
Volume 321 Number 4, February 15, 2009 ISSN 0021-8693
ELSEVIER
JOURNAL OF
ALGEBRA
Editor-in-Chief
Michel Broué
Editor of the section on Computational Algebra
Gerhard Hiss
Founding Editor
G. Higman, 1964–1984
Editor-in-Chief
W. Feit, 1985–2000
Editorial Board
Nicolás Andruskiewitsch
Luchezar L. Avramov
Eva Bayer-Fluckiger
Steven Dale Cutkosky
Dihua Jiang
Masaki Kashiwara
E.I. Khukhro
Kazuhiko Kurano
Gus I. Lehrer
Martin Liebeck
Peter Littelmann
Laurent Moret-Bailly
Leonard L. Scott, Jr.
Vera Serganova
Aner Shalev
Ronald Solomon
J.T. Stafford
Gernot Stroth
Michel Van den Bergh
Changchang Xi
Efim Zelmanov
Section on Computational Algebra
Jon Carlson
John Cremona
Patrick Dehornoy
Harm Derksen
Meinolf Geck
Derek Holt
William M. Kantor
Reinhard Laubenbacher
Gunter Malle
Eamonn O'Brien
Bruno Salvy
Jean-Yves Thibon

Journal of Algebra 321 (2009) 1256–1284

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the arithmetic of Krull monoids with finite Davenport constant ☆

Alfred Geroldinger *, David J. Grynkiewicz

Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstraße 36, 8010 Graz, Austria

ARTICLE INFO

Article history:
Received 7 June 2008
Available online 17 December 2008
Communicated by Luchezar L. Avramov

Keywords:
Non-unique factorizations
Sets of lengths
Krull monoids

ABSTRACT

Let H be a Krull monoid with class group G, $G_P \subset G$ the set of classes containing prime divisors and $\mathsf{D}(G_P)$ the Davenport constant of G_P. We show that the finiteness of the Davenport constant implies the Structure Theorem for Sets of Lengths. More precisely, if $\mathsf{D}(G_P) < \infty$, then there exists a constant M—for which we derive an explicit upper bound in terms of $\mathsf{D}(G_P)$—such that the set of lengths of every element $a \in H$ is an almost arithmetical multiprogression with bound M.

1. Introduction

Krull monoids, defined as completely integrally closed monoids satisfying the ascending chain condition for divisorial ideals, were introduced by L.G. Chouinard [4] in order to study the Krull ring property of semigroup rings. It followed that a monoid (that is, a commutative cancellative semigroup with identity) is a Krull monoid if and only if it has a divisor theory, and that a domain is a Krull domain if and only if its multiplicative monoid is a Krull monoid [19,22]. Regular congruence monoids in Krull domains are Krull [14, Proposition 2.11.6], and Krull monoids also occur when studying direct sum decompositions of modules [5,6,26]. Further examples and a thorough presentation of the theory of Krull monoids may be found in each of the monographs [14,17,20].

The theory of non-unique factorizations grew out of algebraic number theory [1,2,13,14,18,25]. Starting with rings of integers of algebraic number fields, the arithmetic of general noetherian domains became the center of interest. If a noetherian domain is integrally closed, then it is a Krull domain and thus its multiplicative monoid is Krull. Moreover, many arithmetical problems in a Krull monoid can be studied in the associated monoid of zero-sum sequences over the class group of the

☆ This work was supported by the Austrian Science Fund FWF (Project Number M1014-N13).

* Corresponding author.
E-mail addresses: alfred.geroldinger@uni-graz.at (A. Geroldinger), diambri@hotmail.com (D.J. Grynkiewicz).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.020

original monoid. The monoid of zero-sum sequences is a Krull monoid again, and it represents the link between general Krull monoids and combinatorial number theory.

Let H be a Krull monoid with class group G, $G_P \subset G$ the set of classes containing prime divisors, and let $\mathsf{D}(G_P)$ denote the Davenport constant of G_P (this is, the supremum of the lengths of minimal zero-sum sequences over G_P). C-monoids and weakly C-monoids generalize Krull monoids with finite class group to v-noetherian monoids which are not necessarily completely integrally closed. More precisely, the Krull monoid H is a weakly C-monoid if and only if G_P is finite (see [15, Proposition 4.8]).

In the present paper, we continue with Krull monoids but weaken the condition of the finiteness of G_P to that of the finiteness of $\mathsf{D}(G_P)$. It is well known that the finiteness of the Davenport constant implies the finiteness of a large variety of arithmetical invariants. For example, if $\mathsf{D}(G_P) < \infty$, then H is a tame, finite factorization monoid with finite elasticity, finite catenary degree, finite set of distances, and unions of sets of lengths are almost arithmetical progressions with some universal bound (see [14, Chapter 3] and [11]). All further results on the structure of sets of lengths have up to now been restricted to the case where G_P is finite. In the present paper, we remove this additional assumption. Our main result states, among others, that in a Krull monoid with finite Davenport constant there exists a constant M such that every set of lengths is an AAMP (almost arithmetical multiprogression) with bound M (see Theorem 4.4). Moreover, we derive an explicit upper bound for the constant M in terms of the Davenport constant. Such an explicit upper bound was not previously known even in the setting of a finite class group G.

The paper is organized as follows. After fixing notation in Section 2, we study transfer homomorphisms and pattern ideals in Section 3. It is well known that, if all pattern ideals of a monoid H are tamely generated (and some further very mild finiteness conditions are satisfied), then sets of lengths in H are AAMPs (see Proposition 3.4). In Section 3, we show that if $\theta : H \to B$ is a transfer homomorphism with a global bound on the tame degrees in the fibres, then patterns ideals of H are tamely generated if and only if pattern ideals of B are tamely generated. In Section 4, we apply this to a Krull monoid H and its block homomorphism—which is a transfer homomorphism—$\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$, where $\mathcal{B}(G_P)$ is the monoid of zero-sum sequences over the set of classes containing prime divisors. These general algebraic concepts reduce the question on the structure of sets of lengths to a combinatorial problem on zero-sum sequences (first statement of Proposition 4.3). In Remark 4.5, we discuss some striking features of our main result and some open questions. The proof of the crucial Proposition 4.3 is postponed to Section 5.

2. Preliminaries

Our notation and terminology is consistent with [14]. We briefly gather some key notions. We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\}$, and we define $\sup \emptyset = \max \emptyset = \min \emptyset = 0$.

Let $A, B \subset \mathbb{Z}$ be finite nonempty subsets. Then $A + B = \{a + b \mid a \in A,\ b \in B\}$ is their *sumset*, and we use

$$\operatorname{diam}(A) = \max A - \min A$$

to denote the diameter of A. A set $A' \subset \mathbb{Z}$ is called an *interval* of A if A' is nonempty and $A' = A \cap [\min A', \max A']$. We denote by $\Delta(A)$ the *set of* (*successive*) *distances* of A, that is, if $A = \{a_1, \ldots, a_t\}$ with $t \in \mathbb{N}$ and $a_1 < \cdots < a_t$, then $\Delta(A) = \{a_{\nu+1} - a_\nu \mid \nu \in [1, t-1]\}$. Moreover, we set $\Delta(\emptyset) = \emptyset$.

Let $d \in \mathbb{N}$, $l, M \in \mathbb{N}_0$ and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. A subset $L \subset \mathbb{Z}$ is called an

- *arithmetical multiprogression* (AMP for short) with *difference* d, *period* $\mathcal{D}$ and *length* l, if L is an interval of $\min L + \mathcal{D} + d\mathbb{Z}$ (in particular, $L \neq \emptyset$), and l is maximal such that $\min L + ld \in L$.
- *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, *length* l and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z}, \tag{2.1}$$

where L^* is an AMP with difference d (whence $L^* \ne \emptyset$), period $\mathcal{D}$ and length l such that $\min L^* = 0$, $L' \subset [-M, -1]$, $L'' \subset \max L^* + [1, M]$ and $y \in \mathbb{Z}$.

We call $y + L'$ the *initial part*, $y + L^*$ the *central part* and $y + L''$ the *end part* of L.

Note that an AAMP is a finite nonempty subset of $\mathbb{Z}$, and a finite nonempty subset $L \subset \mathbb{Z}$ is an arithmetical progression with difference $d \in \mathbb{N}$ if and only if it is an AAMP with period $\{0, d\}$, difference d and bound 0. The interested reader is referred to [14] for more information on the structure of AAMPs.

By a *monoid*, we always mean a commutative semigroup with identity which satisfies the cancellation law (that is, if a, b, c are elements of the monoid with $ab = ac$, then $b = c$ follows). If R is an integral domain and $R^\bullet = R \setminus \{0\}$ its multiplicative semigroup of non-zero elements, then $R^\bullet$ is a monoid.

Let H be a monoid. We denote by $H^\times$ the set of invertible elements of H, and we say that H is *reduced* if $H^\times = \{1\}$. Let $H_{\mathrm{red}} = H/H^\times = \{aH^\times \mid a \in H\}$ be the associated reduced monoid, and $\mathsf{q}(H)$ a quotient group of H. For a subset $H_0 \subset H$, we denote by $[H_0] \subset H$ the submonoid generated by H_0. For elements $a, b \in H$, we say that a *divides* b (and we write $a \mid b$) if there is an element $c \in H$ such that $b = ac$.

A monoid F is called *free* (*abelian, with basis* $P \subset F$) if every $a \in F$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with } \mathsf{v}_p(a) \in \mathbb{N}_0 \text{ and } \mathsf{v}_p(a) = 0 \text{ for almost all } p \in P.$$

We set $F = \mathcal{F}(P)$ and call

$$|a|_F = |a| = \sum_{p \in P} \mathsf{v}_p(a) \quad \text{the } \textit{length} \text{ of } a.$$

Next we recall some basic arithmetical notions from factorization theory. We denote by $\mathcal{A}(H)$ the *set of atoms* of H, and we call $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ the *factorization monoid* of H. Further, $\pi : \mathsf{Z}(H) \to H_{\mathrm{red}}$ denotes the natural homomorphism. For $a \in H$, the set

$$\mathsf{Z}(a) = \mathsf{Z}_H(a) = \pi^{-1}\big(aH^\times\big) \subset \mathsf{Z}(H) \quad \text{is called the } \textit{set of factorizations} \text{ of } a,$$

$$\mathsf{L}(a) = \mathsf{L}_H(a) = \big\{|z| \mid z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0 \quad \text{is called the } \textit{set of lengths} \text{ of } a, \quad \text{and}$$

$$\mathcal{L}(H) = \big\{\mathsf{L}(a) \bigm| a \in H\big\} \quad \text{is called the } \textit{system of sets of lengths} \text{ of } H.$$

The monoid H is called

- *atomic* if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$ (equivalently, every non-unit of H may be written as a finite product of atoms of H);
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$;
- a BF-*monoid* (a bounded factorization monoid) if $\mathsf{L}(a)$ is finite and nonempty for all $a \in H$.

All v-noetherian monoids (these include Krull monoids) are BF-monoids. If H is a BF-monoid, then

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \subset \mathbb{N} \quad \text{and} \quad \rho(H) = \sup\left\{\frac{\max L}{\min L} \;\middle|\; L \in \mathcal{L}(H)\right\} \in \mathbb{R}_{\geqslant 1} \cup \{\infty\}$$

denote the *set of distances* of H and the *elasticity* of H.

We recall the concept of the *distance* of two factorizations and the concept of *local tameness*, which is a basic finiteness property in factorization theory. Let $z, z' \in \mathsf{Z}(H)$. Then we can write

$$z = u_1 \cdot \ldots \cdot u_l v_1 \cdot \ldots \cdot v_m \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_l w_1 \cdot \ldots \cdot w_n,$$

where $l, m, n \in \mathbb{N}_0$, $u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathcal{A}(H_{\mathrm{red}})$ are such that

$$\{v_1, \ldots, v_m\} \cap \{w_1, \ldots, w_n\} = \emptyset.$$

We call $\mathsf{d}(z, z') = \max\{m, n\} \in \mathbb{N}_0$ the *distance* of z and z'. For a factorization $x \in \mathsf{Z}(H)$ and $a \in H$, we define the *tame degree* $\mathsf{t}(a, x)$ to be the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

> If $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \neq \emptyset$ and $z \in \mathsf{Z}(a)$, then there exists some factorization $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\mathsf{d}(z, z') \leqslant N$.

For any subsets $H' \subset H$ and $X \subset \mathsf{Z}(H)$, we set $\mathsf{t}(H', X) = \sup\{\mathsf{t}(a, x) \mid a \in H',\ x \in X\}$. The monoid H is called *locally tame* if it is atomic and $\mathsf{t}(H, u) < \infty$ for all $u \in \mathcal{A}(H_{\mathrm{red}})$, and it is called *tame* if it is atomic and

$$\mathsf{t}(H) = \sup\bigl\{\mathsf{t}(H, u) \bigm| u \in \mathcal{A}(H_{\mathrm{red}})\bigr\} < \infty.$$

Every locally tame monoid is a BF-monoid [14, Theorem 1.6.7].

3. Transfer homomorphisms and arithmetical properties

Definition 3.1. A monoid homomorphism $\theta : H \to B$ is called a *transfer homomorphism* if it has the following properties:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
(T2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

The following lemma gathers the main properties of transfer homomorphisms which will be used in the sequel without further mention (see [14, Proposition 3.2.3]).

Lemma 3.2. *Let $\theta : H \to B$ be a transfer homomorphism and $u \in H$.*

1. *u is an atom of H if and only if $\theta(u)$ is an atom of B.*
2. *There is a unique homomorphism $\bar\theta : \mathsf{Z}(H) \to \mathsf{Z}(B)$ satisfying*

$$\bar\theta\bigl(uH^\times\bigr) = \theta(u)B^\times \quad \textit{for all } u \in \mathcal{A}(H).$$

   This map is surjective and has the following properties:
   (a) *If $z, z' \in \mathsf{Z}(H)$, then $|\bar\theta(z)| = |z|$ and $\mathsf{d}(\bar\theta(z), \bar\theta(z')) \leqslant \mathsf{d}(z, z')$.*
   (b) *$\bar\theta(\mathsf{Z}_H(u)) = \mathsf{Z}_B(\theta(u))$ and $\mathsf{L}_H(u) = \mathsf{L}_B(\theta(u))$.*
   (c) *If $z \in \mathsf{Z}(u)$ and $\bar y \in \mathsf{Z}(\theta(u))$, then there exists some $y \in \mathsf{Z}(u)$ such that $\bar\theta(y) = \bar y$, $\bar\theta(\gcd(z, y)) = \gcd(\bar\theta(z), \bar y)$ and $\mathsf{d}(z, y) = \mathsf{d}(\bar\theta(z), \bar y)$.*
3. *H is atomic if and only if B is atomic, and if this holds, then $\mathcal{L}(H) = \mathcal{L}(B)$.*

The homomorphism $\bar\theta : \mathsf{Z}(H) \to \mathsf{Z}(B)$ occurring in Lemma 3.2.2 is called *the extension of θ to the factorization monoids*.

Definition 3.3. Let H be atomic, $\mathfrak{a} \subset H$ and $A \subset \mathbb{Z}$ be a finite nonempty subset.

1. We say that a subset $L \subset \mathbb{Z}$ *contains the pattern A* if there exists some $y \in \mathbb{Z}$ such that $y + A \subset L$. We denote by $\varPhi(A) = \varPhi_H(A)$ the set of all $a \in H$ for which $\mathsf{L}(a)$ contains the pattern A.

2. $\mathfrak{a}$ is called a *pattern ideal* if $\mathfrak{a} = \Phi(B)$ for some finite, nonempty subset $B \subset \mathbb{Z}$.
3. A subset $E \subset H$ is called a *tame generating set* of $\mathfrak{a}$ if $E \subset \mathfrak{a}$ and there exists some $N \in \mathbb{N}$ with the following property:
   for every $a \in \mathfrak{a}$, there exists some $e \in E$ such that
   $$e \mid a, \qquad \sup \mathsf{L}(e) \leqslant N \quad \text{and} \quad \mathsf{t}\big(a, \mathsf{Z}(e)\big) \leqslant N.$$
   In this case, we call E a *tame generating set with bound N*, and we say that $\mathfrak{a}$ is called *tamely generated*.
4. If $\mathfrak{a}$ is tamely generated, then we denote by $\varphi(\mathfrak{a})$ the smallest $N \in \mathbb{N}_0$ such that $\mathfrak{a}$ has a tame generating set with bound N. Otherwise, we define $\varphi(\mathfrak{a}) = \infty$, and we set $\varphi(A) = \varphi(\Phi(A))$.

