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Results in Mathematics

The Monotone Catenary Degree of Krull Monoids

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Abstract. Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. The monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ of H is the smallest integer m with the following property: for each $a \in H$ and each two factorizations z, z' of a with length $|z| \le |z'|$, there exist factorizations $z = z_0, \ldots, z_k = z'$ of a with increasing lengths—that is, $|z_0| \le \cdots \le |z_k|$—such that, for each $i \in [1, k]$, z_i arises from z_{i-1} by replacing at most m atoms from z_{i-1} by at most m new atoms. Up to now there was only an abstract finiteness result for $\mathsf{c}_{\mathrm{mon}}(H)$, but the present paper offers the first explicit upper and lower bounds for $\mathsf{c}_{\mathrm{mon}}(H)$ in terms of the group invariants of G.

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1. Introduction

In an atomic monoid (this setting includes the multiplicative monoid of non-zero elements from a noetherian domain), every non-unit can be written as a finite product of irreducible elements. Catenary and monotone catenary degrees are arithmetical invariants describing the non-uniqueness of such factorizations. We recall the definition of these concepts.

Let H be an atomic monoid. Then H is not factorial if and only if there is an element $a \in H$ having at least two (essentially) distinct factorizations, say $a = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n$ where $l, m, n \in \mathbb{N}_0$, all u_i, v_j, w_k are atoms (irreducible elements), and the v_j and w_k are pairwise

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non-associated. Then $l+m$ and $l+n$ are called the lengths of the respective factorizations, and $\max\{m,n\}$ is called their distance. If $\max\{m,n\}>0$ and $N\in\mathbb{N}$, then, checking the factorizations of a^N, we see that an element may have factorizations with distance greater than the given N. Now the catenary degree $\mathsf{c}(H)$ of H is the smallest $m\in\mathbb{N}_0\cup\{\infty\}$ with the following property: for each $a\in H$ and each two factorizations z,z' of a there is a chain $z=z_0,z_1,\ldots,z_k=z'$ of factorizations of a between z and z' such that the distance between two successive factorizations is bounded by m. Thus, roughly speaking, the finiteness of the catenary degree means that general phenomena of non-uniqueness are not wilder than phenomena occurring already at powers of elements.

The catenary degree is finite, among others, for orders in number fields, Krull monoids with finite class group, and for C-monoids (these include higher dimensional noetherian domains R with non-trivial conductor $\mathfrak{f}$, finite class group and finite residue field $R/\mathfrak{f}$; see [15, Theorem 2.11.9], [27]). For recent progress on the catenary degree we refer to [3,14,16,19,23], where Krull monoids, higher dimensional noetherian domains, and numerical monoids are studied (the software package [6] includes tools for calculating the catenary degree of numerical monoids).

Suppose H is an atomic monoid with finite catenary degree. Only little is known so far on the structure of the concatenating chains. The monotone catenary degree asks for concatenating chains of factorizations whose lengths are monotone. More precisely, the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ of H is the smallest $m\in\mathbb{N}_0\cup\{\infty\}$ with the following property: for each $a\in H$ and each two factorizations z,z' of a, where the length $|z|$ of z is smaller than or equal to the length $|z'|$ of z', there is a chain $z=z_0,z_1,\ldots,z_k=z'$ of factorizations of a between z and z' such that the distance between two successive factorizations is bounded by m, as well as the associated sequence of lengths is increasing (this means $|z_0|\le|z_1|\le\cdots\le|z_k|$). By definition, the catenary degree $\mathsf{c}(H)$ is smaller than or equal to the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$.

The additional requirement of monotonicity has a strong impact on the problem. It seems that even the finiteness of the monotone catenary degree is a rare phenomenon (inside the class of objects having finite catenary degree). More on monotone and near monotone chains of factorizations in various classes of monoids and noetherian domains can be found in [8–10,13,18,24,25].

The focus of the present paper is on Krull monoids with finite class group such that every class contains a prime divisor (this setting includes rings of integers in number fields; more examples are given in Sect. 2). Let H be such a Krull monoid, G its class group, and $\mathsf{D}(G)$ the Davenport constant of G. Then the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ of H is finite (this was first proved by Foroutan in [8]), and it is well-known that both the catenary and the monotone catenary degree of H depend only on the class group G (see Lemma 2.1 for details). It is not difficult to show that $\mathsf{c}(H)\le\mathsf{D}(G)$, and that equality holds if and only if G is cyclic or an elementary 2-group (see [15, Section 6.4], and [14] for recent progress).

In contrast to the results on the catenary degree, there are so far no explicit upper and lower bounds for the monotone catenary degree, let alone precise results for certain types of class groups (the above mentioned theorem by Foroutan is an abstract finiteness result). The goal of the present paper is to obtain such explicit bounds for the monotone catenary degree of H in terms of the class group G. We formulate our main result, whose proof follows immediately from the slightly stronger Theorems 3.4, 4.5 and 5.1.

Theorem 1.1. *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor.*

1. *If $\mathsf{D}(G) = 1$, then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = 0$. If $\mathsf{D}(G) \in [2, 4]$, then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = \mathsf{D}(G)$.*
2. *If $\mathsf{D}(G) > 4$, then*
$$\mathsf{c}_{\mathrm{mon}}(H) \le \frac{|G|+1}{2}(2|G|(|G|+1)(\mathsf{D}(G)+1))^{|G|}.$$
3. *If G is neither cyclic nor isomorphic to any of the groups in $\{C_2^3, C_2^4, C_3^2, C_3^3, C_3^4, C_3^5, C_2 \oplus C_4, C_2 \oplus C_6\}$, then $\mathsf{D}^*(G) < \mathsf{c}_{\mathrm{mon}}(H)$.*

In Sect. 2 we introduce the required concepts, and the proofs of each of the three statements will be given in sections of their own (Sects. 3, 4, and 5). Here we only discuss the relevance of the lower bound. Suppose that G is isomorphic to $C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r$. Then $\mathsf{D}^*(G) = 1 + \sum_{i=1}^r (n_i - 1)$, $\mathsf{D}^*(G) \le \mathsf{D}(G)$, where equality holds for p-groups and others (see Sect. 2). If G is such a group, and in addition neither cyclic nor an elementary 2-group, then in combination with the above mentioned result on the catenary degree we have
$$\mathsf{c}(H) < \mathsf{D}(G) = \mathsf{D}^*(G) < \mathsf{c}_{\mathrm{mon}}(H).$$
Cyclic class groups seem to have an exceptional behavior. We end with the following conjecture.

Conjecture 1.2. *Let H be a Krull monoid with non-trivial, finite cyclic class group G, and suppose that every class contains a prime divisor. Then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = \mathsf{D}(G)$.*

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For every $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let $L, L' \subset \mathbb{Z}$. We denote by $L + L' = \{a + b \mid a \in L, b \in L'\}$ their *sumset*. Two distinct elements $k, l \in L$ are called *adjacent* if $L \cap [\min\{k, l\}, \max\{k, l\}] = \{k, l\}$. A positive integer $d \in \mathbb{N}$ is called a *distance* of L if there exist adjacent elements $k, l \in L$ with $d = |k - l|$, and we denote by $\Delta(L)$ the *set of distances*.

Monoids and factorizations. By a *monoid*, we mean a commutative, cancellative semigroup with unit element. Let H be a monoid. We denote by $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H, by $H^\times$ the group of invertible elements, by $H_{\rm red} = \{aH^\times \mid a \in H\}$ the associated reduced monoid of H, and by $\mathsf{q}(H)$ a quotient group of H with $H \subset \mathsf{q}(H)$.

For a set P, we denote by $\mathcal{F}(P)$ the *free* (*abelian*) *monoid* with basis P. Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \text{ with } \mathsf{v}_p(a) \in \mathbb{N}_0 \text{ and } \mathsf{v}_p(a) = 0 \quad \text{for almost all } p \in P.$$

We call $|a| = \sum_{p \in P} \mathsf{v}_p(a)$ the *length* of a and $\mathrm{supp}(a) = \{p \in P \mid \mathsf{v}_p(a) > 0\} \subset P$ the *support* of a.

The free (abelian) monoid $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\rm red}))$ is called the *factorization monoid* of H, the unique homomorphism

$$\pi \colon \mathsf{Z}(H) \to H_{\rm red} \text{ satisfying } \pi(u) = u \quad \text{for each } u \in \mathcal{A}(H_{\rm red})$$

is called the *factorization homomorphism* of H. For $a \in H$ and $k \in \mathbb{N}$,

$\mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H)$ is the *set of factorizations* of a,
$\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\}$ is the *set of factorizations* of a of length k, and
$\mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\} \subset \mathbb{N}_0$ is the *set of lengths* of a.

Then

$$\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a))$$

is called the *set of distances* of H, and H is said to be half-factorial if $\Delta(H) = \emptyset$.

Let $z, z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l, m, n \in \mathbb{N}_0$ and $u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathcal{A}(H_{\rm red})$ are such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

Then $\gcd(z, z') = u_1 \cdot \ldots \cdot u_l$, and we call

$$\mathsf{d}(z, z') = \max\{m, n\} = \max\{|z \gcd(z, z')^{-1}|, |z' \gcd(z, z')^{-1}|\} \in \mathbb{N}_0$$

the *distance* between z and z'. If $\pi(z) = \pi(z')$ and $z \ne z'$, then

$$2 + ||z| - |z'|| \le \mathsf{d}(z, z') \tag{2.1}$$

by [15, Lemma 1.6.2]. For subsets $X, Y \subset \mathsf{Z}(H)$, we set

$$\mathsf{d}(X, Y) = \min\{\mathsf{d}(x, y) \mid x \in X, y \in Y\},$$

and thus $\mathsf{d}(X, Y) = 0$ if and only if ($X \cap Y \ne \emptyset$ or $X = \emptyset$ or $Y = \emptyset$).

Chains of factorizations. Let $a \in H$ and $N \in \mathbb{N}_0 \cup \{\infty\}$. A finite sequence $z_0, \ldots, z_k \in \mathsf{Z}(a)$ is called a (*monotone*) *N-chain of factorizations* if $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$ (and $|z_0| \le \cdots \le |z_k|$ or $|z_0| \ge \cdots \ge |z_k|$). We denote by $\mathsf{c}(a) \in \mathbb{N}_0 \cup \{\infty\}$ (or by $\mathsf{c}_{\mathrm{mon}}(a)$ resp.) the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations $z, z' \in \mathsf{Z}(a)$ can be concatenated by an N-chain (or by a monotone N-chain resp.). Then

$$\mathsf{c}(H) = \sup\{\mathsf{c}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\} \quad \text{and}$$
$$\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{mon}}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

denote the *catenary degree* and the *monotone catenary degree* of H. The monotone catenary degree is studied by using the two auxiliary notions of the equal and the adjacent catenary degrees. Let $\mathsf{c}_{\mathrm{eq}}(a)$ denote the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations $z, z' \in \mathsf{Z}(a)$ with $|z| = |z'|$ can be concatenated by a monotone N-chain. We call

$$\mathsf{c}_{\mathrm{eq}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

the *equal catenary degree* of H. We set

$$\mathsf{c}_{\mathrm{adj}}(a) = \sup\{\mathsf{d}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \mid k, l \in \mathsf{L}(a) \text{ are adjacent}\},$$

and the *adjacent catenary degree* of H is defined as

$$\mathsf{c}_{\mathrm{adj}}(H) = \sup\{\mathsf{c}_{\mathrm{adj}}(b) \mid b \in H\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Obviously, we have

$$\mathsf{c}(a) \le \mathsf{c}_{\mathrm{mon}}(a) = \sup\{\mathsf{c}_{\mathrm{eq}}(a), \mathsf{c}_{\mathrm{adj}}(a)\} \le \sup \mathsf{L}(a) \quad \text{for all } a \in H,$$

and hence

$$\mathsf{c}(H) \le \mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{eq}}(H), \mathsf{c}_{\mathrm{adj}}(H)\}. \tag{2.2}$$

Note that $\mathsf{c}_{\mathrm{adj}}(H) = 0$ if and only if H is half-factorial, and if H is not half-factorial, then (2.1) shows that $2 + \sup \Delta(H) \le \mathsf{c}(H)$. Moreover, $\mathsf{c}_{\mathrm{eq}}(H) = 0$ if and only if for all $a \in H$ and all $k \in \mathsf{L}(a)$ we have $|\mathsf{Z}_k(a)| = 1$. Thus a recent result of Coykendall and Smith implies that for the multiplicative monoid H of non-zero elements from a domain we have $\mathsf{c}_{\mathrm{eq}}(H) = 0$ if and only if H is factorial ([5, Corollary 2.12]).

Krull monoids. A monoid homomorphism $\varphi \colon H \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies that $a \,|\, b$ for all $a, b \in H$.
- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \,|\, \varphi(u)$.
- a *divisor theory* (for H) if $D = \mathcal{F}(P)$ for some set P, φ is a divisor homomorphism, and for every $p \in P$ (equivalently for every $p \in \mathcal{F}(P)$), there exists a finite subset $\emptyset \ne X \subset H$ satisfying $p = \gcd(\varphi(X))$.

The quotient group $\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ is called the *class group* of φ. For $a \in \mathsf{q}(D)$, we denote by $[a] = [a]_\varphi = a\,\mathsf{q}(\varphi(H)) \in \mathsf{q}(D)/\mathsf{q}(\varphi(H))$ the class containing a. If $\varphi \colon H \to \mathcal{F}(P)$ is a cofinal divisor homomorphism, then

$$G_P = \{[p] = p\,\mathsf{q}(\varphi(H)) \mid p \in P\} \subset \mathcal{C}(\varphi)$$

is called the *set of classes containing prime divisors*. The monoid H is called a *Krull monoid* if it satisfies one of the following equivalent properties ([15, Theorem 2.4.8] or [17, Chapter 22]):

(a) H is v-noetherian and completely integrally closed,
(b) H has a divisor theory,
(c) H has a divisor homomorphism into a free monoid.

If H is a Krull monoid, then a divisor theory is essentially unique and the associated class group depends only on H (it is called the class group of H). An integral domain R is a Krull domain if and only if its multiplicative monoid $R\setminus\{0\}$ is a Krull monoid, and thus Property (a) shows that a noetherian domain is Krull if and only if it is integrally closed. Rings of integers, holomorphy rings in algebraic function fields, and regular congruence monoids in these domains are Krull monoids with finite class group such that every class contains a prime divisor ([15, Section 2.11]). Monoid domains and power series domains that are Krull and have prime divisors in all classes are discussed in [2,20,21].

Main portions of the arithmetic of a Krull monoid—including catenary and monotone catenary degrees—can be studied in the associated monoid of zero-sum sequences over its class group. We introduce the necessary concepts, and summarize the relationship between a Krull monoid and its associated monoid of zero-sum sequences in Lemma 2.1.

