 260–295.
2. P. J. Davis, *Spirals from Theodorus to Chaos*, A. K. Peters, Wellesley, MA, 1993.
3. A. Gray, *Modern Differential Geometry of Curves and Sufaces with Mathematica*, 2nd ed., CRC Press, Boca Raton, FL, 1998.
4. D. Gronau, Why is the gamma function so as it is? *Teaching Mathematics and Computer Sciences* **1** (2003) 43–54.
5. J. Heitzer, *Spiralen, ein Kapitel phänomenaler Mathematik,* Lesehefte Mathematik, Ernst Klett, Schulbuchverlag, Leipzig, 1998.
6. K. J. Heuvers, D. S. Moak, and B. Boursaw, The functional equation of the square root spiral, in *Functional Equations and Inequalities*, T. M. Rassias, ed., Kluwer, Dordrecht, 2000, pp. 111–117.
7. F. John, Special solutions of certain difference equations, *Acta Math.* **71** (1939) 175–189.
8. W. Krull, Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$, *Math. Nachr.* **1**(1948) 365–376.
9. ———, Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$. II, *Math. Nachr.* **2** (1949) 251–262.
10. J. D. Lawrence, *A Catalog of Special Plane Curves,* Dover, New York, 1973.
11. N. E. Nörlund, *Vorlesung über Differenzenrechung*, Springer-Verlag, Berlin, 1924.
12. W. Rudin, *Principles of Mathematical Analysis,* 3rd ed., McGraw-Hill, New York, 1985.

DETLEF GRONAU received his Ph.D. at the University of Innsbruck under the direction of Wolfgang Gröbner. He is professor of mathematics at the University of Graz. His main field of interest in mathematics is the theory of functional equations and iteration theory. Lately he has become more and more interested in the history of mathematics, especially the history of functional equations.
Institut für Mathematik, Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria
gronau@uni-graz.at