The significance of tamely generated pattern ideals stems from the following result whose proof can be found in [14, Theorem 4.3.11].

Proposition 3.4. *Let H be a* BF*-monoid with finite nonempty set of distances $\Delta(H)$, and suppose that all pattern ideals of H are tamely generated. Then there exists a bound $M \in \mathbb{N}_0$ such that every $L \in \mathcal{L}(H)$ is an* AAMP *with some difference $d \in \Delta(H)$ and bound M.*

More precisely, for every $\theta \in \mathbb{N}$ with $\theta \geqslant 2 \max \Delta(H) - 1$, let

$$M_1 = \max\big\{\varphi(A) \bigm| A \subset [0, \theta]\big\} + \max \Delta(H),$$
$$M_2 = \max\big\{\varphi(A) \bigm| A \subset \big[0, M_1 + \max \Delta(H)\big]\big\},$$

and

$$M(\theta) = 2M_1 + 2M_2 + \max \Delta(H).$$

Then, for every $a \in H$, the set $\mathsf{L}(a)$ is an AAMP *with some difference $d \in \Delta(H)$ and bound $M(\theta)$, and if $\mathsf{L}(a) \not\subset \min \mathsf{L}(a) + [0, M(\theta)]$, then $\mathsf{L}(a)$ is an* AAMP *with some difference $d \in \Delta(H)$ and bound M_1.*

We continue with some results on tame monoids. Proposition 4.1 will reveal that Krull monoids with finite Davenport constant are tame. But this class of monoids also includes all finitely generated monoids and certain weakly Krull domains which are not Krull (see [14, Theorem 3.7.1]).

Lemma 3.5. *Let H be a tame monoid and $\mathfrak{a} \subset H$ an s-ideal. Then the following statements are equivalent*:

(a) *$\mathfrak{a}$ is tamely generated.*
(b) *There is an $M \in \mathbb{N}$ such that $\{e \in \mathfrak{a} \mid \max \mathsf{L}(e) \leqslant M\}H = \mathfrak{a}$.*

If (b) holds, then $\varphi(\mathfrak{a}) \leqslant 2M\mathsf{t}(H)$.

Proof. (a) $\Rightarrow$ (b) This follows from the definition.

(b) $\Rightarrow$ (a) We set $E = \{e \in \mathfrak{a} \mid \max \mathsf{L}(e) \leqslant M\}$ and have to verify that E is a tame generating set of $\mathfrak{a}$. Let $a \in \mathfrak{a}$. Then there is an $e \in E$ such that $e \mid a$ and $\max \mathsf{L}(e) \leqslant M$. By [14, Lemma 1.6.5.7], it follows that

$$\mathsf{t}\big(a, \mathsf{Z}(e)\big) \leqslant 2 \min \mathsf{L}(e)\mathsf{t}(H) \leqslant 2M\mathsf{t}(H).$$

Thus E is a tame generating set of $\mathfrak{a}$ with bound $2M\mathsf{t}(H)$. $\square$

Next we recall the definition of the successive distance, a strong arithmetical invariant introduced by A. Foroutan in [7], and further studied by A. Foroutan and W. Hassler in [8,21].

Definition 3.6. Let H be atomic and $\pi : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the factorization homomorphism.

1. If $a \in H$, then two positive integers $k, l \in \mathbb{N}$ are called *adjacent lengths* of a if $k, l \in \mathsf{L}(a)$, $k \neq l$, and there is no $m \in \mathsf{L}(a)$ lying strictly between k and l.
2. For $z \in \mathsf{Z}(H)$, we denote by $\delta(z)$ the smallest $N \in \mathbb{N}_0$ with the following property:
   if $k \in \mathbb{N}$ is such that k and $|z|$ are adjacent lengths of $\pi(z)$, then there exists some $y \in \mathsf{Z}(H)$ such that $\pi(y) = \pi(z)$, $|y| = k$ and $\mathsf{d}(z, y) \leqslant N$.

   We call

$$\delta(H) = \sup\{\delta(z) \mid z \in \mathsf{Z}(H)\} \in \mathbb{N}_0 \cup \{\infty\}$$

   the *successive distance* of H.

By definition, we have $\delta(H) = 0$ if and only if H is half-factorial.

Lemma 3.7. *Let H be atomic, $a \in H$, $z, z' \in \mathsf{Z}(a)$, and $l = ||z| - |z'||$. Then there exists some $z'' \in \mathsf{Z}(a)$ such that $|z''| = |z'|$ and $\mathsf{d}(z, z'') \leqslant l\delta(H)$.*

Proof. See [14, Lemma 3.1.3]. $\square$

Proposition 3.8. *Let H be a tame monoid such that $\delta(H) < \infty$. If $A \subset \mathbb{Z}$ is a finite, nonempty subset, then*

$$\varphi(A) \leqslant 2|A| \max \Delta(A)\rho(H)\delta(H)\mathsf{t}(H),$$

and thus all pattern ideals are tamely generated.

Proof. Since $\rho(H) \leqslant \max\{1, \mathsf{t}(H)\}$ (see [14, Theorem 1.6.6]), the right-hand side of the asserted inequality is indeed finite. Let $A \subset \mathbb{Z}$ be a finite nonempty subset and $\mathfrak{a} = \Phi(A)$. By Lemma 3.5, we have to show that, for

$$M = |A| \max \Delta(A)\rho(H)\delta(H) \quad \text{and} \quad E = \{e \in \mathfrak{a} \mid \max \mathsf{L}(e) \leqslant M\},$$

we have $EH = \mathfrak{a}$.

If $|A| = 1$, then $\mathfrak{a} = H$, $M = 0$, $E = H^\times$, and clearly the required property is satisfied. Suppose that $|A| \geqslant 2$. Let $a \in \mathfrak{a}$ and $x \in \mathbb{Z}$ be such that $\{m_1, \ldots, m_l\} = x + A \subset \mathsf{L}(a)$, where $m_1 < \cdots < m_l$. Let $z_1 \in \mathsf{Z}(a)$ with $|z_1| = m_1$ and

$$\psi = \max\{m_i - m_{i-1} \mid i \in [2, l]\} \cdot \delta(H) = \max \Delta(A)\delta(H).$$

Let $i \in [2, l]$ and suppose that $z_{i-1} \in \mathsf{Z}(a)$ is already defined. Then, by Lemma 3.7, there is a $z_i \in \mathsf{Z}(a)$ with $|z_i| = m_i$ and $\mathsf{d}(z_{i-1}, z_i) \leqslant (m_i - m_{i-1})\delta(H) \leqslant \psi$, say $z_{i-1} = x_{i-1}y$ and $z_i = x_iy$, such that $\mathsf{d}(z_{i-1}, z_i) = \max\{|x_{i-1}|, |x_i|\}$. Note that $|x_i| - |x_{i-1}| = m_i - m_{i-1}$.

We continue with the following assertion:

(A) For every $i \in [1, l]$, there are $v_i \in \mathsf{Z}(H)$ and $t_i \in \mathbb{Z}$ such that

$$v_i \mid z_i, t_i + m_i = |v_i| \leqslant i\psi \quad \text{and} \quad t_i + \{m_1, \ldots, m_i\} \subset \mathsf{L}(\pi(v_i)).$$

Suppose that (A) holds and set $e = \pi(v_l)$. Then $\max \mathsf{L}(e) \leqslant |v_l|\rho(H) \leqslant l\psi\rho(H) \leqslant |A|\psi\rho(H)$, $t_l + \{m_1, \ldots, m_l\} \subset \mathsf{L}(e)$ and $e = \pi(v_l)$ divides $a = \pi(z_l)$ because $v_l \mid z_l$. Thus the set

$$E = \bigl\{e \in \mathfrak{a} \bigm| \max \mathsf{L}(e) \leqslant |A|\psi\rho(H) = M\bigr\}$$

satisfies $EH = \mathfrak{a}$ as required.

Proof of (A). We proceed by induction on i. Setting $v_1 = x_1$ and $t_1 = |v_1| - m_1$, we get the assertion for $i = 1$. Let $i \geqslant 2$. By induction hypothesis,

$$z_{i-1} = v_{i-1}w = x_{i-1}y \quad \text{and} \quad z_i = x_i y, \quad \text{where } w, y \in \mathsf{Z}(H).$$

Hence

$$z_{i-1} = x_{i-1}v'_{i-1}y', \quad \text{with } x_{i-1}v'_{i-1} = \operatorname{lcm}(v_{i-1}, x_{i-1}) \quad \text{and} \quad z_i = x_i v'_{i-1}y', \quad \text{with } y' \in \mathsf{Z}(H).$$

We define

$$v_i = x_i v'_{i-1} \quad \text{and} \quad t_i = |v_i| - m_i. \tag{3.1}$$

Clearly, $v_i \mid z_i$,

$$|v_i| = |x_i| + \bigl|v'_{i-1}\bigr| \leqslant |x_i| + |v_{i-1}| \leqslant \psi + (i-1)\psi = i\psi,$$

and

$$|v_i| = t_i + m_i \in \mathsf{L}\bigl(\pi(v_i)\bigr).$$

Since $x_{i-1}v'_{i-1} = v_{i-1}x'_{i-1}$ for some $x'_{i-1} \in \mathsf{Z}(H)$, and since $|x_i| - |x_{i-1}| = m_i - m_{i-1}$, we infer from (3.1) that

$$t_i + m_{i-1} = \bigl|x_{i-1}v'_{i-1}\bigr| = \bigl|x'_{i-1}\bigr| + t_{i-1} + m_{i-1}$$

and thus, because $\pi(x_{i-1}v'_{i-1}) = \pi(x_i v'_{i-1}) = \pi(v_i)$,

$$\begin{aligned} t_i + \{m_1, \ldots, m_{i-1}\} &= \bigl|x'_{i-1}\bigr| + t_{i-1} + \{m_1, \ldots, m_{i-1}\} \in \mathsf{L}\bigl(\pi\bigl(x'_{i-1}\bigr)\pi(v_{i-1})\bigr) \\ &= \mathsf{L}\bigl(\pi\bigl(x_{i-1}v'_{i-1}\bigr)\bigr) = \mathsf{L}\bigl(\pi(v_i)\bigr). \qquad \square \end{aligned}$$

Lemma 3.9. *Let L_1 and L_2 be* AAMPs *with common difference d and respective periods $\mathcal{D}_1$ and $\mathcal{D}_2$, lengths l_1 and l_2, and bounds M_1 and M_2, where $d \in \mathbb{N}$ and $M_i, l_i \in \mathbb{N}_0$ and $\{0, d_i\} \subset \mathcal{D}_i \subset [0, d_i]$, for $i \in [1, 2]$. If $l_1, l_2 \geqslant 1$, then $L_1 + L_2$ is an* AAMP *with difference d, period $\mathcal{D}_1 * \mathcal{D}_2 = (\mathcal{D}_1 + \mathcal{D}_2 + d\mathbb{Z}) \cap [0, d]$, length $l \geqslant l_1 + l_2 - 2$ and bound $M = M_1 + M_2 + d - 1$.*

Proof. Let

$$L_i = y_i + \bigl(L'_i \cup L^*_i \cup L''_i\bigr) \subset y_i + \mathcal{D}_i + d\mathbb{Z},$$

for $i \in [1, 2]$, with all conventions as in (2.1). Without restriction we may assume $y_1 = y_2 = 0$. Then

$$L_1 + L_2 \subset \mathcal{D}_1 * \mathcal{D}_2 + d\mathbb{Z} \quad \text{and} \quad \mathcal{D}_i \subset L_i^*, \quad \text{for all } i \in [1, 2]. \tag{3.2}$$

Let $l_1 d + t_1 = \max L_1^*$ and $l_2 d + t_2 = \max L_2^*$. Let β be the smallest element of $(\mathcal{D}_1 + \mathcal{D}_2 + d\mathbb{Z}) \cap [d, \infty)$ not contained in $L_1^* + L_2^*$, let α be the largest integer less than β contained in $(\mathcal{D}_1 + \mathcal{D}_2 + d\mathbb{Z}) \cap (L_1^* + L_2^*)$, and let $\alpha' = \max(\mathcal{D}_1 \setminus \{d\} \cup \mathcal{D}_2 \setminus \{d\})$. Note $0 \leqslant t_i \leqslant d-1$ with $t_i \in (\mathcal{D}_1 * \mathcal{D}_2) \setminus \{d\}$ by the definition of the length of an AMP, for $i \in [1, 2]$. Let

$$L_1 + L_2 = \alpha' + \left(L_0' \cup L_0^* \cup L_0''\right), \tag{3.3}$$

where $\alpha' + L_0^* = (L_1 + L_2) \cap [\alpha', \alpha]$, $\alpha' + L_0' = (L_1 + L_2) \cap (-\infty, \alpha')$ and $\alpha' + L_0'' = (L_1 + L_2) \cap (\alpha, \infty)$.

Since $0 \in \mathcal{D}_i \subset L_i^*$ for $i \in [1, 2]$, we have $\alpha' \in \mathcal{D}_1 + \mathcal{D}_2 \subset L_1^* + L_2^*$. On the other hand, if $x \in (\mathcal{D}_1 * \mathcal{D}_2) \setminus (\mathcal{D}_1 + \mathcal{D}_2)$, then $x + d = a + b$ for some $a \in \mathcal{D}_1$ and $b \in \mathcal{D}_2$. Consequently, since $0, \alpha' \in \mathcal{D}_1 + \mathcal{D}_2$, it follows from the definition of α' that $x < \alpha'$ (since $\min\{a, b\} \leqslant \alpha'$ and $\max\{a, b\} \leqslant d$). Thus, by definition of α, we see L_0^* is an AMP with difference d and period $\mathcal{D}_1 * \mathcal{D}_2$. Also,

$$\min\left(\alpha' + L_0^*\right) - \min\left(\alpha' + L_0'\right) \leqslant \alpha' - (\min L_1 + \min L_2) \leqslant \alpha' + M_1 + M_2 \leqslant d - 1 + M_1 + M_2. \tag{3.4}$$

It remains to show that α is sufficiently large.

Observe that $(L_1^* + L_2^*) \cap [0, 2d]$ contains the set $\mathcal{D}_1 + \mathcal{D}_2$ (since $\mathcal{D}_i \subset L_i^*$ for $i \in [1, 2]$). By definition, L_i^* is a union of $|\mathcal{D}_i| - 1$ arithmetic progressions with difference d, one for each element of $\mathcal{D}_i \setminus \{d\}$, for $i \in [1, 2]$. Let us denote them by P_{d_i} and Q_{d_j}, where $d_i \in \mathcal{D}_1 \setminus \{d\}$ is the first term in P_{d_i} and $d_j \in \mathcal{D}_2 \setminus \{d\}$ is the first term in Q_{d_j}. From the definitions of t_i and length of an AMP, we have

$$|P_{d_i}| = \begin{cases} l_1 + 1 & \text{for } d_i \leqslant t_1, \\ l_1 & \text{for } d_i > t_1, \end{cases} \qquad |Q_{d_j}| = \begin{cases} l_2 + 1 & \text{for } d_j \leqslant t_2, \\ l_2 & \text{for } d_j > t_1. \end{cases} \tag{3.5}$$

Note that

$$\max(P_{d_i} + Q_{d_j}) = d_i + d_j + \left(|P_{d_i}| + |Q_{d_j}| - 2\right)d \tag{3.6}$$

for all P_{d_i} and Q_{d_j}, and that

$$\beta = d + \max(P_{t_1'} + Q_{t_2'}) = t_1' + t_2' + \left(|P_{t_1'}| + |Q_{t_2'}| - 1\right)d \tag{3.7}$$

for some $P_{t_1'}$ and $Q_{t_2'}$ (since $l_1, l_2 \geqslant 1$). Thus, since $|P_{d_i}| + |Q_{d_j}| \geqslant l_1 + l_2$ for all d_i and d_j (in view of (3.5)), and since $|P_0| + |Q_0| = l_1 + l_2 + 2$, so that $(l_1 + l_2 - 1)d \in P_0 + Q_0$ (in view of (3.6)), it follows that $\beta > (l_1 + l_2 - 1)d$ and $\alpha \geqslant (l_1 + l_2 - 1)d$. Thus the length l of L_0^* is at least $l_1 + l_2 - 2$, as claimed. We consider three cases.