The monoid of zero-sum sequences. Let G be an additively written abelian group and $G_0 \subset G$ a subset. Then $\langle G_0 \rangle \subset G$ denotes the subgroup generated by G_0. A tuple $(e_i)_{i\in I}$ of elements of G is said to be *independent* if all elements are non-zero and if

$$\sum_{i\in I} m_i e_i = 0 \text{ implies } m_i e_i = 0 \quad \text{for all } i \in I, \text{ where } m_i \in \mathbb{Z}.$$

The tuple $(e_i)_{i\in I}$ is called a *basis* if $(e_i)_{i\in I}$ is independent and $\langle \{e_i \mid i \in I\} \rangle = G$, and G_0 is called a basis if the tuple $(g)_{g\in G_0}$ is a basis. If $1 < |G| < \infty$, then we have

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}, \quad \text{and we set} \quad \mathsf{d}^*(G) = \sum_{i=1}^{r}(n_i - 1) \quad \text{and}$$

$$\mathsf{D}^*(G) = \mathsf{d}^*(G) + 1,$$

where $r = \mathsf{r}(G) \in \mathbb{N}, n_1, \ldots, n_r \in \mathbb{N}$ are integers with $1 < n_1 \mid \cdots \mid n_r$ and $n_r = \exp(G)$ is the exponent of G. If $|G| = 1$, then $\mathsf{r}(G) = 0, \exp(G) = 1$, $\mathsf{d}^*(G) = 0$, and $\mathsf{D}^*(G) = 1$.

For our purposes it is convenient to consider sequences over G_0 as elements in the free monoid $\mathcal{F}(G_0)$. Thus sequences will be written multiplicatively. For such a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g\in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0),$$

we set $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ for any homomorphism $\varphi\colon G \to G'$, and in particular, we have $-S = (-g_1) \cdot \ldots \cdot (-g_l)$. We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S,

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ the } \textit{length} \text{ of } S,$$

$$\operatorname{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \text{ the } \textit{support} \text{ of } S,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i \text{ the } \textit{sum} \text{ of } S \quad \text{and}$$

$$\Sigma(S) = \left\{ \sum_{i \in I} g_i \mid \emptyset \ne I \subset [1, l] \right\} \text{ the } \textit{set of subsums} \text{ of } S.$$

The sequence S is said to be a *zero-sum sequence* if $\sigma(S) = 0$, and the monoid $\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$ is called the *monoid of zero-sum sequences* over G_0. Since the embedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, $\mathcal{B}(G_0)$ is a Krull monoid. For every arithmetical invariant $*(H)$ defined for a monoid H, it is usual to write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$ (whenever the meaning is clear from the context). In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ and $\mathsf{c}_{\mathrm{mon}}(G_0) = \mathsf{c}_{\mathrm{mon}}(\mathcal{B}(G_0))$.

The *Davenport constant* $\mathsf{D}(G_0)$, defined as

$$\mathsf{D}(G_0) = \sup\{|U| \mid U \in \mathcal{A}(G_0)\} \in \mathbb{N}_0 \cup \{\infty\},$$

is a central invariant in Combinatorial Number Theory. If G_0 is finite, then $\mathsf{D}(G_0) < \infty$ ([15, Theorem 3.4.2]). Let G be finite. Then $\mathsf{D}^*(G) \le \mathsf{D}(G)$, and equality holds for p-groups, groups having rank at most two, and others (see [12, Section 2], [28, Section 4.1]).

Lemma 2.1. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. Let $\widetilde{\boldsymbol{\beta}}\colon F \to \mathcal{F}(G_P)$ denote the unique homomorphism defined by $\widetilde{\boldsymbol{\beta}}(p) = [p]$ for all $p \in P$.*

1. *The homomorphism $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \circ \varphi\colon H \to \mathcal{B}(G_P)$ is a transfer homomorphism with $\mathsf{c}(H, \boldsymbol{\beta}) \le 2$.*
2. *If H is not half-factorial, then*

$$\mathsf{c}_{\mathrm{eq}}(H) = \mathsf{c}_{\mathrm{eq}}(G_P), \mathsf{c}_{\mathrm{adj}}(H) = \mathsf{c}_{\mathrm{adj}}(G_P), \quad \textit{and} \quad \mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(G_P).$$

Proof. See Lemmas 3.2 and 3.3 in [13]. □

3. An Upper Bound for the Monotone Catenary Degree

The non-uniqueness of factorizations in atomic monoids and domains can be described by a variety of arithmetical invariants, such as sets of lengths, catenary and tame degrees. Only recently these arithmetical invariants have been characterized in abstract semigroup theoretical terms, such as minimal relations and presentations (see, for example, [4,24,26]).

In the present paper we will make use of the abstract upper bounds both for the equal and for the adjacent catenary degrees derived in [1]. Let H be an atomic monoid. Then

$$\sim_{H,\mathrm{eq}} = \{(x,y) \in \mathsf{Z}(H) \times \mathsf{Z}(H) \mid \pi(x) = \pi(y) \text{ and } |x| = |y|\}$$

is called the *monoid of equal-length relations* of H. For a subset $H' \subset H$, we set $\mathsf{Z}(H') = \{z \in \mathsf{Z}(H) \mid \pi(z) \in H'\}$. Let $Z \subset \mathsf{Z}(H)$ be a subset. We say that an element $x \in Z$ is *minimal* in Z if for all elements $y \in Z$ with $y \,|\, x$ it follows that $x = y$. We denote by $\mathrm{Min}(Z)$ the *set of minimal elements* in Z. Let $x \in Z$. Since the number of elements $y \in Z$ with $y \,|\, x$ is finite, there exists an $x^* \in \mathrm{Min}(Z)$ with $x^* \,|\, x$.

Lemma 3.1. *Let H be an atomic monoid.*

1. $\mathsf{c}_{\mathrm{eq}}(H) \le \sup\{|x| \mid (x,y) \in \mathcal{A}(\sim_{H,\mathrm{eq}})$ *for some* $y \in \mathsf{Z}(H) \setminus \{x\}\}$.
2. *For $d \in \Delta(H)$ let $A_d = \{x \in \mathsf{Z}(H) \,|\, |x| - d \in \mathsf{L}(\pi(x))\}$. Then* $\mathsf{c}_{\mathrm{adj}}(H) \le \sup\{|x| \mid x \in \mathrm{Min}(A_d), d \in \Delta(H)\}$.

Proof. See [1, Proposition 4.4]. □

In Sect. 5 we shall see that the above upper bounds are sharp for Krull monoids with small class groups. Here we are going to combine these abstract upper bounds with a result on so-called primitive partition identities due to Diaconis, Graham and Sturmfels. The idea to use this result for getting upper bounds on arithmetical invariants is due to Freeze and Schmid (see [11, Theorem 6.5]). We formulate the theorem on partition identities in a language which is suitable for our applications. Let $d \in \mathbb{N}$ and $A \subset \mathbb{Z}^d$ be a finite subset with $\langle A \rangle = \mathbb{Z}^d$. Then

$$\mathsf{P}(A) = \{(S_1,S_2) \in \mathcal{F}(A) \times \mathcal{F}(A) \mid \sigma(S_1) = \sigma(S_2)\} \subset \mathcal{F}(A) \times \mathcal{F}(A)$$

is a submonoid, and since the embedding $\mathsf{P}(A) \hookrightarrow \mathcal{F}(A) \times \mathcal{F}(A)$ is a divisor homomorphism, it is a Krull monoid. Consider $C = (g)_{g \in A}$ as an integer $|A| \times d$ matrix and define

$$M(A) = \max\{|\det(C')| \mid C' \text{is a } d \times d \text{ minor of } C\}.$$

Lemma 3.2. *If $S = (S_1,S_2)$ is an atom of $\mathsf{P}(A)$, then*

$$|S_1| + |S_2| \le (2d)^d (d+1)^{d+1} M(A).$$

Proof. See [7, Theorem 1], and note that the atoms of $\mathsf{P}(A)$ correspond to the primitive partition identities in that paper. □

Proposition 3.3. *Let G be an abelian group and $G_0 \subset G$ a finite subset. Then*

$$\mathsf{c}_{\mathrm{mon}}(G_0) \le \frac{|G_0|+2}{2}\left((2|G_0|+2)(|G_0|+2)(\mathsf{D}(G_0)+1)\right)^{|G_0|+1}.$$

Proof. Since G_0 is finite, the set $\mathcal{A}(G_0)$ is finite and hence $\mathsf{D}(G_0) < \infty$ ([15, Theorem 3.4.2]). We define a map

$$f\colon \mathcal{A}(G_0) \to \mathbb{N}_0^{G_0} \times \mathbb{N}_0, \quad \text{where } f(U) = ((\mathsf{v}_g(U))_{g\in G_0}, 1) \quad \text{for all } U \in \mathcal{A}(G_0),$$

and we set

$$A = \left\{ ((m_g)_{g\in G_0}, l) \in \mathbb{N}_0^{G_0} \times \mathbb{N}_0 \mid l + \sum_{g\in G_0} m_g \le \mathsf{D}(G_0) + 1 \right\} \subset \mathbb{Z}^{G_0} \times \mathbb{Z}.$$

We observe that $f(U) \in A$ for all $U \in \mathcal{A}(G_0)$, and that $\langle A\rangle = \mathbb{Z}^{G_0} \times \mathbb{Z}$. By [22], it follows that $M(A) \le (\mathsf{D}(G_0)+1)^{|G_0|+1}$. Now we proceed in two steps.

First we show that the equal catenary degree is less than or equal to the asserted bound. We use Lemma 3.1.1. Let $(x, y) \in \mathcal{A}(\sim_{\mathcal{B}(G_0),\mathrm{eq}})$, say $x = U_1 \cdot \ldots \cdot U_k$ and $y = V_1 \cdot \ldots \cdot V_k$ with $U_1, V_1, \ldots, U_k, V_k \in \mathcal{A}(G_0)$. We set $S_1 = f(U_1) \cdot \ldots \cdot f(U_k) \in \mathcal{F}(A)$ and $S_2 = f(V_1) \cdot \ldots \cdot f(V_k) \in \mathcal{F}(A)$. Then $\sigma(S_1) = \sigma(S_2)$, and since (x, y) is an atom of $\sim_{\mathcal{B}(G_0),\mathrm{eq}}$, it follows that (S_1, S_2) is an atom of $\mathsf{P}(A)$. By Lemma 3.2, we infer that

$$\begin{aligned} |x| = k = |S_1| &= \frac{1}{2}(|S_1| + |S_2|) \le \frac{1}{2}(2|G_0|+2)^{|G_0|+1}(|G_0|+2)^{|G_0|+2} M(A) \\ &\le \frac{|G_0|+2}{2}((2|G_0|+2)(|G_0|+2)(\mathsf{D}(G_0)+1))^{|G_0|+1}, \end{aligned}$$

and thus

$$\mathsf{c}_{\mathrm{eq}}(G_0) \le \frac{|G_0|+2}{2}((2|G_0|+2)(|G_0|+2)(\mathsf{D}(G_0)+1))^{|G_0|+1}.$$

In the second step we consider the adjacent catenary degree. We use Lemma 3.1.2. Let $d \in \Delta(G_0)$ and $x \in \mathrm{Min}(A_d)$. Suppose that $x = V_1 \cdot \ldots \cdot V_{k+d}$ and that $U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_{k+d}$, where $U_1, \ldots, U_k, V_1, \ldots, V_{k+d} \in \mathcal{A}(G_0)$. We set $S_1 = f(U_1) \cdot \ldots \cdot f(U_k)(\mathbf{0}, 1)^d \in \mathcal{F}(A)$ and $S_2 = f(V_1) \cdot \ldots \cdot f(V_{k+d}) \in \mathcal{F}(A)$. Since $x \in \mathrm{Min}(A_d)$ and $(\mathbf{0}, 1) \nmid S_2$, it follows that (S_1, S_2) is an atom of $\mathsf{P}(A)$. Using Lemma 3.2 and arguing as above, we infer that

$$|x| = |S_2| = \frac{1}{2}(|S_1| + |S_2|),$$

and thus

$$\mathsf{c}_{\mathrm{adj}}(G_0) \le \frac{|G_0|+2}{2}((2|G_0|+2)(|G_0|+2)(\mathsf{D}(G_0)+1))^{|G_0|+1}.$$

□

Note the upper bound in Theorem 1.1 is a special case of Theorem 3.4.

Theorem 3.4. *Let H be a Krull monoid, $\varphi\colon H \to F = \mathcal{F}(P)$ a cofinal divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group, and $G_P \subset G$ the set of classes containing prime divisors. Suppose that $G_P \subset G$ is finite, and set $G_P^\bullet = G_P \setminus \{0\}$.*

1. *If H is half-factorial, then* $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) \le \mathsf{D}(G_P)$.
2. *Suppose that H is not half-factorial. Then*

$$\mathsf{c}_{\mathrm{mon}}(H) \le \frac{|G_P^\bullet| + 2}{2}((2|G_P^\bullet| + 2)(|G_P^\bullet| + 2)(\mathsf{D}(G_P^\bullet) + 1))^{|G_P^\bullet|+1}.$$

Proof. 1. If H is half-factorial, then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H)$ by definition, and $\mathsf{c}(H) \le \mathsf{D}(G_P)$ by [15, Theorem 3.4.10.5].

2. If $0 \in G_P$, then $\mathcal{B}(G_P) = \mathcal{F}(\{0\}) \times \mathcal{B}(G_P^\bullet)$. This shows that $\mathsf{c}_{\mathrm{mon}}(G_P) = \mathsf{c}_{\mathrm{mon}}(G_P^\bullet)$. Thus the assertion follows from Lemma 2.1 and from Proposition 3.3.

□

The above upper bound does not seem to have the right order of magnitude. However, this is what one gets without solid information on the minimal relations in the monoid $\mathcal{B}(G)$. The complexity of a full discussion of possible minimal relations is extreme, even for small groups. This will be demonstrated by the examples in Sect. 5.

4. A lower bound for the monotone catenary degree

Throughout, we will use the following observation. Let H be an atomic monoid, $a \in H$ and $k \in \mathsf{L}(a)$. If $\mathsf{Z}_k(a) = Z \cup Z'$, then $\mathsf{c}_{\mathrm{eq}}(H) \ge \mathsf{c}_{\mathrm{eq}}(a) \ge \mathsf{d}(Z, Z')$.