Case 1. Suppose $\beta = d + \max(P_{t_1'} + Q_{t_2'})$ with $t_1' > t_1$ and $t_2' > t_2$.

Then $\beta = (l_1 + l_2 - 1)d + t_1' + t_2'$ (in view of (3.5) and (3.7)). In view of (3.5), we have $|P_{t_1}| + |Q_{t_2'}| \geqslant l_1 + l_2 + 1$ (recall $t_1 \in (\mathcal{D}_1 * \mathcal{D}_2) \setminus \{d\}$). Thus (3.6) implies $(l_1 + l_2 - 1)d + t_1 + t_2' \in P_{t_1} + Q_{t_2'}$. Furthermore, since $t_1' > t_1$ and $t_2' > t_2$, we have

$$(l_1 + l_2 - 1)d + t_1' + t_2' > (l_1 + l_2 - 1)d + t_1 + t_2' \geqslant (l_1 + l_2 - 1)d + t_1 + t_2 + 1.$$

As a result, $\alpha \geqslant (l_1 + l_2 - 1)d + t_1 + t_2 + 1$, and hence

$$\begin{aligned}\max\bigl(\alpha' + L_0''\bigr) - \max\bigl(\alpha' + L_0^*\bigr) &= \max L_1 + \max L_2 - \alpha \\ &\leqslant M_1 + M_2 + \max L_1^* + \max L_2^* - \alpha \\ &= M_1 + M_2 + (l_1 + l_2)d + t_1 + t_2 - \alpha \\ &\leqslant M_1 + M_2 + d - 1.\end{aligned}$$

Thus, in view also of (3.4), we see that $L_1 + L_2$ has bound $M = M_1 + M_2 + d - 1$, completing the case.

Case 2. Suppose $\beta = d + \max(P_{t_1'} + Q_{t_2'})$ with $t_1' \leqslant t_1$ and $t_2' \leqslant t_2$.

Then $\beta = (l_1 + l_2 + 1)d + t_1' + t_2'$ (in view of (3.5) and (3.7)), and so $\alpha \geqslant \max(P_{t_1} + Q_{t_2'}) = (l_1 + l_2)d + t_1 + t_2'$ (since $-d + t_1 - t_1' < 0$). Consequently,

$$\begin{aligned}\max\bigl(\alpha' + L_0''\bigr) - \max\bigl(\alpha' + L_0^*\bigr) &= \max L_1 + \max L_2 - \alpha \\ &\leqslant M_1 + M_2 + \max L_1^* + \max L_2^* - \alpha \\ &= M_1 + M_2 + (l_1 + l_2)d + t_1 + t_2 - \alpha \\ &\leqslant M_1 + M_2 + t_2 - t_2' \\ &\leqslant M_1 + M_2 + d - 1.\end{aligned}$$

Thus, in view also of (3.4), we see that $L_1 + L_2$ has bound $M = M_1 + M_2 + d - 1$, completing the case.

Case 3. Suppose $\beta = d + \max(P_{t_1'} + Q_{t_2'})$. Without loss of generality, $t_1' > t_1$ and $t_2' \leqslant t_2$.

Then $\beta = (l_1 + l_2)d + t_1' + t_2'$ (in view of (3.5) and (3.7)), and so $\alpha \geqslant \max(P_{t_1} + Q_{t_2'}) = (l_1 + l_2)d + t_1 + t_2'$. Consequently,

$$\begin{aligned}\max\bigl(\alpha' + L_0''\bigr) - \max\bigl(\alpha' + L_0^*\bigr) &= \max L_1 + \max L_2 - \alpha \\ &\leqslant M_1 + M_2 + \max L_1^* + \max L_2^* - \alpha \\ &= M_1 + M_2 + (l_1 + l_2)d + t_1 + t_2 - \alpha \\ &\leqslant M_1 + M_2 + t_2 - t_2' \\ &\leqslant M_1 + M_2 + d - 1.\end{aligned}$$

Thus, in view also of (3.4), we see that $L_1 + L_2$ has bound $M = M_1 + M_2 + d - 1$, completing the case and the proof. $\square$

Lemma 3.10. *Let $A \subset \mathbb{Z}$ be a finite, nonempty subset, and let L_1 and L_2 be* AAMPs *with respective differences d_1 and d_2, bound $M \in \mathbb{N}_0$, and such that $L_1 + L_2$ contains the pattern A. Then there exist subsets $\widetilde{L_1} \subset L_1$, $\widetilde{L_2} \subset L_2$ such that $\widetilde{L_1} + \widetilde{L_2}$ contains the pattern A and*

$$\mathsf{diam}\,(\widetilde{L_i}) \leqslant \mathsf{diam}\,(A) + 3 \cdot \operatorname{lcm}(d_1, d_2) - 2 + 2M \quad \textit{for all } i \in [1, 2].$$

Proof. By definition, every AAMP with difference d, period $\mathcal{D}$ and bound M is an AAMP with difference kd, period $(\mathcal{D} + d\mathbb{Z}) \cap [0, kd]$ and bound M, for every $k \in \mathbb{N}$. Thus L_1 and L_2 are AAMPs with difference $d = \mathrm{lcm}(d_1, d_2)$, respective periods $\mathcal{D}_1$ and $\mathcal{D}_2$, lengths l_1 and l_2, and with bound M. Therefore, for $i \in [1, 2]$,

$$L_i = y_i + \left(L_i' \cup L_i^* \cup L_i''\right) \subset y_i + \mathcal{D}_i + d\mathbb{Z},$$

with all conventions as in (2.1), and recall that $l_i \in \mathbb{N}_0$ is maximal such that $l_i d \in L_i^*$. We set $\mathcal{D}_1 * \mathcal{D}_2 = (\mathcal{D}_1 + \mathcal{D}_2 + d\mathbb{Z}) \cap [0, d]$. Without restriction, we may assume $y_1 = y_2 = 0$.

Suppose $\min\{l_1, l_2\} = 0$, say $l_2 = 0$. Then $\mathrm{diam}\,(L_2) \leqslant 2M + d - 1$. Since $L_1 + L_2$ contains the pattern A, it follows that $x + A \subset L_1 + L_2$ for some $x \in \mathbb{Z}$. Consequently,

$$x + A \subset L_2 + \left(L_1 \cap [x + \min A - \max L_2, x + \max A - \min L_2]\right),$$

and so, taking $\widetilde{L_2} = L_2$ and $\widetilde{L_1} = L_1 \cap [x + \min A - \max L_2, x + \max A - \min L_2]$, we see that $\mathrm{diam}\,(\widetilde{L_2}) = \mathrm{diam}\,(L_2) \leqslant 2M + d - 1$ and $\mathrm{diam}\,(\widetilde{L_1}) \leqslant \mathrm{diam}\,(L_2) + \mathrm{diam}\,(A) \leqslant 2M + d - 1 + \mathrm{diam}\,(A)$, and the proof is complete. So we may assume $\min\{l_1, l_2\} \geqslant 1$.

Since $L_1 + L_2 \subset \mathcal{D}_1 * \mathcal{D}_2 + d\mathbb{Z}$ contains the pattern A, it follows that $\mathcal{D}_1 * \mathcal{D}_2 + d\mathbb{Z}$ contains the pattern A. Consequently, any AMP with difference d, period $\mathcal{D}_1 * \mathcal{D}_2$ and length $l \geqslant \lceil \frac{\mathrm{diam}\,(A)-1}{d} \rceil + 1$ must also contain the pattern A.

Let $\widetilde{L_1} \subset L_1^*$ and $\widetilde{L_2} \subset L_2^*$ be AMPs with difference d and with respective periods $\mathcal{D}_1$ and $\mathcal{D}_2$ and lengths $\widetilde{l_1} \in [1, l_1]$ and $\widetilde{l_2} \in [1, l_2]$. By Lemma 3.9 (with $M_1 = M_2 = 0$), it follows that $\widetilde{L_1} + \widetilde{L_2}$ is an AAMP with difference d, period $\mathcal{D}_1 * \mathcal{D}_2$ and length $l \geqslant \widetilde{l_1} + \widetilde{l_2} - 2$, and consequently, $\widetilde{L_1} + \widetilde{L_2}$ contains an AMP with difference d, period $\mathcal{D}_1 * \mathcal{D}_2$ and length $l \geqslant \widetilde{l_1} + \widetilde{l_2} - 2$.

As a result, if $l_1 + l_2 \geqslant \lceil \frac{\mathrm{diam}\,(A)-1}{d} \rceil + 3$, then taking $\widetilde{L_1}$ and $\widetilde{L_2}$ such that $\widetilde{l_1} + \widetilde{l_2} = \lceil \frac{\mathrm{diam}\,(A)-1}{d} \rceil + 3$ and $\min \widetilde{L_i} \equiv \max \widetilde{L_i} \mod d$, for $i \in [1, 2]$, it follows that $\widetilde{L_1} + \widetilde{L_2}$ contains the pattern A, and for $i \in [1, 2]$,

$$\mathrm{diam}\,(\widetilde{L_i}) \leqslant \left(\left\lceil \frac{\mathrm{diam}\,(A) - 1}{d} \right\rceil + 2\right) d \leqslant \mathrm{diam}\,(A) + 3d - 2,$$

and the proof is complete.

On the other hand, if $l_1 + l_2 \leqslant \lceil \frac{\mathrm{diam}\,(A)-1}{d} \rceil + 2$, then for $i \in [1, 2]$,

$$\begin{aligned} \mathrm{diam}\,(L_i) &\leqslant \left(\left\lceil \frac{\mathrm{diam}\,(A) - 1}{d} \right\rceil + 2 - l_{3-i}\right) d + (d - 1) + 2M \\ &\leqslant \left(\left\lceil \frac{\mathrm{diam}\,(A) - 1}{d} \right\rceil + 2\right) d + 2M - 1 \\ &\leqslant \mathrm{diam}\,(A) + 3d - 3 + 2M, \end{aligned}$$

and thus the result follows taking $\widetilde{L_1} = L_1$ and $\widetilde{L_2} = L_2$, which completes the proof. $\square$

Let all notations be as in the following Theorem 3.11 and suppose that condition (b) holds. Then Theorem 3.11 and Proposition 3.4 imply that there exists a bound $M \in \mathbb{N}_0$ such that every $L \in \mathcal{L}(H)$ is an AAMP with some difference $d \in \Delta(H)$ and bound M. Thus, since

$$\mathcal{L}(H) = \left\{L_1 + \cdots + L_n \;\middle|\; L_1 \in \mathcal{L}(H_1), \ldots, L_n \in \mathcal{L}(H_n)\right\},$$

Theorem 3.11 is an arithmetic analogue of an addition theorem due to G.A. Freiman and the first author (see [9] and [14, Theorem 4.2.16]).

Theorem 3.11. *Let $H = H_1 \times \cdots \times H_n$, where $n \in \mathbb{N}$ and $H_1, \ldots, H_n$ are monoids. Then the following statements are equivalent*:

(a) *H is a* BF*-monoid with finite set of distances $\Delta(H)$ and all pattern ideals of H are tamely generated.*
(b) *For every $i \in [1, n]$, H_i is a* BF*-monoid with finite set of distances $\Delta(H_i)$ and all pattern ideals of H_i are tamely generated.*

Proof. Without restriction we may suppose that H is reduced, and it suffices to consider the case $n = 2$. Obviously, H is a BF-monoid if and only if both H_1 and H_2 are BF-monoids, and by [14, Proposition 1.4.5] $\Delta(H)$ is finite if and only if $\Delta(H_1)$ and $\Delta(H_2)$ are finite.

Suppose that H_1, H_2 and H are BF-monoids with finite sets of distances. If all pattern ideals of H are tamely generated, then the same is true for all divisor-closed submonoids, and hence all pattern ideals of H_1 and of H_2 are tamely generated.

Now suppose that all pattern ideals of H_1 and of H_2 are tamely generated. We have to show that all pattern ideals of H are tamely generated. Let $A \subset \mathbb{Z}$ be a finite, nonempty subset. We start with the following assertion.

(A) There exist some $s \in \mathbb{N}$ and, for every $i \in [1, s]$, finite nonempty subsets $A_{i,1}, A_{i,2} \subset \mathbb{Z}$, such that $A_{i,1} + A_{i,2}$ contains the pattern A and

$$\Phi_H(A) = \bigcup_{i=1}^{s} \Phi_{H_1}(A_{i,1})\Phi_{H_2}(A_{i,2}).$$

Since the ideals $\Phi_{H_j}(A_{i,j})$ are tamely generated for all $j \in [1, 2]$ and $i \in [1, s]$, the ideal $\Phi_H(A)$ is tamely generated by [14, Proposition 4.3.3].

Proof of (A). Let $A_1, A_2 \subset \mathbb{Z}$ be finite nonempty subsets such that $A_1 + A_2$ contains the pattern A. If $a_i \in \Phi_{H_i}(A_i)$ for $i \in [1, 2]$, then $\mathsf{L}_H(a_1a_2) = \mathsf{L}_{H_1}(a_1) + \mathsf{L}_{H_2}(a_2)$ contains the pattern $A_1 + A_2$. Hence $\mathsf{L}_H(a_1a_2)$ contains the pattern A and $a_1a_2 \in \Phi_H(A)$.

Conversely, let $a = a_1a_2 \in \Phi_H(A)$ with $a_i \in H_i$. Then $\mathsf{L}(a_1)$ and $\mathsf{L}(a_2)$ are finite subsets of $\mathbb{Z}$ whose sumset contains the pattern A.

Now let $a = a_1a_2 \in \Phi(A)$ with $a_i \in H_i$. By Proposition 3.4, there is a constant $M \in \mathbb{N}$ such that every $L \in \mathcal{L}(H_1) \cup \mathcal{L}(H_2)$ is an AAMP with difference $d \in \Delta(H_1) \cup \Delta(H_2)$ and bound M. By Lemma 3.10, there exists a constant N (depending on M, A and $\Delta(H_1) \cup \Delta(H_2)$) and $A_i' \subset L(a_i)$ with $\operatorname{diam}(A_i') \leqslant N$, for every $i \in [1, 2]$, such that $A_1' + A_2'$ contains the pattern A. For $i \in [1, 2]$, we have $a_i \in \Phi_{H_i}(\mathsf{L}(a_i)) \subset \Phi_{H_i}(A_i')$, whence $a = a_1a_2 \in \Phi_{H_1}(A_1')\Phi_{H_2}(A_2')$.

Since, for $i \in [1, 2]$,

$$\Phi_{H_i}(A_i') = \Phi_{H_i}\bigl(-\min(A_i') + A_i'\bigr),$$

the number of required sets is finite (bounded by 2^{N+1}), and thus $\Phi_H(A)$ allows the required representation. $\square$

Next we need the tame degree in the fibres. Its significance for the investigation of pattern ideals will become clear in Proposition 3.14 and then in Theorem 4.4.1.

Definition 3.12. Let $\theta : H \to B$ be a transfer homomorphism of atomic monoids and $\bar{\theta} : \mathsf{Z}(H) \to \mathsf{Z}(B)$ its extension to the factorization monoids.

For $a \in H$ and $x \in \mathsf{Z}(H)$, we denote by $\mathsf{t}(a, x, \theta)$ the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

if $\mathsf{Z}(a) \cap x\mathsf{Z}(H) \neq \emptyset$, $z \in \mathsf{Z}(a)$ and $\bar{\theta}(z) \in \bar{\theta}(x)\mathsf{Z}(B)$, then there exists some $z' \in \mathsf{Z}(a) \cap x\mathsf{Z}(H)$ such that $\bar{\theta}(z') = \bar{\theta}(z)$ and $\mathsf{d}(z, z') \leqslant N$.

We define

$$\mathsf{t}(H,x,\theta)=\sup\bigl\{\mathsf{t}(a,x,\theta)\bigm| a\in H\bigr\}\in\mathbb{N}_0\cup\{\infty\}.$$

Lemma 3.13. *Let $\theta: H\to B$ be a transfer homomorphism, $\bar\theta:\mathsf{Z}(H)\to\mathsf{Z}(B)$ its extension to the factorization monoids, $a\in H$ and $x,x_1,x_2\in\mathsf{Z}(H)$.*

1. *We have either* $\mathsf{t}(a,x)=0$ *or*

$$\mathsf{t}\bigl(\theta(a),\bar\theta(x)\bigr)\leqslant\mathsf{t}(a,x)\leqslant\mathsf{t}\bigl(\theta(a),\bar\theta(x)\bigr)+\mathsf{t}(a,x,\theta).$$

In particular, if $u\in\mathcal{A}(H_{\mathrm{red}})$, then

$$\mathsf{t}\bigl(B,\bar\theta(u)\bigr)\leqslant\mathsf{t}(H,u)\leqslant\mathsf{t}\bigl(B,\bar\theta(u)\bigr)+\mathsf{t}(H,u,\theta).$$

2. $\mathsf{t}(a,x_1x_2,\theta)\leqslant\mathsf{t}(a,x_1,\theta)+\mathsf{t}(H,x_2,\theta)$.
3. $\mathsf{t}(H,x,\theta)\leqslant|x|\sup\{\mathsf{t}(H,u,\theta)\mid u\in\mathcal{A}(H_{\mathrm{red}})\}$.

Proof. Without restriction we may suppose that H is reduced.

1. See [14, Theorem 3.2.5].

2. Let $\mathsf{Z}(a)\cap x_1x_2\mathsf{Z}(H)\neq\emptyset$, $z\in\mathsf{Z}(a)$ and $\bar\theta(z)\in\bar\theta(x_1)\bar\theta(x_2)\mathsf{Z}(B)$. Then there exists a factorization $z_1=x_1y\in\mathsf{Z}(a)\cap x_1\mathsf{Z}(H)$ with $y\in\mathsf{Z}(H)$, $\bar\theta(z_1)=\bar\theta(z)$ and $\mathsf{d}(z,z_1)\leqslant\mathsf{t}(a,x_1,\theta)$. Thus a has the form $a=a_1b$, where $a_1,b\in H$ with $x_1\in\mathsf{Z}(a_1)$, $y\in\mathsf{Z}(b)$, and clearly we get $\bar\theta(z)=\bar\theta(x_1)\bar\theta(y)$. Hence $\mathsf{Z}(b)\cap x_2\mathsf{Z}(H)\neq\emptyset$ and $\bar\theta(y)\in\bar\theta(x_2)\mathsf{Z}(B)$. Thus there is a $y'\in\mathsf{Z}(b)\cap x_2\mathsf{Z}(H)$ with $\bar\theta(y')=\bar\theta(y)$ and $\mathsf{d}(y,y')\leqslant\mathsf{t}(b,x_2,\theta)\leqslant\mathsf{t}(H,x_2,\theta)$. Therefore we obtain $z_2=x_1y'\in\mathsf{Z}(a)$, $\bar\theta(z_2)=\bar\theta(x_1)\bar\theta(y)=\bar\theta(z)$ and

$$\mathsf{d}(z,z_2)\leqslant\mathsf{d}(z,z_1)+\mathsf{d}(z_1,z_2)\leqslant\mathsf{t}(a,x_1,\theta)+\mathsf{d}(x_1y,x_1y')\leqslant\mathsf{t}(a,x_1,\theta)+\mathsf{t}(H,x_2,\theta).$$

3. If $a\in H$ and $x=u_1\cdot\ldots\cdot u_l$ with $l\in\mathbb{N}$ and $u_1,\ldots,u_l\in\mathcal{A}(H)$, then 2 implies that

$$\mathsf{t}(a,x,\theta)\leqslant\sum_{i=1}^{l}\mathsf{t}(H,u_i,\theta)\leqslant|x|\sup\bigl\{\mathsf{t}(H,u,\theta)\bigm| u\in\mathcal{A}(H)\bigr\},$$

and hence the assertion follows. □

Proposition 3.14. *Let $\theta: H\to B$ be a transfer homomorphism of atomic monoids.*

1. *If $\mathfrak{a}\subset H$ and $E\subset\mathfrak{a}$ is a tame generating set with bound $N\in\mathbb{N}$, then $\theta(E)$ is a tame generating set of $\theta(\mathfrak{a})$ with bound N.*
2. *Suppose that $\sup\{\mathsf{t}(H,u,\theta)\mid u\in\mathcal{A}(H_{\mathrm{red}})\}<\infty$. If $\mathfrak{a}\subset B$ and $E\subset\mathfrak{a}$ is a tame generating set with bound $N\in\mathbb{N}$, then $\theta^{-1}(E)$ is a tame generating set of $\theta^{-1}(\mathfrak{a})$ with bound*

$$N\bigl(1+\sup\bigl\{\mathsf{t}(H,u,\theta)\bigm| u\in\mathcal{A}(H_{\mathrm{red}})\bigr\}\bigr).$$

3. *Let $A\subset\mathbb{Z}$ be a finite, nonempty subset. Then $\theta(\Phi_H(A))=\Phi_B(A)$, $\theta^{-1}(\Phi_B(A))=\Phi_H(A)$, and if $\sup\{\mathsf{t}(H,u,\theta)\mid u\in\mathcal{A}(H_{\mathrm{red}})\}<\infty$, then $\Phi_H(A)\subset H$ is tamely generated if and only if $\Phi_B(A)\subset B$ is tamely generated.*
4. $\delta(H)=\delta(B)$.

Proof. Without restriction we may suppose that H is reduced, and we denote by $\bar\theta : \mathsf Z(H) \to \mathsf Z(B)$ the extension of θ to the factorization monoids.

1. Let $E \subset \mathfrak a \subset H$ and $N \in \mathbb N$ be such that for every $a \in \mathfrak a$ there exists some $b \in E$ such that

$$b \mid a, \qquad \sup \mathsf L_H(b) \leqslant N \quad \text{and} \quad \mathsf t\big(a, \mathsf Z_H(b)\big) \leqslant N.$$

We pick an element $\alpha \in \theta(\mathfrak a)$. Thus there exists an $a \in H$ such that $\theta(a) = \alpha$ and an element $b \in E$ with the above properties. Then $\beta = \theta(b) \in \theta(E)$, $\theta(b) \mid \theta(a)$, $\sup \mathsf L_B(\beta) = \sup \mathsf L_H(b) \leqslant N$ and

$$\mathsf t\big(\alpha, \mathsf Z_B(\beta)\big) = \mathsf t\big(\theta(a), \bar\theta\big(\mathsf Z_H(b)\big)\big) \leqslant \mathsf t\big(a, \mathsf Z_H(b)\big) \leqslant N.$$

2. Let $E \subset \mathfrak a \subset B$ and $N \in \mathbb N$ such that for every $\alpha \in \mathfrak a$ there exists some $\beta \in E$ such that

$$\beta \mid \alpha, \qquad \sup \mathsf L_B(\beta) \leqslant N \quad \text{and} \quad \mathsf t\big(\alpha, \mathsf Z_B(\beta)\big) \leqslant N.$$

We pick an element $a \in \theta^{-1}(\mathfrak a)$. Then there exist an element $\beta \in E \subset \mathfrak a$ and an element $\gamma \in B$ such that $\theta(a) = \beta\gamma$, $\sup \mathsf L_B(\beta) \leqslant N$ and $\mathsf t(\theta(a), \mathsf Z_B(\beta)) \leqslant N$. Since θ is a transfer homomorphism, there exist $b, c \in H$ such that $a = bc$, $\theta(b) = \beta$ and $\theta(c) = \gamma$. Then $b \in \theta^{-1}(\beta) \subset \theta^{-1}(E)$. If $x \in \mathsf Z_H(b)$, then $\bar\theta(x) \in \mathsf Z_B(\beta)$, $|x| \leqslant \sup \mathsf L_H(b) \leqslant \sup \mathsf L_B(\beta) \leqslant N$ and, by Lemma 3.13,

$$\begin{aligned}
\mathsf t(a, x) &\leqslant \mathsf t\big(\theta(a), \bar\theta(x)\big) + \mathsf t(a, x, \theta)\\
&\leqslant \mathsf t\big(\theta(a), \mathsf Z_B(\beta)\big) + \mathsf t(H, x, \theta)\\
&\leqslant \mathsf t\big(\theta(a), \mathsf Z_B(\beta)\big) + N \sup\big\{\mathsf t(H, u, \theta) \bigm| u \in \mathcal A(H_{\mathrm{red}})\big\}\\
&\leqslant N\big(1 + \sup\big\{\mathsf t(H, u, \theta) \bigm| u \in \mathcal A(H_{\mathrm{red}})\big\}\big).
\end{aligned}$$

3. Since for every $a \in H$ we have $\mathsf L_H(a) = \mathsf L_B(\theta(a))$ (by Lemma 3.2.2), it follows that

$$\theta\big(\Phi_H(A)\big) = \Phi_B(A) \quad \text{and} \quad \theta^{-1}\big(\Phi_B(A)\big) = \Phi_H(A).$$

Thus the remaining assertions follow from 1 and 2.

4. (a) We assert that $\delta(H) \leqslant \delta(B)$, and for this we have to show that $\delta(z) \leqslant \delta(B)$ for all $z \in \mathsf Z(H)$. Let $z \in \mathsf Z(H)$, $a \in H$ with $z \in \mathsf Z(a)$, and $k \in \mathbb N$ such that k and $|z|$ are adjacent lengths of a. Then k and $|\bar\theta(z)|$ are adjacent lengths of $\theta(a)$. Thus there is a $\bar y \in \mathsf Z(\theta(a))$ with $|\bar y| = k$ and $\mathsf d(\bar y, \bar\theta(z)) \leqslant \delta(B)$. By Lemma 3.2.2(c), there is a $y \in \mathsf Z(a)$ such that $\bar\theta(y) = \bar y$ and $\mathsf d(y, z) = \mathsf d(\bar y, \bar\theta(z))$. Since $|y| = |\bar y| = k$, it follows that $\delta(z) \leqslant \mathsf d(y, z) \leqslant \delta(B)$.

4. (b) We assert that $\delta(B) \leqslant \delta(H)$, and for this we have to show that $\delta(Z) \leqslant \delta(H)$ for all $Z \in \mathsf Z(B)$. Let $Z \in \mathsf Z(B)$, $\alpha \in B$ with $Z \in \mathsf Z(\alpha)$, and $k \in \mathbb N$ such that k and $|Z|$ are adjacent lengths of α. Then there are $a \in H$ and $z \in \mathsf Z(a)$ such that $\theta(a) = \alpha$ and $\bar\theta(z) = Z$. Since k and $|z| = |Z|$ are adjacent lengths of a, there is a $y \in \mathsf Z(a)$ such that $|y| = k$ and $\mathsf d(z, y) \leqslant \delta(H)$. This implies that

$$\mathsf d\big(Z, \bar\theta(y)\big) = \mathsf d\big(\bar\theta(z), \bar\theta(y)\big) \leqslant \mathsf d(z, y) \leqslant \delta(H). \qquad \square$$

4. Krull monoids

We gather some basic terminology for Krull monoids. Let D be a monoid and $H \subset D$ a submonoid with $\mathsf q(H) \subset \mathsf q(D)$. Then $H \subset D$ is called:

- *saturated* if $\mathsf{q}(H) \cap D = H$ (that is, if $a, b \in H$ and a divides b in D, then a divides b in H);
- *cofinal* if for every $a \in D$ there exists some $u \in H$ such that $a \mid u$.

For $a \in \mathsf{q}(D)$, we denote by $[a] = [a]_{D/H} = a\mathsf{q}(H) \in \mathsf{q}(D)/\mathsf{q}(H)$ the class containing a. We call $D/H = \{[a] \mid a \in D\} \subset \mathsf{q}(D)/\mathsf{q}(H)$ the *class group* of D modulo H. Then $H \subset D$ is saturated if and only if $H = \{a \in D \mid [a] = [1]\}$. Furthermore, $H \subset D$ is cofinal if and only D/H is a group (equivalently, $D/H = \mathsf{q}(D)/\mathsf{q}(H)$).

The monoid H is called a *Krull monoid* if H_{red} is a saturated submonoid of a free monoid. If H is a reduced Krull monoid such that $H \subset F = \mathcal{F}(P)$ is a saturated submonoid, then

$$G_P = \big\{[p] = p\mathsf{q}(H) \bigm| p \in P\big\} \subset F/H$$

is called the *set of classes containing prime divisors*.

Many arithmetical problems in general Krull monoids can be reduced to the Krull monoid of zero-sum sequences over subsets of abelian groups (see Proposition 4.1). We shall use this relationship. Let G be an additive abelian group, $G_0 \subset G$ a subset, and $\mathcal{F}(G_0)$ the free monoid with basis G_0. According to the tradition of combinatorial number theory, the elements of $\mathcal{F}(G_0)$ are called *sequences over* G_0. If $S \in \mathcal{F}(G_0)$, then

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S)$ is the g-adic value of S (also called the *multiplicity of g in S*), and $\mathsf{v}_g(S) = 0$ for all $g \in G_0 \setminus \{g_1, \ldots, g_l\}$. Then $|S| = l = \sum_{g \in G_0} \mathsf{v}_g(S)$ is the *length of* S, and we set $-S = (-g_1) \cdot \ldots \cdot (-g_l)$. We call $\mathrm{supp}(S) = \{g_1, \ldots, g_l\}$ the *support* of S, $\sigma(S) = g_1 + \cdots + g_l$ the *sum* of S, and

$$\Sigma(S) = \left\{\sum_{i \in I} g_i \,\middle|\, \emptyset \neq I \subset [1, l]\right\} \subset G \quad \text{the set of subsums of } S.$$

The monoid

$$\mathcal{B}(G_0) = \big\{S \in \mathcal{F}(G_0) \bigm| \sigma(S) = 0\big\}$$

is called the *block monoid* or the *monoid of zero-sum sequences* over G_0, and we have $\mathcal{B}(G_0) = \mathcal{B}(G) \cap \mathcal{F}(G_0)$. Its elements are called *zero-sum sequences* over G_0, and its atoms are the minimal zero-sum sequences (that is, zero-sum sequences without a proper zero-sum subsequence).

For every arithmetical invariant $*(H)$ defined for a monoid H, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ and $\Delta(G_0) = \Delta(\mathcal{B}(G_0))$. We define the *Davenport constant* of G_0 by

$$\mathsf{D}(G_0) = \sup\big\{|U| \bigm| U \in \mathcal{A}(G_0)\big\} \in \mathbb{N}_0 \cup \{\infty\},$$

which is a central invariant in zero-sum theory (see [10], and also [12] for its relevance in factorization theory).

Clearly, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated, and hence $\mathcal{B}(G_0)$ is a Krull monoid. Furthermore, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is cofinal if and only if for every $g \in G_0$ there exists some $B \in \mathcal{B}(G_0)$ with $\mathsf{v}_g(B) > 0$. Suppose that this holds. Then the submonoid $[G_0] \subset G$ is even a subgroup of G, and there is a group isomorphism

$$\Psi : \mathcal{F}(G_0)/\mathcal{B}(G_0) \to [G_0] \subset G, \quad \text{given by } \Psi\big([S]_{\mathcal{F}(G_0)/\mathcal{B}(G_0)}\big) = \sigma(S).$$

In particular, $\Psi^{-1}(G_0) \subset \mathcal{F}(G_0)/\mathcal{B}(G_0)$ is the set of classes containing prime divisors (see [14, Proposition 2.5.6]). Therefore, if $\mathsf{D}(G_0) < \infty$, then $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ satisfies all assumptions of Propositions 4.1, 4.3 and of Theorem 4.4.