Lemma 4.1. *Let $G = C_2 \oplus C_{2n}$ with $n \ge 2$. Then $\mathsf{c}_{\mathrm{eq}}(G) \ge 3n - 2$.*

Proof. Let (e_1, e_2) be a basis of G with $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = 2n$. Then

$$z = (e_1(ne_2)(e_1 + ne_2))^{2n-2}(e_2(-e_2))^n$$

and

$$z' = e_1^{2n-2}(ne_2)^{2n-4}(e_1 + ne_2)^{2n-2}(e_2^n(ne_2))((-e_2)^n(ne_2))$$

are factorizations of the same element $A = \pi(z) = \pi(z')$, and we have $|z| = |z'| = 3n - 2$. We provide the list of all atoms dividing A:

$$e_1^2,\ (ne_2)^2,\ (e_1 + ne_2)^2,\ e_2^n(ne_2),\ (-e_2)e_2,\ (-e_2)^n(ne_2),$$
$$e_1e_2^n(e_1 + ne_2),\ e_1(-e_2)^n(e_1 + ne_2),\ e_1(ne_2)(e_1 + ne_2).$$

Thus, if $e_1e_2^n(e_1 + ne_2)$ or $e_1(-e_2)^n(e_1 + ne_2)$ is a divisor of a factorization $y \in \mathsf{Z}(A)$, then $(-e_2)^n e_1(e_1 + ne_2) \,|\, y$ or $e_2^n(ne_2) \,|\, y$, and so $|y| \le 3n - 3 < |z|$. Therefore z and z' are the only two factorizations of A having length $3n - 2$, and hence

$$\mathsf{c}_{\mathrm{eq}}(A) \ge \mathsf{d}(z, z') = |z| = 3n - 2.$$

□

Proposition 4.2. *For $i \in [1, 2]$, let H_i be an atomic monoid and $a_i \in H_i$ such that $\mathsf{L}(a_i) = \{2, m_i\}$ with $m_i \ge 3$, and $|\mathsf{Z}_{m_i}(a_i)| = 1$. Then*

$$\mathsf{c}_{\mathrm{eq}}(H_1 \times H_2) \ge \frac{m_1m_2 - 4}{d} \quad \textit{where } d = \gcd(m_1 - 2, m_2 - 2).$$

Proof. For $i \in [1,2]$, let $x_i, y_i \in \mathsf{Z}(a_i)$ with $|x_i| = 2$ and $|y_i| = m_i$. Furthermore, let $k, l \in \mathbb{N}$ with $\gcd(k,l) = 1$ and

$$a = a(k,l) = a_1^k a_2^l \in H_1 \times H_2.$$

Then

$$z = x_1^k y_2^l \quad \text{and} \quad z' = y_1^k x_2^l$$

are factorizations of a, and for their lengths we have

$$|z| = 2k + lm_2 \quad \text{and} \quad |z'| = km_1 + 2l.$$

Obviously, $|z| = |z'|$ if and only if $k(m_1-2) = l(m_2-2)$ if and only if $k = \frac{m_2-2}{d}$ and $l = \frac{m_1-2}{d}$. Suppose this holds. Then any factorization $y \in \mathsf{Z}(a)$ has the form

$$y = \widetilde{x_1}^{k-u}\, y_1^u\, \widetilde{x_2}^v\, y_2^{l-v} \quad \text{where } u \in [0,k], v \in [0,l] \text{ and } \widetilde{x_1} \in \mathsf{Z}_2(a_1), \widetilde{x_2} \in \mathsf{Z}_2(a_2),$$

and then

$$|y| = 2(k-u) + um_1 + 2v + (l-v)m_2 = 2k + lm_2 + (m_1-2)u - (m_2-2)v.$$

Then $|y| = |z|$ if and only if $(u,v) \in \{(0,0), (\frac{m_2-2}{d}, \frac{m_1-2}{d})\}$. This shows that $\mathsf{Z}_{|z|}(a) = Z \cup Z'$, where

$$Z = \{\widetilde{x_1}^k y_2^l \mid \widetilde{x_1} \in \mathsf{Z}_2(a_1)\} \quad \text{and} \quad Z' = \{y_1^k \widetilde{x_2}^l \mid \widetilde{x_2} \in \mathsf{Z}_2(a_2)\},$$

and hence

$$\mathsf{c}_{\mathrm{eq}}(H_1 \times H_2) \geq \mathsf{c}_{\mathrm{eq}}(a) = \mathsf{d}(Z, Z') = |z| = 2k + lm_2 = \frac{m_1 m_2 - 4}{d}.$$

□

Corollary 4.3. *Let G be a finite abelian group.*

1. *If $G = C_2^r$ with $r \geq 5$, then $\mathsf{c}_{\mathrm{eq}}(G) \geq \left\lfloor \frac{r+1}{2} \right\rfloor^2 + \left\lfloor \frac{r+1}{2} \right\rfloor - 4 > r + 1 = \mathsf{D}(G)$.*
2. *If $G = C_{n_1} \oplus C_{n_2}$ with $2 < n_1 \mid n_2$ and $d = \gcd(n_1-2, n_2-2)$, then $\mathsf{c}_{\mathrm{eq}}(G) \geq \frac{n_1 n_2 - 4}{d}$.*
3. *If $G = C_2^r \oplus C_n$ with $r \geq 2, n \geq 2$ even and $d = \gcd(r-1, n-2)$, then $\mathsf{c}_{\mathrm{eq}}(G) \geq \frac{(r+1)n-4}{d}$.*
4. *Let $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r, r \geq 2$, and let $k_1, \ldots, k_r \in \mathbb{N}$ be such that $2k_i \leq n_i$ for all $i \in [1,r]$.*
   (i) *Let $I, J \subset [1,r]$ be disjoint subsets such that $a = 1 + \sum_{i \in I} k_i \geq 3$ and $b = 1 + \sum_{j \in J} k_j \geq 3$.*
   (ii) *Let $a = 1 + \sum_{i=1}^{r-1} k_i \geq 3$ and $b \in [3, n_r]$.*
   Then, in both cases, we have

$$\mathsf{c}_{\mathrm{eq}}(G) \geq \frac{ab-4}{d} \quad \textit{where } d = \gcd(a-2, b-2).$$

Proof. 1. Let $(e_1, \ldots, e_r)$ be a basis of $G, t \in [2, r]$ and $s \in [1, t-1]$ with $\gcd(s-1, t-s-1) = 1$. We set $H_1 = \mathcal{B}(\langle e_1, \ldots, e_s \rangle), a_1 = (e_1 \cdot \ldots \cdot e_s (e_1 + \cdots + e_s))^2$, and note that $\mathsf{L}(a_1) = \{2, s+1\}$. We set $H_2 = \mathcal{B}(\langle e_{s+1}, \ldots, e_t \rangle)$, $a_2 = (e_{s+1} \cdot \ldots \cdot e_t(e_{s+1} + \ldots + e_t))^2$, and note that $\mathsf{L}(a_2) = \{2, t-s+1\}$. Since $\gcd(s-1, t-s-1) = 1$, Proposition 4.2 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge (s+1)(t-s+1) - 4$.

If $r = 2m+1$ with $m \ge 2$, then we choose $s = m$ and $t = 2m+1$, and thus

$$\mathsf{c}_{\mathrm{eq}}(C_2^r) \ge (m+1)(m+2) - 4 = m^2 + 3m - 2 = \left(\frac{r+1}{2}\right)^2 + \frac{r+1}{2} - 4 > r+1.$$

If $r = 2m$ with $m \ge 3$, then

$$\mathsf{c}_{\mathrm{eq}}(C_2^r) \ge \mathsf{c}_{\mathrm{eq}}(C_2^{r-1}) \ge \left(\frac{r}{2}\right)^2 + \frac{r}{2} - 4 = \left\lfloor \frac{r+1}{2} \right\rfloor^2 + \left\lfloor \frac{r+1}{2} \right\rfloor - 4.$$

2. Let (e_1, e_2) be a basis of G with $\mathrm{ord}(e_1) = n_1$ and $\mathrm{ord}(e_2) = n_2$. For $i \in [1, 2]$, we set $H_i = \mathcal{B}(\langle e_i \rangle), a_i = ((-e_i)e_i)^{n_i}$, and observe that $\mathsf{L}(a_i) = \{2, n_i\}$. Now Proposition 4.2 implies the assertion.
3. Let $(e_1, \ldots, e_{r+1})$ be a basis of G with $\mathrm{ord}(e_1) = \ldots = \mathrm{ord}(e_r) = 2$ and $\mathrm{ord}(e_{r+1}) = n$. We set $H_1 = \mathcal{B}(\langle e_1, \ldots, e_r \rangle)$, $a_1 = (e_1 \cdot \ldots \cdot e_r(e_1 + \cdots + e_r))^2$, and note that $\mathsf{L}(a_1) = \{2, r+1\}$. We set $H_2 = \mathcal{B}(\langle e_{r+1} \rangle), a_2 = ((-e_{r+1})e_{r+1})^n$, and note that $\mathsf{L}(a_2) = \{2, n\}$. Now Proposition 4.2 implies the assertion.
4. Let $(e_1, \ldots, e_r)$ be a basis of G with $\mathrm{ord}(e_i) = n_i$ for all $i \in [1, r]$. For a subset $\Omega \subset [1, r]$ with $k_\Omega = 1 + \sum_{i \in \Omega} k_i \ge 3$, we set $e_\Omega = -\sum_{i \in \Omega} k_i e_i$ and $a_\Omega = e_\Omega(-e_\Omega) \prod_{i \in \Omega} e_i^{k_i}(-e_i)^{k_i} \in \mathcal{B}(\langle e_i \mid i \in \Omega \rangle)$. Then [15, Lemma 6.4.1] implies that $\mathsf{L}(a_\Omega) = \{2, k_\Omega\}$ and $|\mathsf{Z}_{k_\Omega}(a_\Omega)| = 1$ (however, in general $|\mathsf{Z}_2(a_\Omega)| > 1$).
   (i) We set $H_1 = \mathcal{B}(\langle e_i \mid i \in I \rangle), a_1 = a_I, H_2 = \mathcal{B}(\langle e_j \mid j \in J \rangle)$ and $a_2 = a_J$. Then Proposition 4.2 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge \frac{ab-4}{d}$.
   (ii) We set $I = [1, r-1], H_1 = \mathcal{B}(\langle e_i \mid i \in I \rangle), a_1 = a_I, H_2 = \mathcal{B}(\langle e_r \rangle)$ and $a_2 = (-V)V$ with $V = e_r^{b-1}((b+1)e_r)$. Since $|\mathsf{Z}_b(a_2)| = 1$, Proposition 4.2 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge \frac{ab-4}{d}$.

□

Lemma 4.4. *Let $a, b \in \mathbb{N}$ with $b \ge a \ge 3$, and let $q \ge 3$ be the smallest prime power dividing a (thus, q is either an odd prime or $q = 4$). Then there exists some $v \in [0, \frac{a}{q} + 1]$ such that* $\gcd(a, b-v) = 1$.

Proof. There are uniquely determined positive integers u and t such that

$$b = \frac{a}{q}u + t \quad \text{with} \quad t \in \left[1, \frac{a}{q}\right].$$

If q is an odd prime, then $\gcd(\frac{a}{q}u+1, q) = 1$ or $\gcd(\frac{a}{q}u-1, q) = 1$, which implies that $b-(t-1) = \frac{a}{q}\cdot u+1$ or $b-(t+1) = \frac{a}{q}u-1$ is coprime to a. If $q = 4$ and $\frac{a}{q}u$ is even, then $\gcd(\frac{a}{q}u+1, q) = 1$, which implies that $b-(t-1) = \frac{a}{q}u+1$ is coprime to a. Suppose that $q = 4$ and $\frac{a}{q}u$ is odd. If $t \ge 2$, then $b-(t-2) = \frac{a}{q}u+2$ is coprime to a. If $t = 1$ and $\frac{a}{q} \ge 2$, then $b-3 = \frac{a}{q}-2$ is coprime to a. If $t = 1$ and $\frac{a}{q} = 1$, then $a = 4, b = u+1$, and there is some $v \in [0, 2]$ such that $\gcd(4, b-v) = 1$. □

Theorem 4.5. *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. Suppose that G is neither cyclic nor isomorphic to any of the groups in $\{C_2^3, C_2^4, C_3^2, C_3^3, C_3^4, C_3^5, C_2 \oplus C_4, C_2 \oplus C_6\}$. Then $\mathsf{D}^*(G) < \mathsf{c}_{\mathrm{mon}}(H)$, and in case that G is not isomorphic to $C_n \oplus C_n$ with $n \in [4, 5]$, then we even have $\mathsf{D}^*(G) < \mathsf{c}_{\mathrm{eq}}(H)$.*

Proof. Since G is not cyclic, we may suppose that $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $r \ge 2$ and $1 < n_1 \mid \cdots \mid n_r$, and then $\mathsf{D}^*(G) = 1+\sum_{i=1}^{r}(n_i-1)$. By Lemma 2.1 it is sufficient to consider $\mathcal{B}(G)$ instead of H.

If G is an elementary 2-group of rank r, then by assumption we have $r \ge 5$, and hence the assertion follows from Corollary 4.3.1. From now on we suppose that G is not an elementary 2-group. We consider a partition of $[1, r]$, say $[1, r] = I \uplus J$, and set

$$a = a(I) = 1 + \sum_{i \in I} \left\lfloor \frac{n_i}{2} \right\rfloor \quad \text{and} \quad b = b(J) = 1 + \sum_{j \in J} \left\lfloor \frac{n_j}{2} \right\rfloor \qquad \text{such that } b \ge a.$$

Suppose that in all possible partitions we always get $a = 2$. Then G equals one of the following groups: $C_3^3, C_2 \oplus C_{2n}, C_3 \oplus C_{3n}$ with $n \in \mathbb{N}$. By assumption, G differs from C_3^3. If $G = C_2 \oplus C_{2n}$, then $n \ge 4$ by assumption, and Lemma 4.1 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 3n-2 > 2n+1 = \mathsf{D}^*(G)$. If $G = C_3 \oplus C_{3n}$, then $n \ge 2$ by assumption, and Corollary 4.3.2 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 9n-4 > 3n+2 = \mathsf{D}^*(G)$.

Now we fix an arbitrary partition with $b = b(J) \ge a = a(I) \ge 3$. Before we enter the discussion of various cases, we note that $\mathsf{D}^*(G) \le 2(a+b)-1$, and if n_1 is even, then $2(a+b) = \mathsf{D}^*(G)+r+3$. Moreover, if $\gcd(a-2, b-2) = 1$, then Corollary 4.3.4(i) implies that

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) \ge ab - 4 - 2(a+b) + 1 = (a-2)(b-2) - 7. \qquad (*)$$

Case 1: $a = 3$.

First we handle the case $b = 3$. Then G equals one of the following groups: $C_2 \oplus C_2 \oplus C_4, C_4 \oplus C_4, C_5 \oplus C_5$.

If $G = C_2 \oplus C_2 \oplus C_4$, then Corollary 4.3.3 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 8 > 6 = \mathsf{D}^*(G)$.

If $G = C_4 \oplus C_4$ and (e_1, e_2) is a basis of G, then

$$z = (e_1(-e_1))^4 (e_2^2(2e_2))^2 ((-e_2)^2(2e_2))^2 \quad \text{and} \quad z' = e_1^4(-e_1)^4 e_2^4(-e_2)^4((2e_2)(2e_2))^2$$

are factorizations of the same element $A = \pi(z) = \pi(z')$, and hence $\mathsf{c}_{\mathrm{mon}}(G) \ge \mathsf{c}_{\mathrm{adj}}(A) \ge 8 > \mathsf{D}^*(G)$.

If $G = C_5 \oplus C_5$ and (e_1, e_2) is a basis of G, then

$$z = (e_1(-e_1))^5 (e_2^2(-2e_2))^3 ((-e_2)^2(2e_2))^3 \quad \text{and}$$
$$z' = e_1^5(-e_1)^5((-e_2)e_2)^6((2e_2)(-2e_2))^2$$

are factorizations of the same element $A = \pi(z) = \pi(z')$, and hence $\mathsf{c}_{\mathrm{mon}}(G) \geq \mathsf{c}_{\mathrm{adj}}(A) \geq 11 > \mathsf{D}^*(G)$.

Now we suppose that $b \geq 4$, and we distinguish two subcases.

Case 1.1: $2 \mid n_1$.

Since $\mathsf{D}^*(G) = 2(a+b) - r - 3$, Corollary 4.3.4(i) implies that

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) = b - 7 + r.$$

If $b + r > 7$, then we are done. Suppose that $b + r \leq 7$. Since $b \geq 4$, we get $r \in [2, 3]$. If $r = 3$ and $n_1 = 4$, then $G = C_4 \oplus C_{n_2} \oplus C_{n_3}$ and hence $b > 4$, a contradiction. If $r = 3$ and $n_1 = 2$, then $G = C_2 \oplus C_2 \oplus C_{2n}$, $b = n+1 = 4$, and thus Corollary 4.3.3 implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 14 > \mathsf{D}^*(G)$. Suppose that $r = 2$. Then $(b, r) \in \{(4, 2), (5, 2)\}$, $G = C_4 \oplus C_{4n}$ and hence $b = 2n+1$, which implies $b = 5$ and $n = 2$. Thus $G = C_4 \oplus C_8$, $\mathsf{D}^*(G) = 11$ and $\mathsf{c}_{\mathrm{eq}}(G) \geq 14$ by Corollary 4.3.2.

Case 1.2: $2 \nmid n_1$.

If $b \geq 10$, then $(*)$ implies the assertion. Suppose that $b \in [4, 9]$. Here is the list of possible groups:

- $G = C_3^2 \oplus C_9$: Then $\mathsf{D}^*(G) = 13$, and Corollary 4.3.4(ii) (with $b = 9$) implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 23$.
- $G = C_3^i \oplus C_6^2$ with $i \in [2, 4]$: Then $\mathsf{D}^*(G) = 2i + 11$. If $i = 3$, then Corollary 4.3.4(ii) (with $b = 6$ and $a = i + 4 = 7$) implies the assertion. If $i \in \{2, 4\}$, then Corollary 4.3.4(ii) (with $b = 6$ and $a = i + 3$) implies the assertion.
- $G = C_3^i \oplus C_6$ with $i \in [5, 7]$: Then $\mathsf{D}^*(G) = 2i + 6$. If $i = 6$, then Corollary 4.3.4(ii) (with $b = 6$ and $a = 7$) implies the assertion. If $i \in \{5, 7\}$, then Corollary 4.3.4(ii) (with $b = 6$ and $a = i$) implies the assertion.
- $G = C_3^i$ with $i \in [6, 10]$: Then $\mathsf{D}^*(G) = 2i + 1$. Corollary 4.3.4(ii) (with $b = 3$) implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 3i - 4$, and hence the assertion follows.

Case 2: $a \geq 4$.

We distinguish two cases.

Case 2.1: $\gcd(a - 2, b - 2) = 1$.

If $b \geq 6$, then $(*)$ implies the assertion. If $b < 6$, then $(a, b) = (4, 5)$. Then Corollary 4.3.4(i) implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 16$, and G equals one of the following groups: $C_3^7, C_3^4 \oplus C_6, C_3^3 \oplus C_9, C_2^3 \oplus C_4^2, C_2^4 \oplus C_6$. For all of them we have $\mathsf{D}^*(G) < 16$.

Case 2.2: $\gcd(a - 2, b - 2) > 1$.

First we handle the case $a = 4$. Then $\gcd(a - 2, b - 2) = 2$. We choose $v = 1$ in Lemma 4.4 and obtain that $\mathsf{c}_{\mathrm{eq}}(G) \geq a(b - 1) - 4$. If $2 \mid n_1$, then $2(a + b) = \mathsf{D}^*(G) + r + 3$ and

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) \geq a(b-1) - 4 - 2(a+b) + r + 3 = 2b + r - 13.$$

So it remains to consider the case $b = 4$.

If $2 \nmid n_1$, then $\mathsf{D}^*(G) \leq 2(a+b) - 1$ and

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) \geq a(b-1) - 4 - 2(a+b) + 1 = 2b - 15.$$

So it remains to consider the cases $b \in \{4, 6\}$.

Suppose that $a = b = 4$. We provide the list of possible groups and for each group a lower bound for $\mathsf{c}_{\mathrm{eq}}(G)$ which is strictly larger than $\mathsf{D}^*(G)$:

- $G = C_2^4 \oplus C_4$: Corollary 4.3.3 implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 16$.
- $G = C_2^3 \oplus C_6$: Corollary 4.3.3 implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 10$.
- $G = C_3^6$: Corollary 4.3.4(ii) (with $b = 3$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 14$.
- $G = C_3^3 \oplus C_6$: Corollary 4.3.4(ii) (with $b = 5$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 16$.
- $G = C_2^2 \oplus C_4^2$: Using a new partition with $a' = 3$ and $b' = 5$ Corollary 4.3.4(i) implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 11$.
- $G = C_6 \oplus C_6$: Corollary 4.3.4(ii) (with $b = 5$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 16$.
- $G = C_7 \oplus C_7$: We use Lemma 4.4 with $v = 1$ and 4.3.4(i) to obtain $\mathsf{c}_{\mathrm{eq}}(G) \geq 16$.

Suppose that $a = 4, b = 6$ and that n_1 is odd. We provide the list of possible groups and for each group a lower bound for $\mathsf{c}_{\mathrm{eq}}(G)$ which is strictly larger than $\mathsf{D}^*(G)$:

- $G = C_3 \oplus C_6 \oplus C_9$: Corollary 4.3.4(ii) (with $b = 9$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 41$.
- $G = C_3^5 \oplus C_6$: Using a new partition with $a' = 3$ and $b' = 7$ Corollary 4.3.4(i) implies that $\mathsf{c}_{\mathrm{eq}}(G) \geq 17$.
- $G = C_3^8$: Corollary 4.3.4(ii) (with $b = 3$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 20$.
- $G = C_3^4 \oplus C_9$: Corollary 4.3.4(ii) (with $b = 9$) implies $\mathsf{c}_{\mathrm{eq}}(G) \geq 41$.

Now suppose that $b \geq a \geq 5$. We use Lemma 4.4 with $b - 2 \geq a - 2 \geq 3$. Let $q \geq 3$ be the smallest prime power dividing $a - 2$, and let $v \in [0, \frac{a-2}{q} + 1]$ be such that $\gcd(a-2, b-2-v) = 1$. Then Corollary 4.3.4(i) implies that

$$\mathsf{c}_{\mathrm{eq}}(G) \geq a(b-v) - 4 \geq a\left(b - \frac{a-2}{q} - 1\right) - 4.$$

We set $b = ac$ with $c \in \mathbb{Q}_{\geq 1}$, and distinguish two more cases.

Case 2.2.1: $2 \mid n_1$.

We use that $2(a+b) = \mathsf{D}^*(G) + r + 3$, and consider the lower bound for $\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G)$ as a function f in a. Thus

$$\begin{aligned} f(a) &= a\left(b - \frac{a-2}{q} - 1\right) - 4 - \mathsf{D}^*(G) = a^2c - 2ac - 3a + (r-1) - \frac{a^2 - 2a}{q} \\ &= \frac{cq-1}{q}a^2 - \frac{2(cq-1) + 3q}{q}a + (r-1). \end{aligned}$$

Since

$$\frac{2(2cq-1)+3q}{q}\left(\frac{cq-1}{q}\right)^{-1} = \frac{2(cq-1)+3q}{cq-1} = 2+\frac{3q}{cq-1} \le 7,$$

it follows that $f(a) > 0$ for all $a \ge 7$.

If $a = 6$, then $a-2 = 4 = q$ and $2+\frac{12}{4c-1} \le 6$, which implies that $f(a) > 0$. Suppose that $a = 5$. Then $q = 3$. If $b \ge 7$, then $c \ge \frac{7}{5} > \frac{4}{3}$, $2+\frac{9}{3c-1} < 5$ and $f(a) > 0$. If $b < 7$, then $1 < \gcd(a-2, b-2) = 3$ implies $b = 5$. Now we use Lemma 4.4 with $v = 1$ and obtain

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) \ge a(b-v) - 4 - 2(a+b) + r + 3 = r - 1 > 0.$$

Case 2.2.2: $2 \nmid n_1$.

We use that $\mathsf{D}^*(G) \le 2(a+b)-1$, and again we consider the lower bound for $\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G)$ as a function f in a. Thus

$$\begin{aligned} f(a) &= a\left(b - \frac{a-2}{q} - 1\right) - 4 - \mathsf{D}^*(G) = a^2c - 2ac - 3a - 3 - a\frac{a-2}{q} \\ &= \frac{cq-1}{q}a^2 - \frac{2(cq-1)+3q}{q}a - 3. \end{aligned}$$

Since

$$\frac{2(cq-1)+3q}{cq-1} + \frac{3q}{a(cq-1)} = 2 + \frac{3q}{cq-1} + \frac{3q}{a(cq-1)} < 8,$$

it follows that $f(a) > 0$ for all $a \ge 8$. If $a = 7$, then $a-2 = 5 = q$ and $2+\frac{15}{5c-1}+\frac{15}{7(5c-1)} < 7$, which implies that $f(7) > 0$.

Suppose that $a = 6$. Then $a-2 = 4 = q$. If $b \ge 7$, then $c \ge \frac{7}{6}$, $2+\frac{12}{4c-1}+\frac{12}{6(4c-1)} < 6$ and $f(6) > 0$. If $b = 6$, then we use Lemma 4.4 with $v = 1$ and obtain

$$\mathsf{c}_{\mathrm{eq}}(G) - \mathsf{D}^*(G) \ge a(b-v) - 4 - 2(a+b) + 1 > 0.$$

Suppose that $a = 5$. Then $a-2 = 3 = q$. If $b \ge 8$, then $c \ge \frac{8}{5}$, $2+\frac{9}{3c-1}+\frac{9}{5(3c-1)}) < 5$ and $f(5) > 0$. If $b \le 7$, then $1 < \gcd(3, b-2) = 3$ implies that $b = 5$.

So finally we suppose that $a = b = 5$. Again we provide the list of possible groups and for each group a lower bound for $\mathsf{c}_{\mathrm{eq}}(G)$ which is strictly larger than $\mathsf{D}^*(G)$:

$$C_3^5 \oplus C_6, C_3^2 \oplus C_6^2, C_3^8, C_3^4 \oplus C_9, C_9^2, C_5^4.$$

For the first two groups we use a new partition with $a' = 3$ and $b' = 7$, and then Corollary 4.3.4(i) implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 17$. For the remaining groups we use Corollary 4.3.4(ii) (with $b = \exp(G)$) and obtain lower bounds for $c_{\mathrm{eq}}(G)$ which are strictly larger than $\mathsf{D}^*(G)$. □

Remark 4.6. We briefly discuss the exceptional groups listed in Theorem 4.5. For all of them, apart from C_3^4, C_2^4 and $C_2 \oplus C_4$, we will see that the monotone catenary degree is strictly greater than the catenary degree. Let G be a finite abelian group with $|G| \ge 3$. Recall that $\mathsf{c}(G) \le \mathsf{D}(G)$, and that equality holds if and only if G is cyclic or an elementary 2-group [15, Theorem 6.4.7].

1. Let $G = C_3^r$ with $r \ge 2$. Then $\mathsf{c}(G) \ge r+1$, equality holds for $r \in [2,3]$, and there is known no r with $\mathsf{c}(G) > r+1$ ([14, Corollary 5.1]). If $r = 2$, then Corollary 4.3.2 implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 5$, and hence $\mathsf{c}_{\mathrm{mon}}(G) > \mathsf{c}(G)$. If $r \in [3,5]$, then Corollary 4.3.4(ii) (with $b = 3$ and $a = r$) implies that $\mathsf{c}_{\mathrm{eq}}(G) \ge 3r-4$. Thus, in case $r = 3$, we have $\mathsf{c}_{\mathrm{mon}}(G) \ge 5 > 4 = \mathsf{c}(G)$. If $r = 5$, then $\mathsf{c}_{\mathrm{mon}}(G) \ge 11 = \mathsf{D}^*(G) = \mathsf{D}(G) > \mathsf{c}(G)$.
2. If $G = C_2 \oplus C_6$, then Lemma 4.1 implies that $\mathsf{c}_{\mathrm{mon}}(G) \ge 7 = \mathsf{D}^*(G) = \mathsf{D}(G) > \mathsf{c}(G)$.
3. If $G = C_2^3$, then, in the next section, we will show that $\mathsf{c}(G) = \mathsf{c}_{\mathrm{mon}}(G) = \mathsf{D}(G)$. We have $\mathsf{c}(C_2^4) = \mathsf{D}(G) = 5$ and $\mathsf{c}(C_2 \oplus C_4) = \mathsf{D}(G) - 1 = 4$ ([15, Corollary 6.4.8]), and for these two groups the question, whether $\mathsf{c}(G) = \mathsf{c}_{\mathrm{mon}}(G)$ does hold or not, remains open.

5. Class Groups G with $\mathsf{D}(G) \le 4$

In this section we determine the monotone catenary degree of Krull monoids with class group G satisfying $\mathsf{D}(G) \le 4$. This completes the proof of Theorem 1.1.

Theorem 5.1. *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor.*

1. *If $\mathsf{D}(G) = 1$, then $|G| = 1$ and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = 0$.*
2. *If $\mathsf{D}(G) \in [2,4]$, then $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = \mathsf{D}(G)$.*

Proof. If $\mathsf{D}(G) = 1$, then $|G| = 1$, H is factorial and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = 0$. Similarly, $\mathsf{D}(G) = 2$ implies that $|G| = 2$, hence H is half-factorial and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}(H) = 2$ (for both arguments see [15, Corollary 3.4.12]). □

Suppose that $\mathsf{D}(G) \in [3,4]$. Then H is not half-factorial, and G is isomorphic to one of the following groups: C_3, C_2^2, C_4, C_2^3. Since $\mathsf{c}(H) = \mathsf{D}(G)$ by [15, Theorem 6.4.7] and $\mathsf{c}_{\mathrm{mon}}(H) = \mathsf{c}_{\mathrm{mon}}(G)$ by Lemma 2.1, it remains to prove that $\mathsf{c}_{\mathrm{mon}}(G) \le \mathsf{D}(G)$. Now we consider the four groups. As before we set $G^\bullet = G \setminus \{0\}$ and show that $\mathsf{c}_{\mathrm{mon}}(G^\bullet) \le \mathsf{D}(G)$. For all four groups we have $\Delta(G) = [1, \mathsf{D}(G)-2]$ by [15, Theorem 6.7.1]. For $d \in \Delta(G)$ we set $A_d = \{x \in \mathsf{Z}(G) \mid |x| - d \in \mathsf{L}(\pi(x))\}$, and we will use Lemma 3.1 without further mention.

1. Let $G = C_3$. We set $G = \{0, g, -g\}$, and list the elements of $\mathcal{A}(G^\bullet)$:
$$U = g^3, \quad -U = (-g)^3, \quad V = (-g)g.$$
Since $\mathrm{Min}(A_1) = \{V^3\}$, it follows that $\mathsf{c}_{\mathrm{adj}}(G) \le 3$. Since $\mathcal{A}(\sim_{\mathcal{B}(G^\bullet),\mathrm{eq}}) = \{(U,U), (-U,-U), (V,V)\}$, it follows that $\mathsf{c}_{\mathrm{eq}}(G) = 0$. Thus $\mathsf{c}_{\mathrm{adj}}(G) = \mathsf{c}_{\mathrm{mon}}(G) = 3$.