Let H be a reduced Krull monoid such that $H \subset F = \mathcal{F}(P)$ is a saturated submonoid. The homomorphism $\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$, defined by $\boldsymbol{\beta}(p) = [p]$ for all $p \in P$, is called the *block homomorphism* of $H \subset F$. Now we formulate some of the main properties of Krull monoids needed in the sequel. Recall that for every $k \in \mathbb{N}$,

$$\mathcal{V}_k(H) = \bigcup_{k \in \mathsf{L}(a),\, a \in H} \mathsf{L}(a)$$

denotes the union of all sets of lengths containing k.

Proposition 4.1. *Let H be a reduced Krull monoid, $F = \mathcal{F}(P)$ a free monoid such that $H \subset F$ is a saturated and cofinal submonoid, and $G = F/H$. Let $G_P \subset G$ be the set of all classes containing prime divisors and suppose that $\mathsf{D}(G_P) < \infty$.*

1. *The block homomorphism $\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$ is a transfer homomorphism.*
2. $\max \Delta(H) \leqslant \max\{0, \mathsf{D}(G_P) - 2\}$.
3. $\mathsf{t}(H) \leqslant 1 + \frac{\mathsf{D}(G_P)(\mathsf{D}(G_P)-1)}{2}$.
4. *There is a bound $M \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $\mathcal{V}_k(H)$ is an* AAP *with difference* $\min \Delta(H)$ *and bound M.*

Proof. For 1–3, see [14, Theorems 3.4.10 and 1.6.3], and 4 is proved in [11, Corollary 3.6 and Theorem 4.2]. $\square$

In the following proposition, we give explicit upper bounds for the tame degree in the fibres. Let all notations be as in Proposition 4.2. Note that in order to get the result in 4.2.2, an assumption of that type is necessary. Indeed, if every class in G_P contains precisely one prime divisor, then by definition we get $\mathsf{t}(H, u, \boldsymbol{\beta}) = 0$.

Proposition 4.2. *Let H be a reduced Krull monoid, $F = \mathcal{F}(P)$ a free monoid such that $H \subset F$ is a saturated and cofinal submonoid, and $G = F/H$. Let $G_P \subset G$ be the set of all classes containing prime divisors and $\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$ the block homomorphism of $H \subset F$.*

1. $\mathsf{t}(H, u, \boldsymbol{\beta}) \leqslant 1 + |u| \leqslant 1 + \mathsf{D}(G_P)$ *for all $u \in \mathcal{A}(H)$.*
2. *If $G_P = -G_P$ and every class contains at least two distinct prime divisors, then* $\mathsf{t}(H, u, \boldsymbol{\beta}) = 1 + |u|$ *for all $u \in \mathcal{A}(H)$ with $|u| \geqslant 3$, and thus if $\mathsf{D}(G_P) \geqslant 3$, then*

$$\max\{\mathsf{t}(H, u, \boldsymbol{\beta}) \mid u \in \mathcal{A}(H)\} = \mathsf{D}(G_P) + 1.$$

Proof. 1. It suffices to show that, for every $a \in H$ and every $u \in \mathcal{A}(H)$, we have $\mathsf{t}(a, u, \boldsymbol{\beta}) \leqslant 1 + |u|$. Let $u = p_1 \cdot \ldots \cdot p_l \in \mathcal{A}(H)$, where $l = |u|$ and $p_1, \ldots, p_l \in P$, let $a \in uH$, let $z = u_0 u_1 \cdot \ldots \cdot u_m \in \mathsf{Z}(a)$, where $m \in \mathbb{N}$ and $u_0, \ldots, u_m \in \mathcal{A}(H)$, and let $\boldsymbol{\beta}(z) = U_0 U_1 \cdot \ldots \cdot U_m$, where $\boldsymbol{\beta}(u_i) = U_i \in \mathcal{A}(G_P)$ for all $i \in [1, m]$ and $\boldsymbol{\beta}(u_0) = \boldsymbol{\beta}(u) = U_0$. Then $u_0 = p_1' \cdot \ldots \cdot p_l'$, where $p_i' \in P \cap [p_i]$ for all $i \in [1, l]$. After renumbering if necessary, there is some $k \in [1, l]$ such that $u = p_1 \cdot \ldots \cdot p_l$ divides $u_0 u_1 \cdot \ldots \cdot u_k$ (in F and thus also in H). Exchanging p_i and p_i' for all $i \in [1, l]$, we obtain a factorization $z' = u u_1' \cdot \ldots \cdot u_k' u_{k+1} \cdot \ldots \cdot u_m$, where $\boldsymbol{\beta}(u_i') = U_i$ for all $i \in [1, k]$, and clearly $\mathsf{d}(z, z') \leqslant 1 + k \leqslant 1 + l$.

2. Suppose that $G_P = -G_P$ and that every class contains at least two distinct prime divisors. Let $u = p_1 \cdot \ldots \cdot p_l \in \mathcal{A}(H)$ with $|u| = l \geqslant 3$ and $p_1, \ldots, p_l \in P$. It suffices to show that $1 + |u| \leqslant \mathsf{t}(H, u, \boldsymbol{\beta})$.

Let $i \in [1, l]$. We set $g_i = [p_i]$ and choose $\overline{p_i} \in (P \cap g_i) \setminus \{p_i\}$ and $q_i \in (P \cap (-[p_i])) \setminus \{p_i\}$. Then $U = \boldsymbol{\beta}(u) = g_1 \cdot \ldots \cdot g_l \in \mathcal{A}(G_P)$, $\bar{u} = \overline{p_1} \cdot \ldots \cdot \overline{p_l} \in \mathcal{A}(H)$, $v_i = p_i q_i \in \mathcal{A}(H)$, $\overline{v_i} = \overline{p_i} q_i \in \mathcal{A}(H)$, and setting $V_i = (-g_i) g_i$, we clearly get that $\boldsymbol{\beta}(v_i) = \boldsymbol{\beta}(\overline{v_i}) = V_i$.

Let $a \in H$ with $z = v_1 \cdot \ldots \cdot v_l \bar{u} \in \mathsf{Z}(a)$. Then $\overline{\boldsymbol{\beta}}(z) = V_1 \cdot \ldots \cdot V_l U$, and we pick any $z' \in \mathsf{Z}(a) \cap u\mathsf{Z}(H)$ such that $\overline{\boldsymbol{\beta}}(z') = V_1 \cdot \ldots \cdot V_l U$, say $z' = v_1' \cdot \ldots \cdot v_l' u$ with $v_i' \in \mathcal{A}(H)$, such that $\boldsymbol{\beta}(v_i') = V_i$ for all $i \in [1, l]$. We have

$$\begin{aligned} v_1 \cdot \ldots \cdot v_l \bar{u} &= (p_1 q_1) \cdot \ldots \cdot (p_l q_l)(\overline{p_1} \cdot \ldots \cdot \overline{p_l}) \\ &= (\overline{p_1} q_1) \cdot \ldots \cdot (\overline{p_l} q_l)(p_1 \cdot \ldots \cdot p_l) \in \mathcal{F}(P), \end{aligned}$$

and hence, by the choice of q_i and $\overline{p_i}$, no p_i divides $u^{-1}a = (\overline{p_1} q_1) \cdot \ldots \cdot (\overline{p_l} q_l)$. This implies that $\mathsf{d}(v_1 \cdot \ldots \cdot v_l, v_1' \cdot \ldots \cdot v_l') = l$, $\mathsf{d}(z, z') = l + 1$ and therefore $1 + l = 1 + |u| \leqslant \mathsf{t}(a, u, \boldsymbol{\beta}) \leqslant \mathsf{t}(H, u, \boldsymbol{\beta})$. $\square$

Proposition 4.3 is crucial for the proof of Theorem 4.4, which is the main result of this paper. We postpone the proof of Proposition 4.3 to Section 5, and give the short proof of Theorem 4.4, which is based on Proposition 4.3 and on all the machinery developed so far. We remark that the bounds present in Proposition 4.3 and Theorem 4.4 have been chosen in part due to their simplified presentation, and slightly more accurate bounds can be obtained from the proof.

Proposition 4.3. *Let G be an abelian group, $G_0 \subset G$ a subset with $\mathsf{D}(G_0) < \infty$ and $A \subset \mathbb{Z}$ a finite, nonempty subset. If $S \in \Phi(A)$, then there exists an $S' \in \Phi(A)$ with*

$$S' \mid S \quad \text{and} \quad |S'| \leqslant \big(9 \cdot \mathrm{diam}\,(A) + \mathsf{D}(G_0)^4\big)\left(\mathrm{diam}\,(A) \cdot \mathsf{D}(G_0)^2 + \frac{1}{2}\mathsf{D}(G_0)^4\right)^{\frac{(\mathsf{D}(G_0)-1)(\mathsf{D}(G_0)-2)}{2}}. \tag{4.1}$$

In particular, the pattern ideal $\Phi(A) \subset \mathcal{B}(G_0)$ is tamely generated with bound

$$\varphi(A) \leqslant \left(\mathrm{diam}\,(A) \cdot \mathsf{D}(G_0)^2 + \frac{1}{2}\mathsf{D}(G_0)^4\right)^{\frac{\mathsf{D}(G_0)(\mathsf{D}(G_0)-1)}{2}} - \mathsf{D}(G_0) + 1.$$

Theorem 4.4. *Let H be a reduced Krull monoid, $F = \mathcal{F}(P)$ a free monoid such that $H \subset F$ is a saturated and cofinal submonoid, and $G = F/H$. Let $G_P \subset G$ be the set of all classes containing prime divisors and suppose that $\mathsf{D}(G_P) < \infty$.*

1. *All pattern ideals of H are tamely generated.*
2. *There exists a bound $M \in \mathbb{N}$ such that every $L \in \mathcal{L}(H)$ is an* AAMP *with some difference $d \in \Delta(H)$ and bound M.*
   Moreover, the bound M depends only on $\mathsf{D}(G_P)$. If $\mathsf{D}(G_P) \geqslant 3$ and $L \in \mathcal{L}(H)$ is sufficiently large, then L is an AAMP *with some difference $d \in \Delta(H)$ and bound*

$$M_1 = \left(\big(2\mathsf{D}(G_P) - 5\big)\mathsf{D}(G_P)^2 + \frac{1}{2}\mathsf{D}(G_P)^4\right)^{\frac{\mathsf{D}(G_P)(\mathsf{D}(G_P)-1)}{2}}.$$

3. *If G_P is finite, then $\delta(H) < \infty$.*

Proof. 1. By Proposition 4.1, the block homomorphism $\boldsymbol{\beta} : H \to \mathcal{B}(G_P)$ is a transfer homomorphism, and by Proposition 4.3, all pattern ideals of $\mathcal{B}(G_P)$ are tamely generated. Thus Propositions 3.14 and 4.2 imply that pattern ideals of H are tamely generated.

2. This follows from 1 and from Proposition 3.4.

In order to give an explicit bound, note that it suffices to do this for $\mathcal{B}(G_P)$. Suppose that $\mathsf{D}(G_P) \geqslant 3$. Then Proposition 4.1 implies $\max \Delta(G_0) \leqslant \mathsf{D}(G_p) - 2$, and we may apply Proposition 3.4 with $H = \mathcal{B}(G_P)$ and $\theta = 2\mathsf{D}(G_P) - 5 \geqslant 1$. We calculate the bound M_1. Then all $L \in \mathcal{L}(H)$ with $\mathrm{diam}\,(L) > M(\theta)$ are AAMPs with bound M_1. Using Propositions 3.4 and 4.3, we obtain that

$$\begin{aligned} M_1 &= \max\bigl\{\varphi(A) \bigm| A \subset [0,\theta]\bigr\} + \max \Delta(H) \\ &\leqslant \max\bigl\{\varphi(A) \bigm| A \subset \bigl[0, 2\mathsf{D}(G_P)-5\bigr]\bigr\} + \mathsf{D}(G_P) - 2 \\ &\leqslant \left(\bigl(2\mathsf{D}(G_P)-5\bigr)\mathsf{D}(G_P)^2 + \frac{1}{2}\mathsf{D}(G_P)^4\right)^{\frac{\mathsf{D}(G_P)(\mathsf{D}(G_P)-1)}{2}}. \end{aligned}$$

3. Suppose that G_P is finite. Then $\mathcal{B}(G_P)$ is a reduced and finitely generated monoid, and thus $\delta(G_P)$ is finite by [14, Theorems 3.1.4 and 3.4.2]. Thus Propositions 3.14 and 4.1 imply that $\delta(H) < \infty$. $\square$

Remarks 4.5. Let $H \subset F = \mathcal{F}(P)$ be a saturated and cofinal submonoid, $G = F/H$ and $G_P \subset G$ the set of all classes containing prime divisors.

1. Clearly, the submonoid $[G_P] \subset G$ generated by G_P equals G. Conversely, let G be an abelian group and $G_0 \subset G$ a subset with $[G_0] = G$ and $\mathsf{D}(G_0) < \infty$. Then $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated and cofinal, and—as outlined at the beginning of Section 4—there is an isomorphism from G to $\mathcal{F}(G_0)/\mathcal{B}(G_0)$ mapping G_0 onto the set of classes containing prime divisors. Thus $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ satisfies all assumptions of Theorem 4.4. Moreover, by Claborn's Realization Theorem, for any G and any nonempty $G_0 \subset G$ with $[G_0] = G$, there exists a Dedekind domain having ideal class group isomorphic to G with G_0 corresponding to the set of ideal classes containing prime ideals (see [14, Theorem 3.7.8]). In Example 4.6, we discuss a very simple case of an infinite subset G_0 with $[G_0] = G$ and $\mathsf{D}(G_0) < \infty$. By [14, Theorem 3.4.2], every such G must have infinite total rank.

2. If G_P is finite, then $\mathsf{D}(G_P) < \infty$, and statements 1 and 2 of Theorem 4.4 have been known before (see [14, Section 4.7]). But even in the finite case, there had not been any explicit result for the bounds of the AAMPs.

3. By Schmid's Realization Theorem (see [24]), statement 2 of Theorem 4.4 is sharp in the following sense: For every $M \in \mathbb{N}_0$ and every finite nonempty subset $\Delta \subset \mathbb{N}$, there exists a Krull monoid H with finite class group such that the following holds: for every AAMP L with difference $d \in \Delta$ and bound M, there is some $y_{H,L} \in \mathbb{N}$ such that

$$y + L \in \mathcal{L}(H), \qquad \text{for all } y \geqslant y_{H,L}.$$

4. Suppose that $G_P = G$ is infinite. Then $\mathsf{D}(G_P) = \infty$, and by Kainrath's Theorem, every finite set $L \subset \mathbb{N}_{\geqslant 2}$ occurs as a set of lengths [14, Theorem 7.4.1]. Thus clearly statement 2 of Theorem 4.4 does not hold. In that case, the Krull monoid H is not locally tame (see [16, Theorem 4.4]) and $\Delta(H)$ is infinite. But there are also locally tame Krull monoids with finite set of distances for which statement 2 of Theorem 4.4 does not hold (see [14, Theorem 4.8.4]).

5. By Proposition 3.14, we have $\delta(H) = \delta(G_P)$, and rephrasing the definition in terms of zero-sums, $\delta(G_P)$ is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

If $A = U_1 \cdot \ldots \cdot U_l \in \mathcal{B}(G_P)$, with $U_1, \ldots, U_l \in \mathcal{A}(G_P)$ and $l + d$ is an adjacent length, then there are a subset $I \subset [1, l]$, say $I = [1, k]$, and a factorization $A = V_1 \cdot \ldots \cdot V_{k+d} U_{k+1} \cdot \ldots \cdot U_l$, where $V_1, \ldots, V_{k+d} \in \mathcal{A}(G_P)$, such that $\max\{k, k+d\} \leqslant N$.

Suppose that $\delta(H) < \infty$. Then, by Proposition 3.8, all pattern ideals of H are tamely generated, and moreover, H is monotone tame and its monotone catenary degree is finite (apply Lemmas 3.5 and 3.7 in [7] with $\mathsf{Y} = \mathsf{Z}(H)$).

If G_P is finite, then $\delta(H) < \infty$ by Theorem 4.5.3, but it remains open whether the finiteness of the Davenport constant $\mathsf{D}(G_P)$ implies the finiteness of $\delta(H)$ or the finiteness of the monotone catenary degree. Recall that there are tame monoids with $\delta(H) = \infty$ (see [7, Example 4.5 and Remark 4.6]).

6. If G_P is finite, then every set of lengths is an AAMP with difference

$$d \in \Delta^*(G_P) = \bigl\{\min \Delta(G_1) \bigm| G_1 \subset G_P \text{ with } \Delta(G_1) \neq \emptyset\bigr\} \subset \Delta(H).$$

The set $\Delta^*(G)$ has found considerable interest (see [14, Sections 4.3 and 6.8], [3,23]), and in general, $\Delta^*(G)$ is a proper subset of $\Delta(H)$. If G_P is infinite, then the occurring differences of the AAMPs are actually in $\Delta(H)$ and not in the subset $\Delta^*(G_P)$ (see [14, Example 4.8.10]).

Example 4.6. Let G be a direct sum of cyclic groups, say $G = \oplus_{n \geqslant 0} \langle e_n \rangle$, such that $\mathrm{ord}(e_0) < \infty$, and define

$$G_0 = \{e_n \mid n \geqslant 0\} \cup \{-e_0 - e_n \mid n \geqslant 1\}.$$

1. $[G_0] = G$, and $\mathcal{B}(G_0)$ is neither finitely generated nor a product of non-trivial submonoids.
2. The following statements are equivalent:
   (a) $\mathsf{D}(G_0) < \infty$.
   (b) There is an $N \in \mathbb{N}$ such that $Ng = 0$ for all torsion elements $g \in G_0$.

Proof. 1. Obviously, $[G_0] = G$, and since, for every $n \geqslant 1$,

$$U_n = e_0(-e_0 - e_n)e_n \in \mathcal{A}(G_0),$$

$\mathcal{B}(G_0)$ is not finitely generated.

Suppose that $\mathcal{B}(G_0) = H_1 \times H_2$ where H_1, H_2 are submonoids of $\mathcal{B}(G_0)$. Then each H_i is a divisor-closed submonoid of $\mathcal{B}(G)$. Thus by [14, Proposition 2.5.6.4], $H_i = \mathcal{B}(G_i)$ for every $i \in [1,2]$, where G_i is the set of all $g \in G_0$ with $\mathsf{v}_g(A) > 0$ for some $A \in H_i$. Without restriction suppose that $U_1 \in H_1$. Then $\{e_0, e_1, -e_0 - e_1\} \subset G_1$, whence $e_0^{\mathrm{ord}(e_0)} \in H_1 \setminus H_2$, and thus $e_0 \notin G_2$. Therefore, for every $n \geqslant 2$, we have $U_n \notin \mathcal{B}(G_2)$, whence $U_n \in \mathcal{B}(G_1)$. This implies that $G_0 \subset G_1$, $G_2 = \emptyset$ and $H_2 = \{1\}$ is the trivial monoid.

2. (a) $\Rightarrow$ (b) If $g \in G_0$ is a torsion element, then $g^{\mathrm{ord}(g)} \in \mathcal{A}(G_0)$ and hence $\mathrm{ord}(g) \leqslant \mathsf{D}(G_0)$. This implies that $\mathrm{lcm}\{\mathrm{ord}(g) \mid g \in G_0 \text{ with } \mathrm{ord}(g) < \infty\}$ is an integer with the required property.

(b) $\Rightarrow$ (a) We pick a $U \in \mathcal{A}(G_0)$. After renumbering if necessary, it can be written in the form

$$U = e_0^{s_0} e_1^{s_1} \cdot \ldots \cdot e_l^{s_l} (-e_0 - e_1)^{t_1} \cdot \ldots \cdot (-e_0 - e_l)^{t_l},$$

where $s_0, s_1, \ldots, s_l, t_1, \ldots, t_l \in \mathbb{N}_0$, $e_1, \ldots, e_k$ are torsion elements and $\mathrm{ord}(e_{k+1}) = \cdots = \mathrm{ord}(e_l) = \infty$. Clearly, we have $s_0 \leqslant N$. For every $i \in [k+1, l]$, we have $s_i = t_i$. Since $Ne_0 = 0$, it follows that

$$\sum_{i=k+1}^{l} t_i \leqslant N, \quad \text{and hence} \quad \sum_{i=k+1}^{l} (s_i + t_i) \leqslant 2N.$$

For every $i \in [1,k]$, we clearly have $s_i + t_i \leqslant 2N$ and $s_i \equiv t_i \mod \mathrm{ord}(e_i)$. If $k > N$, then

$$e_1^{s_1} \cdot \ldots \cdot e_N^{s_N} (-e_0 - e_1)^{t_1} \cdot \ldots \cdot (-e_0 - e_N)^{t_N}$$

would be a proper zero-sum subsequence of U. Thus we get $k \leqslant N$, $\sum_{i=1}^{k} (s_i + t_i) \leqslant 2kN \leqslant 2N^2$ and

$$|U| = s_0 + \sum_{i=1}^{k} (s_i + t_i) + \sum_{i=k+1}^{l} (s_i + t_i) \leqslant N + 2N^2 + 2N.$$

This shows that $\mathsf{D}(G_0) \leqslant 3N + 2N^2$. $\square$

5. Proof of Proposition 4.3

We start with a simple lemma.

Lemma 5.1. *Let $M, N, d \in \mathbb{N}$ and let $B \subset \mathbb{Z}$ be a finite, nonempty subset with $-B = B$, $\max(B) = dN$ and $\gcd(B) = d$. Then there exist $R, R' \in \mathcal{F}(B \setminus \{0\})$ such that*

$$d\mathbb{Z} \cap [-dM, dM] \subset \Sigma(R) \quad \text{and} \quad |R| \leqslant 2\left\lfloor \frac{M+1}{N} \right\rfloor + 3N - 5 \leqslant \max\{2M, 3N-3\}, \tag{5.1}$$

$$d\mathbb{Z} \cap [d, dM] \subset \Sigma(R') \quad \text{and} \quad |R'| \leqslant \left\lfloor \frac{M+1}{N} \right\rfloor + 2N - 3 \leqslant \max\{M, 2N-2\}. \tag{5.2}$$

Moreover, if $N \geqslant 2$, then

$$|R| \leqslant 2\left\lfloor \frac{M+1}{N} \right\rfloor + 3N - 5 \leqslant \max\{M+2, 3N-1\}. \tag{5.3}$$

Proof. Since $\gcd(B) = d \in \mathbb{N}$, it follows that $B \neq \{0\}$. Thus we may assume without restriction that $0 \notin B$ and that $d = 1$. First observe that the second inequalities in (5.1), (5.2) and (5.3) follow by simple calculations. If $|B| = 2$, then $B = \{-1, 1\}$, and the lemma follows using $R = 1^M(-1)^M$ and $R' = 1^M$.

Suppose that $|B| \geqslant 4$. We set $B_0 = B \setminus \{N, -N\}$, $B_0^- = B_0 \cap (-\mathbb{N})$, and let $\psi : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ denote the canonical homomorphism. Since $\gcd(B \cap (-\mathbb{N})) = \gcd(B) = 1$, it follows that $\langle \psi(B_0^-) \rangle = \mathbb{Z}/N\mathbb{Z}$.

Let $T \in \mathcal{F}(B_0^- \cup \{0\})$ be a sequence with $\Sigma(\psi(T)) \neq \mathbb{Z}/N\mathbb{Z}$. We assert that there is some $b \in B_0^-$ such that $|\Sigma(\psi(Tb))| > |\Sigma(\psi(T))|$. Assume to the contrary that this does not hold. Then $\psi(b) + \Sigma(\psi(T)) \subset \Sigma(\psi(T))$ for all $b \in B_0^-$, which implies that

$$\mathbb{Z}/N\mathbb{Z} + \Sigma\big(\psi(T)\big) = \big\langle \psi\big(B_0^-\big)\big\rangle + \Sigma\big(\psi(T)\big) \subset \Sigma\big(\psi(T)\big) \subsetneq \mathbb{Z}/N\mathbb{Z},$$

a contradiction.

Hence, beginning with the single term 0, one may construct a sequence $0R_0$ with $R_0 \in \mathcal{F}(B_0^-)$, $|0R_0| \leqslant N$ and $\Sigma(\psi(0R_0)) = \mathbb{Z}/N\mathbb{Z}$. Consequently, $|R_0| \leqslant N-1$ and $\Sigma(R_0) \cup \{0\}$ contains a full system of representatives modulo N.

Now let $\alpha \in [-M, M]$ be given. Since $\Sigma(R_0) \cup \{0\}$ contains a full system of representatives modulo N, we can add at most $N-1$ terms from $-R_0$ to α such that the sum α' satisfies $\alpha' \in N\mathbb{Z}$ and

$$|\alpha'| \leqslant |\alpha| + (N-1) \cdot \max B_0 \leqslant |\alpha| + (N-1)^2 \leqslant M + (N-1)^2.$$

Thus

$$\alpha' \in [-\beta N, \beta N] \quad \text{with } \beta = \left\lfloor \frac{M+1}{N} \right\rfloor + N - 2.$$

Consequently, (5.1) and (5.3) follow by taking $R = R_0 N^\beta (-N)^\beta$, and since all terms of R_0 are negative, (5.2) holds using $R' = R_0 N^\beta$. $\square$

Proof of Proposition 4.3. Let $S \in \Phi(A)$ be given. If $G_1 = \operatorname{supp}(S)$, then $\mathcal{B}(G_1) \subset \mathcal{B}(G_0)$ is a divisor-closed submonoid with $\mathsf{D}(G_1) \leqslant \mathsf{D}(G_0)$, and we could work in $\mathcal{B}(G_1)$ instead of $\mathcal{B}(G_0)$. Thus, changing notation if necessary, we may suppose without restriction that for every $g \in G_0$ there is a $B \in \mathcal{B}(G_0)$

with $g \mid B$. Thus $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated and cofinal, and hence all statements of Proposition 4.1 hold for $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$. In particular, $\mathcal{B}(G_0)$ is tame with bound

$$\mathsf{t}(G_0) \leqslant 1 + \frac{\mathsf{D}(G_0)(\mathsf{D}(G_0)-1)}{2}.$$

If $\mathsf{D}(G_0) \leqslant 2$, then Proposition 4.1.2 implies that $\mathcal{B}(G_0)$ is half-factorial, and thus $|A| = 1$. If $|A| = 1$, then $\Phi(A) = \mathcal{B}(G_0)$, the trivial sequence $S' = 1$ with $|S'| = 0$ has the required property, and $\Phi(A)$ is tamely generated with bound $\varphi(A) = 0$.

Thus we may suppose that $|A| \geqslant 2$ and $\mathsf{D}(G_0) \geqslant 3$. Since $0^{-\mathsf{v}_0(S)} S \in \Phi(A)$, we may suppose without restriction that $\mathsf{v}_0(S) = 0$.

Suppose that we have found a subsequence S' of S satisfying (4.1), and let M_0 denote the upper bound on $|S'|$ from (4.1). Since $\mathsf{v}_0(S') = \mathsf{v}_0(S) = 0$, it follows that $\max \mathsf{L}(S') \leqslant \frac{1}{2}|S'| \leqslant \frac{1}{2}M_0$. Thus Lemma 3.5 implies that

$$\begin{aligned}
\varphi(A) &\leqslant 2\left(\frac{1}{2}M_0\right)\mathsf{t}(G_0) \\
&\leqslant M_0\left(1 + \frac{\mathsf{D}(G_0)(\mathsf{D}(G_0)-1)}{2}\right) \\
&\leqslant M_0 \frac{1}{2}\mathsf{D}(G_0)^2 - \mathsf{D}(G_0) + 1 \\
&= \frac{1}{2}\mathsf{D}(G_0)^2\big(9 \cdot \mathrm{diam}\,(A) + \mathsf{D}(G_0)^4\big)\left(\mathrm{diam}\,(A) \cdot \mathsf{D}(G_0)^2 + \frac{1}{2}\mathsf{D}(G_0)^4\right)^{\frac{(\mathsf{D}(G_0)-1)(\mathsf{D}(G_0)-2)}{2}} - \mathsf{D}(G_0) + 1 \\
&\leqslant \left(\mathrm{diam}\,(A) \cdot \mathsf{D}(G_0)^2 + \frac{1}{2}\mathsf{D}(G_0)^4\right)^{\frac{\mathsf{D}(G_0)(\mathsf{D}(G_0)-1)}{2}} - \mathsf{D}(G_0) + 1,
\end{aligned}$$

where we have used that $\mathsf{D}(G_0) \geqslant 3$, $M_0 \geqslant 4$, and $\frac{\mathsf{D}(G_0)(\mathsf{D}(G_0)-1)}{2} - \frac{(\mathsf{D}(G_0)-1)(\mathsf{D}(G_0)-2)}{2} \geqslant 2$. Thus the proof is complete. So it remains to establish (4.1).

Since $\Phi(A) = \Phi(y + A)$ for all $y \in \mathbb{Z}$, we may suppose that

$$A = \{m, m+d_1, m+d_2, \ldots, m+d_s\} \subset \mathsf{L}(S),$$

where $s, d_1, \ldots, d_s \in \mathbb{N}$ and $|A| = s + 1$. We set $d_0 = 0$, and for $i \in [0, s]$, we choose factorizations

$$w_i \in \mathsf{Z}(S) \quad \text{with} \quad |w_i| = m + d_i.$$

We have to find a zero-sum subsequence S' of S such that $|S'| \leqslant M_0$ and $n + A \subset \mathsf{L}(S')$ for some $n \in \mathbb{Z}$. The proof is divided into three parts.

Part 1: *The setting and key ideas.*

We set

$$D = \mathsf{D}(G_0), \qquad M = \mathrm{diam}\,(A) \quad \text{and} \quad N = \frac{(D-1)(D-2)}{2}. \tag{5.4}$$

For a sequence $T \in \mathcal{F}(G_0)$, let $\Delta_{\mathrm{cat}}(T)$ be all those integers $d \in [-N, N]$ for which there exists a zero-sum subsequence T_0 of T such that $|T_0| \leqslant D^2$ and $|z| - |z'| = d$ for some $z, z' \in \mathsf{Z}(T_0)$. By symmetry, we have

$$\Delta_{\mathrm{cat}}(T) = -\Delta_{\mathrm{cat}}(T). \tag{5.5}$$

For a sequence $T \in \mathcal{F}(G_0)$ and some $t \in \mathbb{N}_0$, we say that $d \in [-N, N]$ is *t-deficient* (in T) if there exists a subsequence T_0 of T such that $|T_0| \leqslant t$ and $d \notin \Delta_{\mathrm{cat}}(T_0^{-1}T)$. In view of (5.5), it follows that d is t-deficient if and only if $-d$ is t-deficient.