2. Let $G = C_2 \oplus C_2$. We set $G = \{0, e_1, e_2, e_1 + e_2\}$ and list the elements of $\mathcal{A}(G^\bullet)$:
$$U_1 = e_1^2, \quad U_2 = e_2^2, \quad U_3 = (e_1 + e_2)^2, \quad V = e_1 e_2 (e_1 + e_2).$$
Since $\mathrm{Min}(A_1) = \{U_1 U_2 U_3\}$, it follows that $\mathsf{c}_{\mathrm{adj}}(G) \le 3$. Since $\mathcal{A}(\sim_{\mathcal{B}(G^\bullet),\mathrm{eq}}) = \{(U_1, U_1), (U_2, U_2), (U_3, U_3), (V, V)\}$, it follows that $\mathsf{c}_{\mathrm{eq}}(G) = 0$. Thus $\mathsf{c}_{\mathrm{adj}}(G) = \mathsf{c}_{\mathrm{mon}}(G) = 3$.
3. Let $G = C_4$. We set $G = \{0, g, 2g, -g\}$ and list the elements of $\mathcal{A}(G^\bullet)$:
$$\begin{aligned} &U_1 = g^4, -U_1 = (-g)^4, U_2 = (2g)^2, U_3 = g(-g), U_4 = g^2(2g), \\ &\qquad -U_4 = (-g)^2(2g). \end{aligned}$$
We have $U_1(-U_1) = U_3^4, U_4(-U_4) = U_2 U_3^2, U_1(-U_4) = U_3^2 U_4$ and $(-U_1)U_4 = U_3^2(-U_4)$ which shows that $\mathrm{Min}(A_2) = \{U_3^4\}$ and $\mathrm{Min}(A_1) = \{U_2 U_3^2, U_3^2 U_4, U_3^2(-U_4)\}$. Thus we obtain that $\mathsf{c}_{\mathrm{adj}}(G) \le 4$. Since
$$\begin{aligned} &\mathcal{A}(\sim_{\mathcal{B}(G^\bullet),\mathrm{eq}}) \setminus \{(V, V) \mid V \in \mathcal{A}(G^\bullet)\} \\ &\quad = \{(U_1 U_2, U_4^2), ((-U_1)U_2, (-U_4)^2), (U_1(-U_4)^2, (-U_1)U_4^2)\}, \end{aligned}$$
it follows that $\mathsf{c}_{\mathrm{eq}}(G) \le 3$. Thus $\mathsf{c}_{\mathrm{adj}}(G) = \mathsf{c}_{\mathrm{mon}}(G) = 4$.
4. Let $G = C_2^3$. We choose a basis (e_1, e_2, e_3) of G and list the elements of $\mathcal{A}(G^\bullet)$:
$$\begin{aligned} &e_1^2, e_2^2, e_3^2, (e_1 + e_2)^2, (e_1 + e_3)^2, (e_2 + e_3)^2, (e_1 + e_2 + e_3)^2; \\ &e_1 e_2 (e_1 + e_2), e_1 e_3 (e_1 + e_3), e_2 e_3 (e_2 + e_3), (e_1 + e_2)(e_1 + e_3)(e_2 + e_3), \\ &e_1 (e_2 + e_3)(e_1 + e_2 + e_3), e_2 (e_1 + e_3)(e_1 + e_2 + e_3), \\ &\quad e_3 (e_1 + e_2)(e_1 + e_2 + e_3); \\ &e_1 (e_1 + e_2)(e_1 + e_3)(e_1 + e_2 + e_3), e_2 (e_1 + e_2)(e_2 + e_3)(e_1 + e_2 + e_3), \\ &\quad e_3 (e_1 + e_3)(e_2 + e_3)(e_1 + e_2 + e_3), \\ &e_1 e_2 e_3 (e_1 + e_2 + e_3), e_1 e_2 (e_1 + e_3)(e_2 + e_3), e_1 e_3 (e_1 + e_2)(e_2 + e_3), \\ &\quad e_2 e_3 (e_1 + e_2)(e_1 + e_3). \end{aligned}$$

Some simple facts

F1. Let $U_1, U_2 \in \mathcal{A}(G^\bullet)$ be distinct with $|U_1| = |U_2| = 3$. Then
$$|\gcd(U_1, U_2)| = 1 \quad \text{and} \quad U_1 U_2 = g^2 V,$$
where $g \in G$ and $V \in \mathcal{A}(G^\bullet)$ with $|V| = 4$.

F2. Let $U_1, U_2, U_3 \in \mathcal{A}(G^\bullet)$ be pairwise distinct with $|U_1| = |U_2| = |U_3| = 4$. Then $|\gcd(U_1, U_2)| = 2$, and if $\mathrm{supp}(U_1 U_2 U_3) = G^\bullet$, then $|\gcd(U_1, U_2, U_3)| = 1$.

F3. Let $U_1, U_2 \in \mathcal{A}(G^\bullet)$ be distinct with $|U_1| = |U_2| = 4$ and $g \in G^\bullet$ with $g \nmid U_1 U_2$. We determine the structure of $U_1 U_2 g^2$, and then discuss its factorizations.

We set $\mathrm{supp}(U_1)\cap\mathrm{supp}(U_2) = \{g_1, g_2\}$ and $U_1 = g_1g_2g_3(g_1+g_2+g_3)$. Then (g_1, g_2, g_3) is a basis of $G, \mathrm{supp}(U_1U_2g) = G^\bullet$, and $g \in \{g_1+g_2, g_1+g_3, g_2+g_3\}$. If $g = g_1+g_2$, then $\mathrm{supp}(U_1U_2g) = G^\bullet$ implies that $U_2 = g_1g_2(g_1+g_3)(g_2+g_3)$. The cases that $g = g_1+g_3$ and $g = g_2+g_3$ are similar.

Now we set $U_3 = (g_1+g_2)^2$ and list some factorizations of $U_1U_2U_3$:

$$\begin{aligned}
z_1 &= (g_1+g_2)^2(g_1g_2g_3(g_1+g_2+g_3))(g_1g_2(g_1+g_3)(g_2+g_3)),\\
z_2 &= (g_1g_2(g_1+g_2))^2(g_3(g_1+g_3)(g_2+g_3)(g_1+g_2+g_3)),\\
z_3 &= (g_1g_2(g_1+g_2))(g_2g_3(g_2+g_3))(g_1(g_1+g_2)(g_1+g_3)(g_1+g_2+g_3)),\\
z_4 &= (g_1g_2(g_1+g_2))(g_1(g_2+g_3)(g_1+g_2+g_3))(g_2g_3(g_1+g_3)(g_1+g_2)),\\
z_5 &= (g_1g_2(g_1+g_2))(g_2(g_1+g_3)(g_1+g_2+g_3))(g_1g_3(g_1+g_2)(g_2+g_3)),\\
z_6 &= (g_1g_2(g_1+g_2))(g_3(g_1+g_2)(g_1+g_2+g_3))(g_1g_2(g_1+g_3)(g_2+g_3)) \quad \text{and}\\
z_7 &= (g_1g_2(g_1+g_2))(g_1g_3(g_1+g_3))(g_2(g_1+g_2)(g_2+g_3)(g_1+g_2+g_3)).
\end{aligned}$$

Obviously, we have $|z_1| = \cdots = |z_7| = 3, \mathsf{d}(z_i, z_{i+1}) = 3$ for all $i \in [1,6]$, and for all $W \in \mathcal{A}(G^\bullet)$ with $|W| = 4$ there is an $i \in [1,7]$ with $W \mid z_i$.

F4. Let $U, V \in \mathcal{A}(G^\bullet)$ with $|U| = 4, |V| = 3$ and $\gcd(U,V) = 1$ (equivalently, $\mathrm{supp}(UV) = G^\bullet$). Then

$$\mathsf{Z}(UV) = \{W(W^{-1}UV) \mid W \in \mathcal{A}(G^\bullet) \text{ with } |W| = 4\}.$$

Note that $W^{-1}UV \in \mathcal{A}(G^\bullet), |W^{-1}UV| = 3$ and $W^{-1}UV$ runs through all $T \in \mathcal{A}(G^\bullet)$ with $|T| = 3$.

F5. Let $U_1, U_2, U_3 \in \mathcal{A}(G^\bullet)$ be pairwise distinct with $|U_1| = |U_2| = |U_3| = 4$ and $\mathrm{supp}(U_1U_2U_3) \neq G^\bullet$. Then $\gcd(U_1, U_2, U_3) = 1$, and for any $V \in \mathcal{A}(G^\bullet)$ with $|V| = 3$ and $\mathrm{supp}(U_1U_2U_3V) = G^\bullet$, there exists some $i \in [1,3]$ such that $\gcd(U_i, V) = 1$.

F6. We set $A = \prod_{g\in G^\bullet} g^2$ and list some factorizations of length 5:

$$\begin{aligned}
z_1 &= (e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3))^2e_2^2e_3^2(e_2+e_3)^2,\\
z_2 &= (e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3))^2e_1^2e_2^2(e_1+e_2)^2,\\
z_3 &= (e_2(e_1+e_2)(e_2+e_3)(e_1+e_2+e_3))^2e_1^2e_3^2(e_1+e_3)^2,\\
z_4 &= (e_1e_2(e_1+e_3)(e_2+e_3))^2e_3^2(e_1+e_2)^2(e_1+e_2+e_3)^2,\\
z_5 &= (e_1e_3(e_1+e_2)(e_2+e_3))^2e_2^2(e_1+e_3)^2(e_1+e_2+e_3)^2,\\
z_6 &= (e_2e_3(e_1+e_2)(e_1+e_3))^2e_1^2(e_2+e_3)^2(e_1+e_2+e_3)^2 \quad \text{and}\\
z_7 &= (e_1e_2e_3(e_1+e_2+e_3))^2(e_1+e_2)^2(e_1+e_3)^2(e_2+e_3)^2.
\end{aligned}$$

Obviously, we have $|z_1| = \cdots = |z_7| = 5, \mathsf{d}(z_i, z_{i+1}) = 4$ for all $i \in [1,6]$, and

$$\begin{aligned}\mathsf{Z}_5(A) \supset \{U^2g_1^2g_2^2g_3^2 \mid U \in \mathcal{A}(G^\bullet) \text{ with } |U| = 4 \text{ and } G^\bullet\setminus\mathrm{supp}(U)\\ = \{g_1, g_2, g_3\}\}.\end{aligned}$$

F7. If $A = (e_1e_2e_3(e_1+e_2+e_3))^2(e_2e_3(e_2+e_3))^2(e_1+e_2)^2(e_1+e_3)^2$, then

$$\mathsf{Z}_6(A) \supset \{e_2^2e_3^2U^2V^2 \mid U,V \in \mathcal{A}(G^\bullet) \text{ with } |U|=4,\ |V|=3, \text{ and } \gcd(U,V)=1\}.$$

F8. If $A = \prod_{g \in G^\bullet} g^2$, then

$$\mathsf{Z}_6(A) = \{U^2g_1^2g_2^2g_3^2g_4^2 \mid U \in \mathcal{A}(G^\bullet) \text{ with } |U|=3 \text{ and } G^\bullet \setminus \operatorname{supp}(U) = \{g_1,g_2,g_3,g_4\}\}.$$

The adjacent catenary degree $\mathsf{c}_{\mathrm{adj}}(G)$

Let $A \in \mathcal{B}(G^\bullet)$. We show that $\mathsf{c}_{\mathrm{adj}}(A) \le \max\{4, \mathsf{c}_{\mathrm{eq}}(G)\}$. To do so we choose a factorization $\xi \in \mathsf{Z}(A)$ with $|\xi| < \max \mathsf{L}(A)$ and find factorizations $\xi', \xi'' \in \mathsf{Z}(A)$ such that $\mathsf{d}(\xi', \xi'') \le 4$, $|\xi''| = |\xi|$, and

$$|\xi'| = \begin{cases} |\xi|+1 & \text{if } |\xi|+1 \in \mathsf{L}(A), \\ |\xi|+2 & \text{otherwise.} \end{cases}$$

Suppose that $\xi = U_1 \cdot \ldots \cdot U_l V_1 \cdot \ldots \cdot V_m W_1 \cdot \ldots \cdot W_n$ where all U_i, V_j, W_k are atoms, $|U_1| = \cdots = |U_l| = 4, |V_1| = \cdots = |V_m| = 3$ and $|W_1| = \cdots = |W_n| = 2$.

Suppose that $m \ge 2$. If two of the V_j's are equal, say $V_1 = V_2 = g_1g_2g_3$, then we set $V_i' = g_i^2 \in \mathcal{A}(G^\bullet)$ for $i \in [1,3], \xi' = (V_1V_2)^{-1}V_1'V_2'V_3'\xi$, and $\xi'' = \xi$. If all V_j's are pairwise distinct, then **F1** gives us a factorization ξ'' with $|\xi''| = |\xi|$ where the number of atoms having length 3 is smaller. Thus we may suppose that $m \le 1$. Note that $|\xi| < \max \mathsf{L}(A)$ implies that $l + m \ge 2$.

Suppose that there exist two distinct U_i's, say (by **F2**) $U_1 = e_1e_2e_3(e_1 + e_2 + e_3)$ and $U_2 = e_1e_2(e_1+e_3)(e_2+e_3)$. Then we set $\xi'' = \xi$ and find a $\xi' = (U_1U_2)^{-1}e_1^2e_2^2(e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3))\xi$ as required. From now on we suppose that $U_1 = \cdots = U_l$. If $m = 0$, then $l \ge 2$, there is no factorization of length $|\xi|+1$, and since $4 \in \mathsf{L}(U_1^2)$, it is clear how to define ξ' and ξ''.

Suppose that $m = 1$. After a renumbering we have $\xi = U^lVW_1 \cdot \ldots \cdot W_n$ with $U, V \in \mathcal{A}(G^\bullet), |U| = 4$ and $|V| = 3$. Suppose that $|\gcd(U,V)| \ge 1$, say $U = e_1e_2e_3(e_1+e_2+e_3)$ and $V = e_1gh$ with $g, h \in G$. Then it follows that $|\gcd(U,V)| = 2$, say $V = e_1e_2(e_1+e_2)$, and it is clear how to define ξ' and ξ''. From now on we suppose that $\gcd(U,V) = 1$. Then $\operatorname{supp}(UV) = G^\bullet$, say $U = e_1e_2e_3(e_1+e_2+e_3)$ and $V = (e_1+e_2)(e_1+e_3)(e_2+e_3)$. If $l = 1$, then it follows that $|\xi| = \max \mathsf{L}(A)$, a contradiction. Thus we have $l \ge 2$. Since

$$U^2V = (e_1+e_2+e_3)^2((e_1+e_2)e_1e_2)((e_1+e_3)e_1e_3)((e_2+e_3)e_2e_3)$$

and hence $4 \in \mathsf{L}(U^2V)$, it is clear how to define ξ' and ξ''.

The equal catenary degree $\mathsf{c}_{\mathrm{eq}}(G)$

Let $(\xi, \xi') \in \mathcal{A}(\sim_{\mathcal{B}(G^\bullet),\mathrm{eq}})$. By Lemma 3.1 we have to show that there exists a monotone 4-chain of factorizations between ξ and ξ'. We proceed by induction on $|\xi|$. If $|\xi| = 1$, then $\xi = \xi' = U \in \mathcal{A}(G^\bullet)$, and the assertion is clear. Suppose that $|\xi| > 1$, and that the assertion holds for all $(\zeta, \zeta') \in \mathcal{A}(\sim_{\mathcal{B}(G^\bullet),\mathrm{eq}})$ with $|\zeta| < |\xi|$.

We need some additional notation. First, we set $\pi(\xi) = A$. If $\langle \mathrm{supp}(A) \rangle$ is a proper subgroup of G, then $\mathsf{c}_{\mathrm{eq}}(A) \le \mathsf{c}_{\mathrm{eq}}(C_2^2) \le 3$. Thus we may suppose that $\langle \mathrm{supp}(A) \rangle = G$. Furthermore, for $\nu \in [1, 4]$ and a factorization $z \in \mathsf{Z}(G)$, we denote by $\omega_\nu(z)$ the number of distinct atoms $U \in \mathcal{A}(G)$ with $|U| = \nu$ and $U \mid z$. If ξ is divisible only by atoms of length 2, then $|\xi'| = |\xi| = |A|/2$, ξ' is divisible only by atoms of lengths 2, and hence $\xi = \xi'$, a contradiction. Thus both, ξ and ξ', are divisible by at least one atom of length 3 or 4. By **F1**, we may suppose that neither ξ nor ξ' is divisible by two distinct atoms of length 3. So, in more technical terms, we have $\omega_3(\xi) + \omega_4(\xi) \ge 1, \omega_3(\xi') + \omega_4(\xi') \ge 1, \omega_3(\xi) \le 1, \omega_3(\xi') \le 1$. By symmetry, we may suppose that $\omega_4(\xi) \ge \omega_4(\xi')$.

We start with four assertions which handle some special cases.

A0. Suppose that there exists a monotone 4-chain of factorization $y = y_0, \ldots, y_k = \xi'$ of A such that $|\gcd(y, \xi)| > 1$. Then there exists a monotone 4-chain of factorizations between ξ and ξ'.

A1. Suppose that $\xi = U^a x_1, \xi' = V^b y_1$, where $U, V \in \mathcal{A}(G^\bullet)$ with $|U| = 4, |V| = 3, a, b \in \mathbb{N}$, and x_1, y_1 are products of length 2 atoms. Then there exists a monotone 4-chain of factorizations between ξ and ξ'.

A2. Suppose that $\omega_4(\xi') \le 1$, and $\mathrm{supp}(A) = G^\bullet$. Then there exists a monotone 4-chain of factorizations between ξ and ξ'.

A3. Suppose that $\omega_4(\xi') = 2$ and $\mathrm{supp}(A) = G^\bullet$. Then there exists a monotone 4-chain of factorizations between ξ and ξ'.

Proof of **A0**. Let $W \in \mathcal{A}(G^\bullet)$ be such that $W \mid \gcd(y, \xi)$. By induction hypothesis there exists a monotone 4-chain of factorizations between $W^{-1}y$ and $W^{-1}\xi$ and hence a monotone 4-chain between y and ξ. Combining this chain with the monotone 4-chain between between y and ξ' we obtain a monotone 4-chain between ξ and ξ'. $\square$

Proof of **A1**. After a base change if necessary, we may assume that $U = e_1 e_2 e_3 (e_1 + e_2 + e_3)$. We distinguish two cases.

First suppose that $\gcd(U, V) = 1$. Then we obtain $V = (e_1 + e_2)(e_1 + e_3)(e_2 + e_3)$. The minimality of (ξ, ξ') implies that

$$\xi = U^a (e_1 + e_2)^b (e_1 + e_3)^b (e_2 + e_3)^b \quad \text{and} \quad \xi' = V^b e_1^a e_2^a e_3^a (e_1 + e_2 + e_3)^a$$

where a, b are even with $a + 3b/2 = b + 2a$. This shows that $a = 2$ and $b = 4$. Setting

$$z_1 = e_1^2 e_2^2 (e_1+e_2)^4 (e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3))^2 (e_1+e_3)^2 (e_2+e_3)^2 \quad \text{and}$$
$$z_2 = e_1^2 e_2^2 (e_3(e_1+e_2)(e_1+e_2+e_3))^2 ((e_1+e_2)(e_1+e_3)(e_2+e_3))^2 (e_1+e_3)^2 (e_2+e_3)^2$$

we infer that $\xi = z_0, z_1, z_2, z_3 = \xi'$ are factorizations of A with $|z_1| = |z_2| = |\xi| = 8$ and $\mathsf{d}(z_i, z_{i+1}) = 4$ for all $i \in [0, 2]$.

Now suppose that $\gcd(U, V) \ne 1$. Then $|\gcd(U, V)| = 2$, and we may assume that $V = e_1 e_2 (e_1 + e_2)$. Again by the minimality of (ξ, ξ') it follows that

$$\xi = e_1^2 e_2^2 (e_1+e_2)^4 (e_1e_2e_3(e_1+e_2+e_3))^2 \quad \text{and}$$
$$\xi' = (e_1e_2(e_1+e_2))^4 e_3^2 (e_1+e_2+e_3)^2.$$

If

$$z_1 = e_1^2 e_2^2 (e_1e_2(e_1+e_2))^2 (e_3(e_1+e_2)(e_1+e_2+e_3))^2,$$

then $z_0 = \xi, z_1, z_2 = \xi'$ are factorizations of A with $|z_1| = |\xi| = 6$ and $\mathsf{d}(z_i, z_{i+1}) = 4$ for all $i \in [0,1]$. $\square$

Proof of **A2**. We distinguish three cases.

First suppose that $\omega_3(\xi') = 0$. Then ξ' is divisible by an atom V of length 4, say (after a base change if necessary) $V = e_1e_2e_3(e_1+e_2+e_3)$, and $\operatorname{supp}(A) = G^\bullet$ implies that

$$\xi' = (e_1e_2e_3(e_1+e_2+e_3))^a (e_1+e_2)^2 (e_1+e_3)^2 (e_2+e_3)^2 x,$$

where $a \in \mathbb{N}$ and x is a product of length 2 atoms. If $a \ge 2$, then $(e_1e_2e_3(e_1+e_2+e_3))^2(e_1+e_2)^2(e_1+e_3)^2(e_2+e_3)^2 \mid \xi'$, and since $\omega_4(\xi) \ge \omega_4(\xi') = 1$, the assertion follows from **F6** and **A0**.

If $a = 1$, then

$$\begin{aligned} z_1 &= Vx_1 \\ z_2 &= (e_1e_2(e_1+e_2))(e_3(e_1+e_2)(e_1+e_2+e_3))x_2 \\ z_3 &= (e_1e_3(e_1+e_3))(e_2(e_1+e_3)(e_1+e_2+e_3))x_3 \quad \text{and} \\ z_4 &= (e_2e_3(e_2+e_3))(e_1(e_2+e_3)(e_1+e_2+e_3))x_4, \end{aligned}$$

where x_1, x_2, x_3, x_4 are products of length 2 atoms, are all factorizations of length $|\xi'|$. Now the assertion follows from the fact that $\mathsf{d}(z_i, z_j) = 3$ for all distinct $i, j \in [1,4]$.

Second, suppose that $\omega_4(\xi') = 0$. Then we may assume that $\xi' = V^b y$, where $V = e_1e_2(e_1+e_2), b \in \mathbb{N}$, and y is a product of length 2 atoms, and $\operatorname{supp}(A) = G^\bullet$ implies that

$$\xi' = (e_1e_2(e_1+e_2))^b e_3^2 (e_1+e_2+e_3)^2 (e_1+e_3)^2 (e_2+e_3)^2 x,$$

where $b \in \mathbb{N}$, and x is a product of length 2 atoms. If $b = 1$, then A has only one factorization z such that $|z| = |\xi'|$ and this is ξ'. If $b = 2$, by **F8**, the only factorizations of $\pi(\xi')$ are of the form $V_1^2 h$, where $V_1 \in \mathcal{A}(G^\bullet)$ with $|V_1| = 3$ and h is a product of length 2 atoms. If $b = 3$, then the only factorizations of A are of the form $VV_1^2 h$, where $V_1 \in \mathcal{A}(G^\bullet)$ with $|V_1| = 3$ and h is a product of length 2 atoms. Hence we are done for $b \le 3$.

For $b \ge 4$, **F8** shows that the factorizations of the element $(e_1e_2(e_1+e_2))^2 e_3^2 (e_1+e_2+e_3)^2 (e_1+e_3)^2 (e_3+e_2)^2 \in \mathcal{B}(G)$ have the form

$$U^2 g_1^2 g_2^2 g_3^2 g_4^2, \quad \text{where } U \in \mathcal{A}(G^\bullet), g_1, \ldots, g_4 \in G^\bullet \text{ such that } |U| = 3, \text{ and } \operatorname{supp}(Ug_1g_2g_3g_4) = G^\bullet.$$

For every such $U \in \mathcal{A}(G^\bullet)$ with $|U| = 3$, the factorizations of $(e_1e_2(e_1 + e_2))^2U^2 \in \mathcal{B}(G^\bullet)$ have the form

$U_1^2 g^4$ where $U_1 \in \mathcal{A}(G^\bullet)$ with $|U_1| = 4$ and $U_1 \neq e_3(e_3+e_1)(e_2+e_3)(e_1+e_2+e_3)$.

If ξ is divisible by an atom of length 3, then there is a factorization z of A such that $\mathsf{d}(z, \xi') \le 4$ and $\gcd(\xi, z)$ is divisible by an atom W of length 3. Therefore the assertion follows from **A0**.

If ξ is divisible by an atom W of length 4 with $W \neq e_3(e_1 + e_3)(e_2 + e_3)(e_1 + e_2 + e_3)$, then, by the above argument, there is a monotone 4-chain of factorizations between z and ξ' such that $\gcd(\xi, z)$ is divisible by an atom W of length 4. Therefore the assertion follows from **A0**.

If $\xi = (e_3(e_1 + e_3)(e_2 + e_3)(e_1 + e_2 + e_3))^a x$, where $a \in \mathbb{N}$ and x is a product of length 2 atoms, then the assertion follows from **A1**.

Finally, suppose that there are $U, V \in \mathcal{A}(G^\bullet)$ with $|U| = 4, |V| = 3$ and $UV \mid \xi'$. If $\gcd(U, V) = 1$, then **F4** and **A0** imply the assertion. Otherwise, we may assume that $U = e_1e_2e_3(e_1 + e_2 + e_3)$ and $V = e_1e_2(e_1 + e_2)$. Then

$$\xi' = U^a V^b (e_1 + e_3)^2 (e_2 + e_3)^2,$$

where $a, b \in \mathbb{N}$, and x is a product of length 2 atoms.

If $a \ge 2$ and $b \ge 2$, then $U^2V^2(e_1 + e_3)^2(e_2 + e_3)^2 \mid \xi'$, and **F7** and **A0** imply the assertion.

If $b = 1$ and $a > 1$, then

$$U^2V(e_1 + e_3)^2(e_2 + e_3)^2 \mid \xi'.$$

Since $\mathsf{Z}(U^2(e_1+e_3)^2(e_2+e_3)^2) \supset \{w = U^2(e_1+e_3)^2(e_2+e_3)^2, w' = e_1^2e_2^2(e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3))^2\}$, and $\operatorname{supp}(V(e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3)) = G^\bullet$, the assertion follows from **F4** and **A0**.

If $a = 1$ and $b > 1$, then

$$UV^2(e_1 + e_3)^2(e_2 + e_3)^2 \mid \xi'.$$

Since $\mathsf{Z}(V^2(e_1+e_3)^2(e_2+e_3)^2) \supset \{w = V^2(e_1+e_3)^2(e_2+e_3)^2, w' = e_1^2e_2^2((e_1+e_2)(e_1 + e_3)(e_2 + e_3))^2\}$, and $\operatorname{supp}(U((e_1 + e_2)(e_1 + e_3)(e_2 + e_3)) = G^\bullet$, the assertion follows from **F4** and **A0**. □

Proof of **A3**. After a base change if necessary and by **F2**, we may assume that $V_1V_2 \mid \xi'$ where $V_1 = e_1e_2e_3(e_1 + e_2 + e_3)$ and $V_2 = e_1(e_1 + e_2)(e_1 + e_3)(e_1 + e_2 + e_3)$. We distinguish two cases.

First suppose that $\omega_3(\xi') = 0$. Then $\xi' = V_1^a V_2^b y$, where $a, b \in \mathbb{N}$, and y is a product of length 2 atoms. Since $\operatorname{supp}(A) = G^\bullet$, it follows $V_1V_2(e_3 + e_2)^2 \mid \xi'$. Then **F3** implies that, for any $W \in \mathcal{A}(G^\bullet)$ with $W \mid \xi$ and $|W| = 4$, there is a monotone 3-chain of factorizations $z_0, z_1, \ldots, z_k$ of the element $V_1V_2(e_2+e_3)^2 \in \mathcal{B}(G)$ such that $z_0 = V_1V_2(e_2+e_3)^2$ and $W \mid z_k$. Therefore the assertion follows by **A0**.

Now suppose that $\omega_3(\xi')=1$, say $V_3 \in \mathcal{A}(G^\bullet)$ with $|V_3|=3$ divides ξ'. By **F3** and **A0**, we may suppose that $(e_2+e_3)^2 \nmid \xi'$, and by **F4** and **A0**, we may suppose that $V_3 = e_1(e_2+e_3)(e_1+e_2+e_3)$. We set

$$\begin{aligned}
&U_1 = e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3), \quad U_2 = e_1e_2e_3(e_1+e_2+e_3),\\
&U_3 = e_1e_2(e_1+e_3)(e_2+e_3), \quad U_4 = e_1e_3(e_1+e_2)(e_2+e_3),\\
&U_5 = e_2e_3(e_1+e_2)(e_1+e_3),
\end{aligned}$$

and distinguish two cases.

First suppose that $\omega_3(\xi)=0$. Then we set

$$\xi = U_1^aU_2^bU_3^cU_4^dU_5^ex \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hy,$$

where $f,g,h \in \mathbb{N}, a,b,c,d,e \in \mathbb{N}_0$, and x,y are products of length 2 atoms. Since

$$\begin{aligned}
&a+b+c+d+e+\frac{|A|-4(a+b+c+d+e)}{2}\\
&\quad = |\xi| = |\xi'| = f+g+h+\frac{|A|-4(f+g)-3h}{2},
\end{aligned}$$

we obtain

$$a+b+c+d+e = f+g+\frac{h}{2}. \tag{5.1}$$

It follows that

$$a+b+c+d < f+g+h,$$

which implies that $e_1^2 \,|\, \xi$ and $(e_1+e_2+e_3)^2 \,|\, \xi$.

If $e_1^2|\xi$ and $a+b>0$, say $a>0$, then we observe that $\mathsf{Z}(e_1^2U_1) \supset \{w = e_1^2U_1, w' = V_3(e_1e_3(e_1+e_3))\}$, and hence the assertion follows by **A0**. If $(e_1+e_2+e_3)^2 \,|\, \xi$ and $c+d>0$, the argument is similar. Since $a+b+c+d>0$, we are done for this case.

Now we suppose that $\omega_3(\xi)=1$. Then we set

$$\xi = U_1^aU_2^bU_3^cU_4^dU_5^eU^ix \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hy,$$

where $f,g,h,i \in \mathbb{N}, a,b,c,d,e \in \mathbb{N}_0, U \in \mathcal{A}(G^\bullet)$ with $|U|=3$ and x,y are products of length 2 atoms.