For a subsequence T of S and $z, z' \in \mathsf{Z}(S)$, a *catenary chain $\mathcal{C}$ from z to z' that greedily eats T* is an ordered sequence of factorizations of S, say $z = z_0, z_1, z_2, \ldots, z_r = z'$, where each z_j is obtained from the previous $z_{j-1} \neq z'$ by the following process. For $j \in [1, r]$, set

$$z_{j-1} = x_{j-1} y_{j-1} \quad \text{and} \quad z' = x_{j-1} y'_{j-1}, \quad \text{where } x_{j-1} = \gcd\bigl(z_{j-1}, z'\bigr) \text{ and } y_{j-1}, y'_{j-1} \in \mathsf{Z}(G_0).$$

Choose an element $g \in G_0$ such that $g \mid T \gcd(T, \pi(x_{j-1}))^{-1}$ if $T \nmid \pi(x_{j-1})$, and otherwise such that $g \mid \pi(y_{j-1})$. Furthermore, choose an atom $V^{(j)} \in \mathcal{A}(G_0)$ such that $V^{(j)} \mid y'_{j-1}$ and $g \mid V^{(j)}$. Let $n_j \in \mathbb{N}$ be minimal such that there are $U_1^{(j)}, \ldots, U_{n_j}^{(j)} \in \mathcal{A}(G_0)$ with

$$V^{(j)} \bigm| U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)} \quad \text{in } \mathcal{B}(G_0) \quad \text{and} \quad U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)} \bigm| y_{j-1} \quad \text{in } \mathsf{Z}(G_0),$$

and choose $w^{(j)} \in \mathsf{Z}(U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)}) \cap V^{(j)}\mathsf{Z}(G_0)$. Set $n'_j = |w^{(j)}|$ and

$$z_j = z_{j-1}\bigl(U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)}\bigr)^{-1} w^{(j)} \in \mathsf{Z}(S).$$

Since $V^{(j)} x_{j-1} \mid x_j$, we see that $|x_{j-1}| < |x_j|$, and hence the process terminates with some $z_r = z'$, and we set $x_r = z'$. Let $|\mathcal{C}| = r$ denote the length of the chain $\mathcal{C}$, and let the truncation number $tr(\mathcal{C})$ be the smallest $j \in [0, r]$ such that $T \mid \pi(x_j)$. Let $T_{\mathrm{int}} = \gcd(T, \pi(x_0))$, and let $n_0 \in \mathbb{N}_0$ be minimal such that there are $U_1^{(0)}, \ldots, U_{n_0}^{(0)} \in \mathcal{A}(G_0)$ with

$$T_{\mathrm{int}} \bigm| U_1^{(0)} \cdot \ldots \cdot U_{n_0}^{(0)} \quad \text{in } \mathcal{B}(G_0) \quad \text{and} \quad U_1^{(0)} \cdot \ldots \cdot U_{n_0}^{(0)} \bigm| x_0 \quad \text{in } \mathsf{Z}(G_0).$$

Note $n_0 \leqslant |T_{\mathrm{int}}|$, and so

$$\left| \pi\left(\prod_{i=1}^{n_0} U_i^{(0)} \right) \right| \leqslant n_0 \cdot D \leqslant |T_{\mathrm{int}}| \cdot D. \tag{5.6}$$

Since $|\gcd(T, \pi(x_{j-1}))| < |\gcd(T, \pi(x_j))|$ for all $j \in [1, tr(\mathcal{C})]$, it follows that

$$tr(\mathcal{C}) \leqslant |T| - |T_{\mathrm{int}}|. \tag{5.7}$$

Let $j \in [1, r]$. Since $\mathsf{v}_0(S) = 0$, $D \geqslant 3$ and $V^{(j)} \nmid y_{j-1}$, we obtain that

$$2 \leqslant n_j \leqslant \bigl|V^{(j)}\bigr| \leqslant D, \tag{5.8}$$

$$\bigl|U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)}\bigr| \leqslant n_j D \leqslant D^2, \tag{5.9}$$

$$2 \leqslant n'_j \leqslant 1 + \frac{1}{2}\bigl(\bigl|U_1^{(j)} \cdot \ldots \cdot U_{n_j}^{(j)}\bigr| - \bigl|V^{(j)}\bigr|\bigr) \leqslant 1 + \frac{1}{2}\bigl(n_j D - \bigl|V^{(j)}\bigr|\bigr),$$

$$n'_j - n_j \leqslant 1 + \frac{1}{2}\bigl(n_j D - \bigl|V^{(j)}\bigr|\bigr) - n_j \leqslant 1 + \frac{1}{2}\bigl(\bigl|V^{(j)}\bigr| D - \bigl|V^{(j)}\bigr|\bigr) - \bigl|V^{(j)}\bigr| \leqslant 1 + \frac{D(D-3)}{2} = N, \tag{5.10}$$

and

$$n'_j - n_j \geqslant -(D-2) \geqslant -N. \tag{5.11}$$

We define

$$TR(\mathcal{C}) = \gcd\left(z, \prod_{j=0}^{tr(\mathcal{C})} \prod_{i=1}^{n_j} U_i^{(j)}\right) \in \mathsf{Z}(G_0), \tag{5.12}$$

$$TR'(\mathcal{C}) = TR(\mathcal{C})\left(\prod_{i=1}^{n_0} U_i^{(0)}\right)^{-1} \in \mathsf{Z}(G_0), \tag{5.13}$$

and continue with the following assertion:

(A1) $T \mid \pi(TR(\mathcal{C}))$.

Proof of (A1). Clearly, (A1) can be seen by a close look at the construction, but we also offer a formal proof. Since $T_{\text{int}} \mid \prod_{i=1}^{n_0} U_i^{(0)}$, it suffices to check that $TT_{\text{int}}^{-1} \mid \pi(TR'(\mathcal{C}))$.

Let $TR''(\mathcal{C}) \in \mathsf{Z}(S)$ be the minimal divisor of z such that $z \cdot (TR''(\mathcal{C}))^{-1}$ also divides $z_{tr(\mathcal{C})}$, in other words

$$TR''(\mathcal{C}) = z \gcd(z, z_{tr(\mathcal{C})})^{-1}.$$

Thus $TR''(\mathcal{C})$ is the product of all those atoms of z that have been involved in one of the first $tr(\mathcal{C})$ iterations. Consequently, we see that

$$TR''(\mathcal{C}) \text{ divides } z\left(\prod_{i=1}^{n_0} U_i^{(0)}\right)^{-1}$$

and that

$$TR''(\mathcal{C}) \text{ divides } \prod_{j=1}^{tr(\mathcal{C})} \prod_{i=1}^{n_j} U_i^{(j)}.$$

As a result (in view of (5.12) and (5.13)), we find that

$$TR''(\mathcal{C}) \text{ divides } TR'(\mathcal{C}). \tag{5.14}$$

Since TT_{int}^{-1} is relatively prime with $\pi(\gcd(z, z'))T_{\text{int}}^{-1}$ (this follows from the definition of T_{int}), and since TT_{int}^{-1} divides $\pi(x_{tr(\mathcal{C})})T_{\text{int}}^{-1} = \pi(\gcd(z_{tr(\mathcal{C})}, z'))T_{\text{int}}^{-1}$ (this follows from the definition of $tr(\mathcal{C})$), we see that

$$TT_{\text{int}}^{-1} \text{ divides } \frac{\pi(\gcd(z_{tr(\mathcal{C})}, z'))}{\pi(\gcd(z, z'))} \quad \text{which divides } \pi\left(z \gcd(z, z_{tr(\mathcal{C})})^{-1}\right) = \pi\left(TR''(\mathcal{C})\right),$$

and the assertion follows from (5.14). $\square$

Now observe (in view of (5.6), (5.8) and (5.7)) that

$$\begin{aligned} \left|\pi\left(TR(\mathcal{C})\right)\right| &\leqslant D \cdot |T_{\text{int}}| + D \cdot \sum_{j=1}^{tr(\mathcal{C})} n_j \leqslant D \cdot |T_{\text{int}}| + tr(\mathcal{C}) \cdot D^2 \\ &\leqslant D\left(|T_{\text{int}}| + D\left(|T| - |T_{\text{int}}|\right)\right) \leqslant |T| \cdot D^2. \end{aligned} \tag{5.15}$$

For $\alpha, \beta \in [0, r]$ with $\alpha \leqslant \beta$, we will use $\mathcal{C}[\alpha, \beta]$ to refer to the ordered subsequence of $\mathcal{C}$ given by $z_\alpha, z_{\alpha+1}, \ldots, z_\beta$, and we let $\mathcal{C}[\alpha] = z_\alpha$. We associate to $\mathcal{C}[\alpha, \beta]$ the sequence $Di \circ \mathcal{C}[\alpha, \beta] \in \mathcal{F}(\mathbb{Z})$ defined as (in view of (5.10) and (5.11))

$$Di \circ \mathcal{C}[\alpha, \beta] = \prod_{j=\alpha+1}^{\beta} (n'_j - n_j) \in \mathcal{F}\big([-N, N]\big). \tag{5.16}$$

Moreover, from the definitions of $\mathcal{C}$ and $Di \circ \mathcal{C}[\alpha, \beta]$, we have that

$$\sigma\big(Di \circ \mathcal{C}[\alpha, \beta]\big) = |z_\beta| - |z_\alpha|. \tag{5.17}$$

Let T be a subsequence of S, $z, z'_1, \ldots, z'_s \in \mathsf{Z}(S)$, and for every $i \in [1, s]$, let $\mathcal{C}_i$ be a catenary chain from z to z'_i that greedily eats T. For $\mathfrak{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_s\}$, we define

$$TR(\mathfrak{C}) = \operatorname{lcm}\big(\big\{TR(\mathcal{C}_i) \bigm| i \in [1, s]\big\}\big) \quad \text{and} \quad tr(\mathfrak{C}) = \max\big\{tr(\mathcal{C}_i) \bigm| i \in [1, s]\big\}.$$

In view of (5.7) and (5.15), we have

$$tr(\mathfrak{C}) \leqslant |T|, \quad \text{and} \tag{5.18}$$

$$\big|\pi\big(TR(\mathfrak{C})\big)\big| \leqslant |T| + \sum_{i=1}^{s}\big(\big|\pi\big(TR(\mathcal{C}_i)\big)\big| - |T|\big) \leqslant s|T|\big(D^2 - 1\big) + |T|, \tag{5.19}$$

where for (5.19) we use that $T \mid \pi(TR(\mathcal{C}_i))$ for all $i \in [1, s]$. (We remark, as a technical point, that to conclude (5.19), we must actually assume the $\mathcal{C}_i$ were chosen so that the 'same' sequence T was eaten by each $\mathcal{C}_i$; that this is always possible is trivially obvious if one considers all sequences ordered. However, for such a minor and subtle point, only affecting the value of the constants computed, mentioning more details would only exacerbate notation already complicated enough.)

Part 2: *The definition of the sequence S'.*

We now set $t_{-1} = 0$, and for $j \in [0, N]$, we define $t_j \in \mathbb{N}_0$ recursively as follows:

$$t_0 = \left(\left\lfloor \frac{M+1}{N} \right\rfloor + 2N - 4\right) D^2, \tag{5.20}$$

$$t_j = \big(s\big(D^2 - 1\big) + 1\big)t_{j-1} + (M + 1 + t_{j-1}N)D^2 \quad \text{for } j \in [1, N-2], \tag{5.21}$$

$$t_{N-1} = \big(s\big(D^2 - 1\big) + 1\big)t_{N-2} + (2M - 1 + 2t_{N-2}N)D^2 \quad \text{for } N \geqslant 2, \tag{5.22}$$

$$t_N = \big(s\big(D^2 - 1\big) + 1\big)t_{N-1}. \tag{5.23}$$

Observe that

$$0 = t_{-1} \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_N. \tag{5.24}$$

Furthermore, since $D \geqslant 3$ and $s \geqslant 1$, we have

$$t_0 \geqslant D^2 \geqslant 9, \tag{5.25}$$

when $M \geqslant 2$ or $N \geqslant 2$. We continue with the following assertion:

(A2) If $|S| \geqslant t_N$, then there exist $k \in [0, N]$, a subset $B' \subset [1, N]$ with $|B'| = k$, and a subsequence T of S with $|T| = t_{k-1}$, such that the following two conditions are satisfied:
(a) $B' \cap \Delta_{\mathrm{cat}}(T^{-1}S) = \emptyset$.
(b) No $d \in [1, N] \setminus B'$ is $(t_k - t_{k-1})$-deficient in $T^{-1}S$.

Proof of (A2). Note that (a) holds trivially for $k = 0$, $B' = \emptyset$ and $T = 1$, and that (b) holds trivially for $k = N$, $B' = [1, N]$ and any sequence T with $|T| = t_{N-1}$. Thus, to establish the claim, it suffices (by an inductive argument) to show that if there is a $k \in [0, N]$, a subset B' and a sequence T, as above, such that (a) holds but (b) fails, then there is a subset $B'' \subset [1, N]$ and a subsequence T' of T, with $|B''| = k + 1$ and $|T'| = t_k$, such that (a) holds.

Let $k \in [0, N]$, let $B' \subset [1, N]$ with $|B'| = k$, and let T be a subsequence of S with $|T| = t_{k-1}$ such that (a) holds but (b) fails. Then $k < N$ and there is a $d \in [1, N] \setminus B'$ and a subsequence T' of $T^{-1}S$ such that $|T'| = t_k - t_{k-1}$ (here we use that $|S| \geqslant t_N \geqslant t_k$) and $d \notin \Delta_{\mathrm{cat}}(T'^{-1}T^{-1}S)$. This implies that (a) holds for the integer $k + 1 \in [0, N]$, the set $B' \cup \{d\}$ and the sequence $T'T$. $\square$

If $|S| < t_N$, then we define $S' = S$. Otherwise, let k, B' and T be as given by (A2). We define

$$B^+ = [1, N] \setminus B', \qquad B = B^+ \cup \left(-B^+\right), \qquad d = \gcd(B), \qquad N' = \frac{1}{d} \max B \quad \text{and} \quad M' = M + t_{k-1}N,$$

and we apply Lemma 5.1 using B, M' and N' in place of B, M and N. Let $R = r_1 \cdot \ldots \cdot r_l$ and $R' = r'_1 \cdot \ldots \cdot r_{l'}$ be the resulting sequences (for $k = N$, take R and R' to be trivial). We distinguish two cases.

Case 1. $k = 0$.

Then $B = [-N, N] \setminus \{0\}$, $d = 1$, $N' = N$ and $M' = M$. Let $r \in [0, |R'|]$ be maximal with the following property:

for every $i \in [1, r]$, there exist $S_i \in \mathcal{B}(G_0)$ and $z_i, z'_i \in \mathsf{Z}(S_i)$ such that $|S_i| \leqslant D^2$, $|z'_i| - |z_i| = r'_i$ and $S_1 \cdot \ldots \cdot S_r \mid S$.

Note that $r'_i \in \Delta_{\mathrm{cat}}(S)$ for every $i \in [1, r]$.

Suppose $r < |R'|$. Since r is maximal, it follows that $r'_{r+1} \in B = [-N, N] \setminus \{0\}$ is $|S_1 \cdot \ldots \cdot S_r|$-deficient in S (use the definition of deficiency with $T_0 = S_1 \cdot \ldots \cdot S_r$). However, by (5.2), we know

$$|S_1 \cdot \ldots \cdot S_r| \leqslant rD^2 \leqslant \left(|R'| - 1\right)D^2 \leqslant \left(\left\lfloor \frac{M+1}{N} \right\rfloor + 2N - 4\right)D^2 = t_0, \tag{5.26}$$

while, by condition (b) in (A2), no $d \in [1, N] \setminus B' = [1, N]$ is t_0-deficient in $T^{-1}S = S$ (recall d is t-deficient if and only if $-d$ is t-deficient), a contradiction. So we conclude that $r = |R'|$.

We define

$$S' = S_1 \cdot \ldots \cdot S_r \quad \text{and} \quad w' = z_1 \cdot \ldots \cdot z_r \in \mathsf{Z}(S').$$

Note, by the same arguments used for (5.26), that

$$|S'| \leqslant t_0 + D^2. \tag{5.27}$$

For a subset $I \subset [1, r]$, let $w'_I \in \mathsf{Z}(S')$ be obtained from w' by replacing z_i by z'_i for each $i \in I$, i.e.,

$$w'_I = \left(\prod_{i \in I} z_i\right)^{-1} w' \left(\prod_{i \in I} z'_i\right).$$

Thus

$$\left|w'_I\right| = |w'| + \sum_{i\in I} r'_i. \tag{5.28}$$

By (5.4) and (5.2), it follows that (recall $d = 1$ and $M' = M$)

$$\{d_1, \dots, d_s\} \subset [1, M] = d\mathbb{Z} \cap [d, dM] \subset \Sigma(R').$$

Thus, letting $I_i \subset [1, r]$ be such that $\sum_{\nu\in I_i} r'_\nu = d_i$, for $i \in [1, s]$, we see (in view of (5.28)) that the factorizations $w', w'_{I_1}, \dots, w'_{I_s} \in \mathsf{Z}(S')$ show that $|w'| - m + A = \{|w'|, |w'| + d_1, \dots, |w'| + d_s\} \subset \mathsf{L}(S')$.

Case 2. $k \geqslant 1$.

Let $\mathfrak{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_s\}$ be a family of catenary chains $\mathcal{C}_i$ from w_0 to w_i that each greedily eats the subsequence T of S for every $i \in [1, s]$. Let

$$S_0 = \pi\big(TR(\mathfrak{C})\big)$$

and let

$$d'_i = d_i - \sigma\big(Di \circ \mathcal{C}_i\big[0, tr(\mathcal{C}_i)\big]\big), \tag{5.29}$$

for $i \in [1, s]$. Note $T \mid S_0$.