If ($e_1^2 \,|\, \xi$ and $a+b>0$) or (($e_1+e_2+e_3)^2 \,|\, \xi$ and $c+d>0$), then we are done by the same argument as above. Since $a+b+c+d>0$, we are left with the cases that ($e_1^2 \,|\, \xi$, $a=b=0$ and $(e_1+e_2+e_3)^2 \nmid \xi$), and (($e_1+e_2+e_3)^2 \,|\, \xi, c=d=0$ and $e_1^2 \nmid \xi$). Since the arguments of these two cases are the same, we consider only the case that $e_1^2 \,|\, \xi, a=b=0$ and $(e_1+e_2+e_3)^2 \nmid \xi$. Thus we have

$$\xi = U_3^cU_4^dU_5^eU^ix \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hy,$$

hence

$$c+d+e+i+\frac{|A|-4(c+d+e)-3i}{2}=|\xi|=|\xi'|$$
$$=f+g+h+\frac{|A|-4(f+g)-3h}{2},$$

and thus

$$2(c+d+e)+i=2(f+g)+h.$$

Since $(e_1+e_2+e_3)^2 \nmid \xi$, we have $(e_1+e_2+e_3) \,|\, U$, $U \in \{e_2(e_1+e_3)(e_1+e_2+e_3), e_3(e_1+e_2)(e_1+e_2+e_3)\}$ and

$$i \geq f+g+h.$$

It follows that either $U=e_2(e_1+e_3)(e_1+e_2+e_3)$ and $(e_1+e_3)^2 \,|\, \xi'$, or that $U=e_3(e_1+e_2)(e_1+e_2+e_3)$ and $(e_1+e_2)^2 \,|\, \xi'$. Since $\mathsf{Z}((e_1+e_3)^2V_1) \supset \{w=(e_1+e_3)^2V_1, w'=U(e_1e_3(e_1+e_3))\}$ and $\mathsf{Z}((e_1+e_2)^2V_1) \supset \{w=(e_1+e_2)^2V_1, w'=U(e_1e_2(e_1+e_2))\}$, the assertion follows by **A0**. $\square$

Now we start the actual proof that there is a monotone 4-chain between ξ and ξ'. If $\omega_4(\xi) \geq 5$, then $\omega_4(\xi') \leq 2, \mathrm{supp}(A)=G^\bullet$, and hence **A2** and **A3** imply the assertion. Thus we assume that $\omega_4(\xi) \in [0,4]$, and we distinguish four cases. Whenever necessary we use a suitable base change without further mention.

Case 1: $\omega_4(\xi) \leq 1$.

We divide this proof into six subcases.

Case 1.1: Suppose that $\xi=U^ax_1, \xi'=V^by_1$, where $U,V \in \mathcal{A}(G^\bullet)$ with $|U|=|V|=3, a,b \in \mathbb{N}$, and x_1,y_1 are products of length 2 atoms.

We may assume that $U=e_1e_2(e_1+e_2)$ and $V=e_1e_3(e_1+e_3)$. Then the minimality of (ξ,ξ') implies that

$$\xi=(e_1e_2(e_1+e_2))^2e_3^2(e_1+e_3)^2 \quad \text{and} \quad \xi'=(e_1e_3(e_1+e_3))e_2^2(e_1+e_2)^2$$

and hence $\mathsf{d}(\xi,\xi')=4$.

Case 1.2: Suppose that $\xi=U^ax_1, \xi'=V^by_1$, where $U,V \in \mathcal{A}(G^\bullet)$ with $|U|=4, |V|=3, a,b \in \mathbb{N}$, and x_1,y_1 are products of length 2 atoms.

This case is settled by **A1**.

Case 1.3: Suppose that $\xi=U_1^aU_2^bx_1, \xi'=V^cy_1$, where $U_1,U_2,V \in \mathcal{A}(G^\bullet)$ with $|U_1|=4, |U_2|=|V|=3, a,b,c \in \mathbb{N}$, and x_1,y_1 are products of length 2 atoms.

If $\gcd(U_1,U_2)=1$, then **F4** shows that there is a $\xi'' \in \mathsf{Z}(A)$ with $\mathsf{d}(\xi,\xi'')=3$ and $V \,|\, \xi''$. Thus the assertion follows by **A0**. So we suppose that $\gcd(U_1,U_2) \neq 1$, say $U_1=e_1e_2e_3(e_1+e_2+e_3)$ and $U_2=e_1e_2(e_1+e_2)$. If $\gcd(U_1,V)=1$, then $V=(e_1+e_2)(e_3+e_2)(e_1+e_3)$, $\mathrm{supp}(A)=G^\bullet$, and thus the assertion follows from **A2**.

Now suppose that $|\gcd(U_1,V)|=2$. Then we may assume that $U_2=e_1e_2(e_1+e_2), V=e_1e_3(e_1+e_3), \xi=(e_1e_2e_3(e_1+e_2+e_3))^{2a}(e_1e_2(e_1+e_2))^{2b}(e_1+e_3)^{2c}e_3^{2c-2a}e_1^{2c-2a-2b}$, and

$$\xi' = e_2^{2a+2b}(e_1+e_2+e_3)^{2a}(e_1+e_2)^{2b}(e_1e_3(e_1+e_3))^{2c}.$$

Since $|\xi| = |\xi'|$, we get $2a+2b+c+(c-a)+(c-a-b) = (a+b)+a+b+2c$, and hence $c = 2a+b$. By the minimality of (ξ, ξ'), it follows that $a = 1, b = 1$, and $c = 3$, whence

$$\begin{aligned} \xi &= (e_1e_2e_3(e_1+e_2+e_3))^2(e_1e_2(e_1+e_2))^2(e_1+e_3)^6e_1^2e_3^4 \quad \text{and} \\ \xi' &= e_2^4(e_1+e_2+e_3)^2(e_1+e_2)^2(e_1e_3(e_1+e_3))^6. \end{aligned}$$

Now we define

$$\begin{aligned} z_1 &= (e_1e_3(e_1+e_3))^2(e_2(e_1+e_3)(e_1+e_2+e_3))^2(e_1e_2(e_1+e_2))^2(e_1+e_3)^2e_1^2e_3^4, \\ z_2 &= (e_1e_3(e_1+e_3))^2(e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3))^2(e_1+e_3)^2e_1^2e_2^4e_3^4, \\ z_3 &= (e_1e_3(e_1+e_3))^4((e_1+e_2)e_3(e_1+e_2+e_3))^2(e_1+e_3)^2e_1^2e_2^4, \end{aligned}$$

and obviously the chain $\xi = z_0, z_1, z_2, z_3, z_4 = \xi'$ satisfies the requirement.

Case 1.4: Suppose that $\xi = U^a x_1, \xi' = V^b y_1$, where $U, V \in \mathcal{A}(G^\bullet)$ with $|U| = |V| = 4, a, b \in \mathbb{N}$, and x_1, y_1 are products of length 2 atoms.

Arguing as in CASE 1.3, it remains to consider the case $U = e_1e_2e_3(e_1 + e_2 + e_3), V = e_1e_2(e_1+e_3)(e_2+e_3)$ and, by a similar computation as above, we infer that

$$\begin{aligned} \xi &= (e_1e_2e_3(e_1+e_2+e_3))^2(e_3+e_2)^2(e_1+e_3)^2 \quad \text{and} \\ \xi' &= e_3^2(e_1+e_2+e_3)^2(e_1e_2(e_1+e_3)(e_2+e_3))^2. \end{aligned}$$

This shows that $\mathsf{d}(\xi, \xi') = 4$.

Case 1.5: Suppose that $\xi = U^a U_1^b x_1, \xi' = V^c y_1$, where $U, U_1, V \in \mathcal{A}(G^\bullet)$ with $|U| = |V| = 4, |U_1| = 3, a, b, c \in \mathbb{N}$, and x_1, y_1 are products of length 2 atoms.

By **F2**, we may suppose that $U = e_1e_2e_3(e_1+e_2+e_3)$ and $V = e_1(e_2 + e_1)(e_1+e_3)(e_1+e_2+e_3)$. Using **F4** and arguing as in CASE 1.3, we may suppose that $\gcd(U, U_1) \neq 1$, and by **A2** we may suppose that $\gcd(V, U_1) \neq 1$. Thus it follows that $U_1 \in \{e_1e_2(e_1+e_2), e_1e_3(e_1+e_3)\}$, say $U_1 = e_1e_2(e_1+e_2)$. By computing the minimal relations as above, we obtain that $e_2^4 \,|\, \xi'$. Since the element $Ve_2^2 \in \mathcal{B}(G)$ has two factorizations $z = Ve_2^2$ and $z' = (e_1e_2(e_1 + e_2))(e_2(e_1+e_3)(e_1+e_2+e_3)) = U_1(e_2(e_1+e_3)(e_1+e_2+e_3))$, the assertion follows from **A0**.

Case 1.6: Suppose that $\xi = U^a U_1^b x_1, \xi' = V^c V_1^d y_1$, where $U, U_1, V, V_1 \in \mathcal{A}(G^\bullet)$ with $|U| = |V| = 4, |U_1| = |V_1| = 3, a, b, c, d \in \mathbb{N}$, and x_1, y_1 are products of length 2 atoms.

We may suppose that $U = e_1e_2e_3(e_1+e_2+e_3)$ and $V = e_1(e_1+e_2)(e_1 + e_3)(e_1+e_2+e_3)$. Using **F4** and **A2**, and arguing as above, we may suppose that $\gcd(U, U_1) \neq 1, \gcd(V, U_1) \neq 1, \gcd(U, V_1) \neq 1$ and $\gcd(V, V_1) \neq 1$. Thus it follows that $U_1, V_1 \in \{e_1e_2(e_1+e_2), e_1e_3(e_1+e_3)\}$, say $U_1 = e_1e_2(e_1+e_2)$. Then $V_1 = e_1e_3(e_1+e_3)$. By computing the minimal relations as above, we obtain that $e_2^4 \,|\, \xi'$. Since the element $Ve_2^2 \in \mathcal{B}(G)$ has two factorizations $z = Ve_2^2$ and

$z' = (e_1e_2(e_1+e_2))(e_2(e_1+e_3)(e_1+e_2+e_3)) = U_1(e_2(e_1+e_3)(e_1+e_2+e_3))$, the assertion follows from **A0**.

Case 2: $\omega_4(\xi) = 2$.

If $\mathrm{supp}(A) = G^\bullet$, then the assertion follows from **A2** and **A3**. Suppose that $\mathrm{supp}(A) \neq G^\bullet$. This implies that $\omega_4(\xi') \le 1$. Let U_1, U_2 denote the length 4 atoms dividing ξ. We may assume that $U_1 = e_1e_2e_3(e_1+e_2+e_3)$ and $U_2 = e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3)$.

Case 2.1: $\omega_3(\xi') = 0$.

Then ξ' is divisible by an atom of length 4, and $\mathrm{supp}(A) \neq G^\bullet$ implies that $\xi' = V^c y$, where $V = e_2e_3(e_1+e_2)(e_1+e_3), c \in \mathbb{N}$, and y is a product of length 2 atoms. Since $|\xi| = |\xi'|$,

$$|\xi| \le 2 + \frac{|A|-8}{2} = \frac{|A|}{2} - 2 \quad \text{and} \quad |\xi'| = c + \frac{|A|-3c}{2} = \frac{A}{2} - \frac{c}{2},$$

it follows that $c \ge 4$, and hence $e_1^2(e_1+e_2+e_3)^2V^2 \,|\, \xi'$. Since $\mathsf{Z}(e_1^2(e_1+e_2+e_3)^2V^2) \supset \{w = e_1^2(e_1+e_2+e_3)^2V^2, w' = (e_1+e_3)^2(e_1+e_2)^2U_1^2\}$, the assertion follows from **A0**.

Case 2.2: $\omega_4(\xi') = 0$.

Then ξ' is divisible by an atom of length 3. Since $\mathrm{supp}(A) \neq G^\bullet$, we may assume that $\xi' = V^c y$, where $V = e_1e_3(e_1+e_3), c \in \mathbb{N}$, and y is a product of length 2 atoms. It follows that $e_3^2U_2 \,|\, \xi'$. Since $\mathsf{Z}(e_3^2U_2) \supset \{w = e_3^2U_2, w' = V(e_3(e_1+e_2)(e_1+e_2+e_3))\}$, the assertion follows from **A0**.

Case 2.3: $\omega_3(\xi') \ge 1$ and $\omega_4(\xi') \ge 1$.

Since $\mathrm{supp}(A) \neq G^\bullet$, we have that $\xi' = V_1^cV_2^dy$, where $V_1 = e_2e_3(e_1+e_2)(e_1+e_3)$,

$$\begin{aligned} V_2 \in \{&e_1e_2(e_1+e_2), e_1e_3(e_1+e_3), e_2(e_1+e_3)(e_1+e_2+e_3),\\ &e_3(e_1+e_2)(e_1+e_2+e_3)\},\end{aligned}$$

$c, d \in \mathbb{N}$, and y is a product of length 2 atoms.

If $V_2 \in \{e_2(e_1+e_3)(e_1+e_2+e_3), e_3(e_1+e_2)(e_1+e_2+e_3)\}$, then $e_1^2 \,|\, \xi'$. Since

$$\mathsf{Z}(e_1^2V_1) \supset \{w = e_1^2V_1, w' = (e_1e_2(e_1+e_2))(e_1e_3(e_1+e_3))\}$$

and $(e_1e_2(e_1+e_2))V_2 \in \{e_2^2U_2, (e_1+e_2)^2U_1\}$, the assertion follows by **A0**.

If $V_2 \in \{e_1e_2(e_1+e_2), e_1e_3(e_1+e_3)\}$, then $(e_1+e_2+e_3)^2 \,|\, \xi'$. Since

$$\begin{aligned}\mathsf{Z}((e_1+e_2+e_3)^2V_1) \supset \{&w = (e_1+e_2+e_3)^2V_1,\\ &w' = (e_2(e_1+e_3)(e_1+e_2+e_3))(e_3(e_1+e_2)(e_1+e_2+e_3))\}\end{aligned}$$

and $(e_2(e_1+e_3)(e_1+e_2+e_3))V_2 \in \{e_2^2U_2, (e_1+e_3)^2U_1\}$, the assertion follows by **A0**.

Case 3: $\omega_4(\xi) = 3$.

Let U_1, U_2, U_3 denote the length 4 atoms dividing ξ. We distinguish two cases.

Case 3.1: $\operatorname{supp}(A) \neq G^\bullet$.

Then $\operatorname{supp}(U_1U_2U_3) \neq G^\bullet$. This implies that $\gcd(U_1, U_2, U_3) = 1$ (see **F5**), that ξ' is not divisible by an atom of length 4 but that it is divisible by some $V \in \mathcal{A}(G^\bullet)$ with $|V| = 3$, and that we must have $\operatorname{supp}(V) \subset \operatorname{supp}(U_1U_2U_3)$. Without restriction we may suppose that

$$U_1 = e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3), \quad U_2 = e_2(e_1+e_2)(e_2+e_3)(e_1+e_2+e_3),$$
$$U_3 = e_1e_2(e_1+e_3)(e_2+e_3) \quad \text{and} \quad V = e_1e_2(e_1+e_2).$$

We set

$$\xi = U_1^aU_2^bU_3^cU_4^dx \quad \text{and} \quad \xi' = V^fy,$$

where $U_4 \in \mathcal{A}(G^\bullet)$ with $|U_4| = 3, a, b, c, f \in \mathbb{N}, d \in \mathbb{N}_0$, and x, y are products of length 2 atoms. Since

$$a+b+c+d+\frac{|A|-4(a+b+c)-3d}{2} = |\xi| = |\xi'| = f + \frac{|A|-3f}{2},$$

it follows that $f = d + 2(a+b+c)$. This shows that $e_2^2 \mid x$. Since $\mathsf{Z}(U_1e_2^2) \supset \{w = e_2^2U_1, w' = (e_2(e_1+e_3)(e_1+e_2+e_3))V\}$, the assertion follows by **A0**.

Case 3.2: $\operatorname{supp}(A) = G^\bullet$.

If $\omega_4(\xi') \le 2$, then the assertion follows by **A2** and **A3**. Since $\omega_4(\xi') \le \omega_4(\xi) \le 3$, we obtain that $\omega_4(\xi') = 3$. Let V_1, V_2, V_3 denote the length 4 atoms dividing ξ'. We distinguish two cases.