Since $B' \cap \Delta_{\text{cat}}(T^{-1}S) = \emptyset$ and $[1, N] \setminus B' = B^+$, it follows from (5.9) and (5.16) that

$$Di \circ \mathcal{C}_i\big[tr(\mathcal{C}_i), |\mathcal{C}_i|\big] \in \mathcal{F}\big(B \cup \{0\}\big), \tag{5.30}$$

and thus in view of (5.17) and (5.29) it follows that

$$\big\{d'_1, \dots, d'_s\big\} \subset \big\langle B \cup \{0\}\big\rangle = d\mathbb{Z}. \tag{5.31}$$

In view of (5.16), (5.18) and (5.4), we see that

$$\left|d'_i\right| \leqslant d_i + tr(\mathcal{C}_i)N \leqslant d_i + t_{k-1}N \leqslant M + t_{k-1}N = M'. \tag{5.32}$$

Case 2.1. $k = N$.

Then $B = \emptyset$, and we see from (5.31) and (5.29) that the factorizations

$$TR(\mathfrak{C}), \quad w^{-1}\mathcal{C}_1\big[tr(\mathcal{C}_1)\big], \quad \dots, \quad w^{-1}\mathcal{C}_r\big[tr(\mathcal{C}_r)\big] \in \mathsf{Z}(S_0),$$

where $w = w_0 TR(\mathfrak{C})^{-1}$, show that $|TR(\mathfrak{C})| - m + A \subset \mathsf{L}(S_0)$ (in view of (5.17) with $\alpha = 0$ and $\beta = tr(\mathcal{C}_i)$). Thus we define $S' = S_0$, and by (5.19) we observe that

$$|S'| \leqslant s|T|\big(D^2 - 1\big) + |T| = \big(s\big(D^2 - 1\big) + 1\big)t_{N-1} = t_N. \tag{5.33}$$

Case 2.2. $k \in [1, N-1]$.

Then $N \geqslant 2$. Let $r \in [0, |R|]$ be maximal with the following property:

for every $i \in [1, r]$, there exist $S_i \in \mathcal{B}(G_0)$ and $z_i, z_i' \in \mathsf{Z}(S_i)$ such that $|S_i| \leqslant D^2$, $|z_i'| - |z_i| = r_i$ and $S_1 \cdot \ldots \cdot S_r \mid S_0^{-1} S$.

Suppose $r < |R|$. Since r is maximal and since $T | S_0$, it follows that $r_{r+1} \in B \subset [-N, N] \setminus \{0\}$ is $(|S_1 \cdot \ldots \cdot S_r| + |S_0| - |T|)$-deficient in $T^{-1}S$ (use the definition of deficiency with $T_0 = T^{-1} S_0 S_1 \cdot \ldots \cdot S_r$). However, for $k \leqslant N-2$, it follows in view of (5.19) and (5.3) (note $k \leqslant N-2$ implies $|B^+| \geqslant 2$ implies $N' \geqslant 2$) that

$$\begin{aligned} |S_1 \cdot \ldots \cdot S_r| + |S_0| - |T| &\leqslant rD^2 + |S_0| - t_{k-1} \\ &\leqslant \big(|R| - 1\big)D^2 + |S_0| - t_{k-1} \\ &\leqslant \max\{M' + 1, 3N' - 2\}D^2 + |S_0| - t_{k-1} \\ &\leqslant \max\{M' + 1, 3N - 2\}D^2 + |S_0| - t_{k-1} \\ &\leqslant \max\{M' + 1, 3N - 2\}D^2 + \big(s\big(D^2 - 1\big) + 1\big)t_{k-1} - t_{k-1}. \end{aligned} \tag{5.34}$$

Furthermore, we have from (5.24) and (5.25) that

$$M' + 1 = M + 1 + t_{k-1}N \geqslant M + 1 + t_0 N \geqslant 9N > 3N - 2,$$

and thus

$$\big|T^{-1} S_0 S_1 \cdot \ldots \cdot S_r\big| \leqslant (M' + 1)D^2 + \big(s\big(D^2 - 1\big) + 1\big)t_{k-1} - t_{k-1} = t_k - t_{k-1},$$

while, by condition (b) in (A2), no $d \in [1, N] \setminus B' = B^+$ is $(t_k - t_{k-1})$-deficient in $T^{-1}S$, a contradiction.

Likewise, for $k = N-1$, it follows, in view of $N \geqslant 2$, (5.19), (5.1), (5.24) and (5.25), that

$$\begin{aligned} |S_1 \cdot \ldots \cdot S_r| + |S_0| - |T| &\leqslant rD^2 + |S_0| - t_{N-2} \\ &\leqslant \big(|R| - 1\big)D^2 + |S_0| - t_{N-2} \\ &\leqslant \max\{2M' - 1, 3N' - 4\}D^2 + |S_0| - t_{N-2} \\ &\leqslant \max\{2M' - 1, 3N - 4\}D^2 + \big(s\big(D^2 - 1\big) + 1\big)t_{N-2} - t_{N-2} \\ &= (2M' - 1)D^2 + \big(s\big(D^2 - 1\big) + 1\big)t_{N-2} - t_{N-2} \\ &= t_{N-1} - t_{N-2}, \end{aligned} \tag{5.35}$$

yielding the same contradiction as in the case $k \leqslant N-2$. So we conclude that $r = |R|$.

We define

$$S' = S_0 S_1 \cdot \ldots \cdot S_r \quad \text{and} \quad w' = TR(\mathfrak{C}) z_1 \cdot \ldots \cdot z_r \in \mathsf{Z}(S').$$

Note that the calculations used in (5.34) and (5.35) show that

$$|S'| \leqslant t_k + D^2. \tag{5.36}$$

For a subset $I \subset [1, r]$ and $j \in [1, s]$, let $w'_{j,I} \in \mathsf{Z}(S')$ be defined by

$$w'_{j,I} = TR(\mathfrak{C})^{-1}\left(\prod_{i\in I} z_i\right)^{-1} w'\left(\prod_{i\in I} z'_i\right)\left(w^{-1}C_j\left[tr(C_j)\right]\right),$$

where $w = w_0 TR(\mathfrak{C})^{-1}$. Thus (in view of (5.17) with $\alpha = 0$ and $\beta = tr(\mathcal{C}_j)$)

$$\left|w'_{j,I}\right| = |w'| + \sigma\left(Di \circ \mathcal{C}_j\left[0, tr(\mathcal{C}_j)\right]\right) + \sum_{i\in I} r_i. \tag{5.37}$$

In view of (5.31), (5.32), and (5.1) and holding for R, it follows that $\{d'_1, \ldots, d'_s\} \subset d\mathbb{Z} \cap [-M', M'] \subset \Sigma(R)$. As a result, letting I_j for $j \in [1, s]$ be such that $\sum_{\nu\in I_j} r_\nu = d'_j$, we see (in view of (5.37) and (5.29)) that the factorizations $w', w'_{1,I_1}, \ldots, w'_{s,I_s} \in \mathsf{Z}(S')$ show that $|w'| - m + A \subset \mathsf{L}(S')$.

Part 3: *An upper bound for the length of S'.*

We have shown that there exists a zero-sum subsequence S' of S such that $n + A \subset \mathsf{L}(S')$ for some $n \in \mathbb{Z}$, and, in view of (5.27), (5.33), (5.36) and (5.24), such that

$$|S'| \leqslant \max\left\{t_{N-1} + D^2, t_N\right\}. \tag{5.38}$$

Thus it remains to verify that

$$\max\left\{t_{N-1} + D^2, t_N\right\} \leqslant \left(9M + D^4\right)\left(MD^2 + \frac{1}{2}D^4\right)^{\frac{(D-1)(D-2)}{2}}.$$

If $M = N = 1$, then (5.38) gives (in view of (5.20) and (5.23))

$$\max\left\{t_{N-1} + D^2, t_N\right\} = t_{N-1} + D^2 = D^2 \leqslant \left(9M + D^4\right)\left(MD^2 + \frac{1}{2}D^4\right)^{\frac{(D-1)(D-2)}{2}}. \tag{5.39}$$

Otherwise, $t_{N-1} \geqslant t_0 \geqslant D^2$ (in view of (5.24) and (5.25)), whence $D \geqslant 3$, $s \geqslant 1$ and (5.23) show $t_N \geqslant t_{N-1} + 8t_{N-1} \geqslant t_{N-1} + 8D^2$. Thus we obtain that

$$\max\left\{t_{N-1} + D^2, t_N\right\} \leqslant t_N.$$

Hence, if $N = 1$ (and thus $D = 3$; see (5.4)) and $M \geqslant 2$, then (5.20) and (5.23) imply

$$\begin{aligned} t_N &\leqslant \left(s\left(D^2 - 1\right) + 1\right)\left(\frac{M+1}{N} + 2N - 4\right)D^2 \leqslant 9M\left(s\left(D^2 - 1\right) + 1\right) \\ &= 9M(8s + 1) \leqslant 9M(8M + 1) \leqslant \left(9M + D^4\right)\left(MD^2 + \frac{1}{2}D^4\right)^{\frac{(D-1)(D-2)}{2}}. \end{aligned}$$

Since $2N = (D-1)(D-2) \neq 4$, we suppose from now on that $N \geqslant 3$, and thus $D \geqslant 4$. The recurrence given by (5.20) and (5.21) is of the form

$$\alpha t_{j-1} + \beta = t_j \quad \text{for all } j \in [1, N-2],$$

where $\alpha = s(D^2 - 1) + 1 + ND^2 > 1$ and $\beta = (M+1)D^2$. The solution to such a recurrence is easily seen to be

$$\alpha_j = \alpha^j t_0 + \beta \cdot \sum_{i=0}^{j-1} \alpha^i = \alpha^j t_0 + \beta \frac{\alpha^j - 1}{\alpha - 1},$$

and hence

$$t_{N-2} = \alpha^{N-2} t_0 + \beta \frac{\alpha^{N-2} - 1}{\alpha - 1}.$$

Thus, since $N \geqslant 3$, $D \geqslant 4$ and $1 \leqslant s \leqslant M$, we derive from (5.4), (5.20), (5.22) and (5.23) that

$$\begin{aligned}
t_N &= \big(\alpha - ND^2\big)\big((\alpha + ND^2)t_{N-2} + (2M-1)D^2\big) \\
&\leqslant \big(\alpha^2 - N^2D^4\big)\left(\alpha^{N-2}\left(\frac{(M+1)D^2}{N} + \big((D-1)(D-2)-4\big)D^2\right) + (M+1)D^2\frac{\alpha^{N-2}-1}{\alpha-1}\right) \\
&\quad + \big(s\big(D^2-1\big)+1\big)(2M-1)D^2 \\
&\leqslant \big(\alpha^2 - N^2D^4\big)\left(\alpha^{N-2}\left(\frac{(M+1)D^2}{N} + \big(D^2-3D-2\big)D^2\right) + (M+1)D^2\frac{\alpha^{N-2}}{ND^2}\right) + 2sMD^4 \\
&= \big(\alpha^2 - N^2D^4\big)\alpha^{N-2}\left(\frac{2(M+1)(D^2+1)}{(D-1)(D-2)} + \big(D^2-3D-2\big)D^2\right) + 2sMD^4 \\
&\leqslant \big(\alpha^2 - N^2D^4\big)\alpha^{N-2}\big(6M + D^4\big) + 2sMD^4 \leqslant \big(6M + D^4\big)\alpha^N \leqslant \big(9M + D^4\big)\alpha^N \\
&= \big(9M + D^4\big)\left(s\big(D^2-1\big) + 1 + \frac{(D-1)(D-2)D^2}{2}\right)^{\frac{(D-1)(D-2)}{2}} \\
&\leqslant \big(9M + D^4\big)\left(MD^2 + \frac{1}{2}D^4\right)^{\frac{(D-1)(D-2)}{2}},
\end{aligned}$$

which completes the proof. □

Acknowledgments

We thank Florian Kainrath for valuable comments on previous versions of this manuscript, and we thank the referee for their careful reading.

References

[1] S.T. Chapman (Ed.), Arithmetical Properties of Commutative Rings and Monoids, Lect. Notes Pure Appl. Math., vol. 241, Chapman & Hall/CRC, 2005.
[2] S.T. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic Publishers, 2000.
[3] S.T. Chapman, W.A. Schmid, W.W. Smith, On minimal distances in Krull monoids with infinite class group, Bull. Lond. Math. Soc. 40 (2008) 613–618.
[4] L.G. Chouinard II, Krull semigroups and divisor class groups, Canad. J. Math. 33 (1981) 1459–1468.
[5] A. Facchini, Direct sum decomposition of modules, semilocal endomorphism rings, and Krull monoids, J. Algebra 256 (2002) 280–307.
[6] A. Facchini, W. Hassler, L. Klingler, R. Wiegand, Direct-sum decompositions over one-dimensional Cohen–Macaulay local rings, in: J.W. Brewer, S. Glaz, W. Heinzer, B. Olberding (Eds.), Multiplicative Ideal Theory in Commutative Algebra, Springer, 2006, pp. 153–168.
[7] A. Foroutan, Monotone chains of factorizations, in: A. Badawi (Ed.), Focus on Commutative Rings Research, Nova Sci. Publ., New York, 2006, pp. 107–130.
[8] A. Foroutan, W. Hassler, Chains of factorizations and factorizations with successive lengths, Comm. Algebra 34 (2006) 939–972.
[9] G.A. Freiman, A. Geroldinger, An addition theorem and its arithmetical application, J. Number Theory 85 (2000) 59–73.
[10] W. Gao, A. Geroldinger, Zero-sum problems in finite abelian groups: A survey, Expo. Math. 24 (2006) 337–369.

[11] W. Gao, A. Geroldinger, On products of k atoms, Monatsh. Math. 156 (2009).
[12] A. Geroldinger, Additive group theory and non-unique factorizations, in: A. Geroldinger, I. Ruzsa (Eds.), Combinatorial Number Theory and Additive Group Theory, in: Adv. Courses Math. CRM Barcelona, Barcelona, Birkhäuser, 2009.
[13] A. Geroldinger, F. Halter-Koch, Non-unique factorizations: A survey, in: J.W. Brewer, S. Glaz, W. Heinzer, B. Olberding (Eds.), Multiplicative Ideal Theory in Commutative Algebra, Springer, 2006, pp. 217–226.
[14] A. Geroldinger, F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, Pure Appl. Math., vol. 278, Chapman & Hall/CRC, 2006.
[15] A. Geroldinger, W. Hassler, Arithmetic of Mori domains and monoids, J. Algebra 319 (2008) 3419–3463.
[16] A. Geroldinger, F. Hassler, Local tameness of v-noetherian monoids, J. Pure Appl. Algebra 212 (2008) 1509–1524.
[17] P.A. Grillet, Commutative Semigroups, Kluwer Academic Publishers, 2001.
[18] F. Halter-Koch, Non-unique factorizations of algebraic integers, Funct. Approx. Comment. Math., in press.
[19] F. Halter-Koch, Halbgruppen mit Divisorentheorie, Expo. Math. 8 (1990) 27–66.
[20] F. Halter-Koch, Ideal Systems. An Introduction to Multiplicative Ideal Theory, Marcel Dekker, 1998.
[21] W. Hassler, Properties of factorizations with successive lengths in one-dimensional local domains, J. Commut. Algebra, in press.
[22] U. Krause, On monoids of finite real character, Proc. Amer. Math. Soc. 105 (1989) 546–554.
[23] M. Radziejewski, W.A. Schmid, Weakly half-factorial sets in finite abelian groups, Forum Math. 19 (2007) 727–747.
[24] W.A. Schmid, A realization theorem for sets of lengths, J. Number Theory, in press.
[25] W.A. Schmid, Characterization of class groups of Krull monoids via their systems of sets of lengths: A status report, in: S.D. Adhikari, B. Ramakrishnan (Eds.), HRI Conference Proceedings, 2008.
[26] R. Wiegand, Direct-sum decompositions over local rings, J. Algebra 240 (2001) 83–97.