First suppose that either $\operatorname{supp}(V_1V_2V_3) \neq G^\bullet$ or that $\operatorname{supp}(U_1U_2U_3) \neq G^\bullet$. By symmetry we may suppose that $\operatorname{supp}(V_1V_2V_3) \neq G^\bullet$. Then $\operatorname{supp}(U_1U_2U_3) = G^\bullet$, and we may assume that

$$\begin{aligned}
&U_1 = e_1e_2e_3(e_1+e_2+e_3), \quad U_2 = e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3),\\
&U_3 = e_2(e_1+e_2)(e_2+e_3)(e_1+e_2+e_3),\\
&U_4 = e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3),\\
&V_1 = e_1e_2(e_1+e_3)(e_2+e_3), \quad V_2 = e_1e_3(e_1+e_2)(e_2+e_3),\\
&V_3 = e_2e_3(e_1+e_2)(e_1+e_3).
\end{aligned}$$

We set

$$\xi = U_1^aU_2^bU_3^cU_5^ex \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hV_4^iy,$$

where $U_5, V_4 \in \mathcal{A}(G^\bullet)$ with $|U_5| = |V_4| = 3, a, b, c, f, g, h \in \mathbb{N}, e, i \in \mathbb{N}_0$, and x, y are products of length 2 atoms. By **F3** and **A0**, it suffices to consider the case

$$(e_1+e_2+e_3)^2 \nmid y, e_3^2 \nmid x, (e_1+e_3)^2 \nmid x \quad \text{and} \quad (e_2+e_3)^2 \nmid x. \tag{5.2}$$

By **F4**, **F5** and **A0**, $e > 0$ implies that

$$U_5 \notin \{(e_1+e_2)(e_1+e_3)(e_2+e_3), e_2e_3(e_2+e_3), e_1e_3(e_1+e_3)\}$$

and $i > 0$ implies that

$$\begin{aligned} V_4 \notin \{&e_1(e_2+e_3)(e_1+e_2+e_3), e_2(e_1+e_3)(e_1+e_2+e_3),\\ &e_3(e_1+e_2)(e_1+e_2+e_3)\}.\end{aligned}$$

Hence $(e_1+e_2+e_3) \notin \operatorname{supp}(V_1V_2V_3V_4)$, and so $(e_1+e_2+e_3)^2 \,|\, y$, which contradicts (5.2).

Second we suppose that $\operatorname{supp}(V_1V_2V_3) = \operatorname{supp}(U_1U_2U_3) = G^\bullet$. We may assume that

$$\begin{aligned}
&U_1 = e_1e_2e_3(e_1+e_2+e_3), \quad U_2 = e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3),\\
&U_3 = e_2(e_1+e_2)(e_2+e_3)(e_1+e_2+e_3),\\
&V_1 = e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3),\\
&V_2 = e_1e_2(e_1+e_3)(e_2+e_3), \quad V_3 = e_1e_3(e_1+e_2)(e_2+e_3),\\
&U_4 = e_2e_3(e_1+e_2)(e_1+e_3),
\end{aligned}$$

and we set

$$\xi = U_1^aU_2^bU_3^cU_5^ex \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hV_4^iy,$$

where $U_5, V_4 \in \mathcal{A}(G^\bullet)$ with $|U_5| = |V_4| = 3, a, b, c, f, g, h \in \mathbb{N}, e, i \in \mathbb{N}_0$, and x, y are products of length 2 atoms. By **F3** and **A0**, it suffices to consider the case

$$\begin{aligned} &e_2^2 \nmid y, (e_1+e_2)^2 \nmid y, (e_1+e_2+e_3)^2 \nmid y, e_3^2 \nmid x, (e_1+e_3)^2 \nmid x \text{ and}\\ &\quad (e_2+e_3)^2 \nmid x. \end{aligned} \tag{5.3}$$

By **F4**, **F5** and **A0**, $e > 0$ implies that

$$U_5 \notin \{(e_1+e_2)(e_1+e_3)(e_2+e_3), e_2e_3(e_2+e_3), e_1e_3(e_1+e_3)\}$$

and $i > 0$ implies that

$$V_4 \notin \{e_1e_2(e_1+e_2), e_2(e_1+e_3)(e_1+e_2+e_3), e_3(e_1+e_2)(e_1+e_2+e_3)\}.$$

Suppose that $(e_1+e_2+e_3) \,|\, U_5$. Since $(e_1+e_2+e_3)^2 \nmid y$, we have

$$\begin{aligned} f + i = \mathsf{v}_{e_1+e_2+e_3}(A) \geq a+b+c+e \quad &\text{and}\\ V_4 = e_1(e_2+e_3)(e_1+e_2+e_3).& \end{aligned} \tag{5.4}$$

Since $(e_2+e_3)^2 \nmid x$, we have

$$f + g + h + i \leq \mathsf{v}_{e_2+e_3}(A) = c + e,$$

which contradicts (5.4).

Now suppose that $(e_1+e_2+e_3) \nmid U_5$. Then $U_5 \neq e_1(e_2+e_3)(e_1+e_2+e_3)$ and $e > 0$ implies that $(e_2+e_3) \nmid U_5$.

If $(e_1+e_2+e_3) \mid V_4$, then $V_4 = e_1(e_2+e_3)(e_1+e_2+e_3)$. Since $(e_1+e_2+e_3)^2 \nmid y$, we have

$$f + i = \mathsf{v}_{e_1+e_2+e_3}(A) \geq a + b + c. \tag{5.5}$$

Since $(e_2 + e_3)^2 \nmid x$, we have

$$f + g + h + i \leq \mathsf{v}_{e_2+e_3}(A) = c,$$

which contradicts (5.5).

Thus we get $(e_1 + e_2 + e_3) \nmid V_4$. Since $(e_1 + e_2 + e_3)^2 \nmid y$, we have

$$f = \mathsf{v}_{e_1+e_2+e_3}(A) \geq a + b + c. \tag{5.6}$$

Since $(e_2 + e_3)^2 \nmid x$, we have

$$f + g + h \leq \mathsf{v}_{e_2+e_3}(A) = c,$$

which contradicts (5.6).

Case 4: $\omega_4(\xi) = 4$.

This implies that $\mathrm{supp}(A) = G^\bullet$. If $\omega_4(\xi') \leq 2$, then the assertion follows by **A2** and **A3**. Since $\omega_4(\xi') \leq 3$, we may suppose that $\omega_4(\xi') = 3$. We denote by U_1, U_2, U_3, U_4 the length 4 atoms dividing ξ, and by V_1, V_2, V_3 the length 4 atoms dividing ξ'.

Case 4.1: $\gcd(U_1, U_2, U_3, U_4) \neq 1$.

We may assume that

$$\begin{aligned}
&U_1 = e_1e_2e_3(e_1 + e_2 + e_3), \quad U_2 = e_1(e_1 + e_2)(e_1 + e_3)(e_1 + e_2 + e_3),\\
&U_3 = e_2(e_1 + e_2)(e_2 + e_3)(e_1 + e_2 + e_3),\\
&U_4 = e_3(e_1 + e_3)(e_2 + e_3)(e_1 + e_2 + e_3),\\
&V_1 = e_1e_2(e_1 + e_3)(e_2 + e_3), \quad V_2 = e_1e_3(e_1 + e_2)(e_2 + e_3),\\
&V_3 = e_2e_3(e_1 + e_2)(e_1 + e_3),
\end{aligned}$$

and we set

$$\xi = U_1^aU_2^bU_3^cU_4^dU_5^ex \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hV_4^iy,$$

where $U_5, V_4 \in \mathcal{A}(G^\bullet)$ with $|U_5| = |V_4| = 3, a, b, c, d, f, g, h \in \mathbb{N}, e, i \in \mathbb{N}_0$, and x, y are products of length 2 atoms. If $(e_1 + e_2 + e_3)^2 \mid y$, then we are done by **F3** and **A0**. Suppose that this is not the case. If $\mathrm{supp}(U_5U_j) = G^\bullet$ for some $j \in [1, 4]$, then we are done by **F4** and **A0**. Thus we may suppose that $U_5 \in \{e_1(e_2+e_3)(e_1+e_2+e_3), e_2(e_1+e_3)(e_1+e_2+e_3), e_3(e_1+e_2)(e_1+e_2+e_3)\}$, which implies that $i = \mathsf{v}_{e_1+e_2+e_3}(A) = a + b + c + d + e$ and $V_4 \in \{e_1(e_2 + e_3)(e_1 + e_2 + e_3), e_2(e_1 + e_3)(e_1 + e_2 + e_3), e_3(e_1 + e_2)(e_1 + e_2 + e_3)\}$. Therefore we get $\mathrm{supp}(V_4V_j) = G^\bullet$ for some $j \in [1, 3]$, and the assertion follows from **F3** and **A0**.

Case 4.2: $\gcd(U_1, U_2, U_3, U_4) = 1$.

We may assume that

$$\begin{aligned}
&U_1 = e_1e_2e_3(e_1+e_2+e_3), \quad U_2 = e_1(e_1+e_2)(e_1+e_3)(e_1+e_2+e_3),\\
&U_3 = e_2(e_1+e_2)(e_2+e_3)(e_1+e_2+e_3), \quad U_4 = e_1e_2(e_1+e_3)(e_2+e_3),\\
&V_1 = e_3(e_1+e_3)(e_2+e_3)(e_1+e_2+e_3), \quad V_2 = e_1e_3(e_1+e_2)(e_2+e_3),\\
&V_3 = e_2e_3(e_1+e_2)(e_1+e_3),
\end{aligned}$$

and we set

$$\xi = U_1^aU_2^bU_3^cU_4^dU_5^ex \quad \text{and} \quad \xi' = V_1^fV_2^gV_3^hV_4^iy,$$

where $U_5, V_4 \in \mathcal{A}(G^\bullet)$ with $|U_5| = |V_4| = 3, a, b, c, d, f, g, h \in \mathbb{N}, e, i \in \mathbb{N}_0$, and x, y are products of length 2 atoms. As above, it suffices to consider the case $\operatorname{supp}(U_5U_j) \neq G^\bullet$ for all $j \in [1, 4]$, and hence $U_5 \in \{e_1(e_2+e_3)(e_1+e_2+e_3), e_2(e_1+e_3)(e_1+e_2+e_3), e_1e_2(e_1+e_2)\}$. If ξ' is not divisible by a length 3 atom, then $e_3^2 \,|\, \xi$, and thus the assertion follows by **F3** and **A0**. If ξ' is divisible by a length 3 atom, then by the same argument as above and since $e_3 \nmid V_4$, we may suppose that $V_4 = (e_1+e_2)(e_1+e_3)(e_2+e_3)$, which implies that $a = \mathsf{v}_{e_3}(A) = f + g + h$. It follows that $e_1^2 \,|\, \xi'$, and the assertion follows by **F3** and **A0**. □

References

[1] Blanco, V., García-Sánchez, P.A., Geroldinger, A.: Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids. Ill. J. Math. (in press)

[2] Chang, G.-W.: Every divisor class of Krull monoid domains contains a prime ideal. J. Algebra **336**, 370–377 (2011)

[3] Chapman, S.T., García-Sánchez, P.A., Llena, D.: The catenary and tame degree of numerical monoids. Forum Math. **21**, 117–129 (2009)

[4] Chapman, S.T., García-Sánchez, P.A., Llena, D., Ponomarenko, V., Rosales, J.C.: The catenary and tame degree in finitely generated commutative cancellative monoids. Manuscr. Math. **120**, 253–264 (2006)

[5] Coykendall, J., Smith, W.W.: On unique factorization domains. J. Algebra **332**, 62–70 (2011)

[6] Delgado, M., García-Sánchez, P.A., Morais, J.: "Numericalsgps": a gap package on numerical semigroups. http://www.gap-system.org/Packages/numericalsgps.html

[7] Diaconis, P., Graham, R.L., Sturmfels, B.: Primitive partition identities, combinatorics, Paul Erdős is eighty. J. Bolyai Math. Soc. **1**, 173–192 (1993)

[8] Foroutan, A.: Monotone chains of factorizations. In: Badawi, A. (ed.) Focus on commutative rings research, pp. 107–130. Nova Science Publishers, New York (2006)

[9] Foroutan, A., Geroldinger, A.: Monotone chains of factorizations in C-monoids. In: Arithmetical Properties of Commutative Rings and Monoids, Lect. Notes Pure Appl. Math., vol. 241, pp. 99–113. Chapman & Hall/CRC, London (2005)

[10] Foroutan, A., Hassler, W.: Chains of factorizations and factorizations with successive lengths. Commun. Algebra **34**, 939–972 (2006)

[11] Freeze, M., Schmid, W.A.: Remarks on a generalization of the Davenport constant. Discrete Math. **310**, 3373–3389 (2010)

[12] Geroldinger, A. : Additive group theory and non-unique factorizations. In: Geroldinger, A., Ruzsa, I. (eds.) Combinatorial Number Theory and Additive Group Theory. Advanced Courses in Mathematics CRM Barcelona, pp. 1–86. Birkhäuser, Basel (2009)

[13] Geroldinger, A., Grynkiewicz, D.J., Schaeffer, G.J., Schmid, W.A.: On the arithmetic of Krull monoids with infinite cyclic class group. J. Pure Appl. Algebra **214**, 2219–2250 (2010)

[14] Geroldinger, A., Grynkiewicz, D.J., Schmid, W.A.: The catenary degree of Krull monoids I. J. Théor. Nombres Bordx. **23**, 137–169 (2011)

[15] Geroldinger, A., Halter-Koch, F.: Non-Unique Factorizations. In: Algebraic, Combinatorial and Analytic Theory. Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, London (2006)

[16] Geroldinger, A., Hassler, W.: Arithmetic of Mori domains and monoids. J. Algebra **319**, 3419–3463 (2008)

[17] Halter-Koch, F.: Ideal Systems. An Introduction to Multiplicative Ideal Theory. Marcel Dekker, New York (1998)

[18] Hassler, W.: Properties of factorizations with successive lengths in one-dimensional local domains. J. Commut. Algebra **1**, 237–268 (2009)

[19] Kainrath, F.: Arithmetic of Mori domains and monoids: the global case (manuscript)

[20] Kim, H.: The distribution of prime divisors in Krull monoid domains. J. Pure Appl. Algebra **155**, 203–210 (2001)

[21] Kim, H., Park, Y.S.: Krull domains of generalized power series. J. Algebra **237**, 292–301 (2001)

[22] Lev, V.F.: The rectifiability threshold in abelian groups. Combinatorica **28**, 491–497 (2008)

[23] Omidali, M.: The catenary and tame degree of numerical monoids generated by generalized arithmetic sequences. Forum Math. (in press)

[24] Philipp, A.: A characterization of arithmetical invariants by the monoid of relations II: The monotone catenary degree and applications to semigroup rings. Semigroup Forum **81**, 424–434 (2010)

[25] Philipp, A.: A precise result on the arithmetic of non-principal orders in algebraic number fields. J. Algebra Appl. (in press)

[26] Philipp, A.: A characterization of arithmetical invariants by the monoid of relations. Semigroup Forum **81**, 424–434 (2010)

[27] Reinhart, A.: On integral domains that are C-monoids. Houston J. Math. (in press)

[28] Schmid, W.A.: The inverse problem associated to the Davenport constant for $C_2 \oplus C_2 \oplus C_{2n}$, and applications to the arithmetical characterization of class groups. Electron. J. Comb. **18**(1) (2011) (Research Paper 33)

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