Tone rows and tropes

Harald Fripertinger
Karl-Franzens-Universität Graz

International Congress on Music and Mathematics
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Joint work with Peter Lackner,
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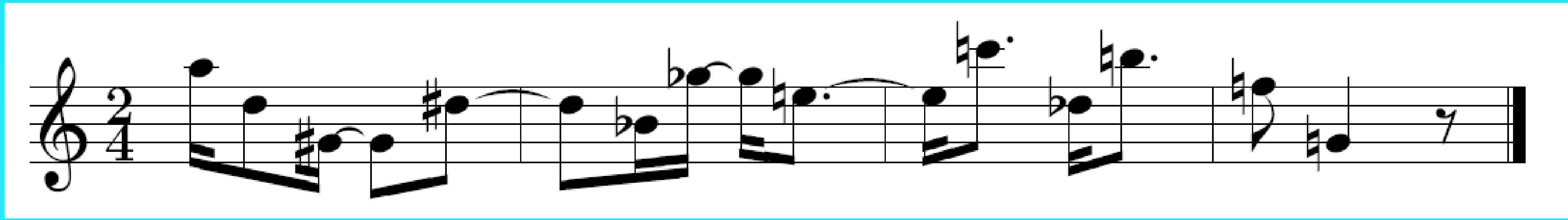

O. Messiaen: Le Merle Noir: A piece for flute and piano composed in 1951. The tone row appears in the coda of the piano part.

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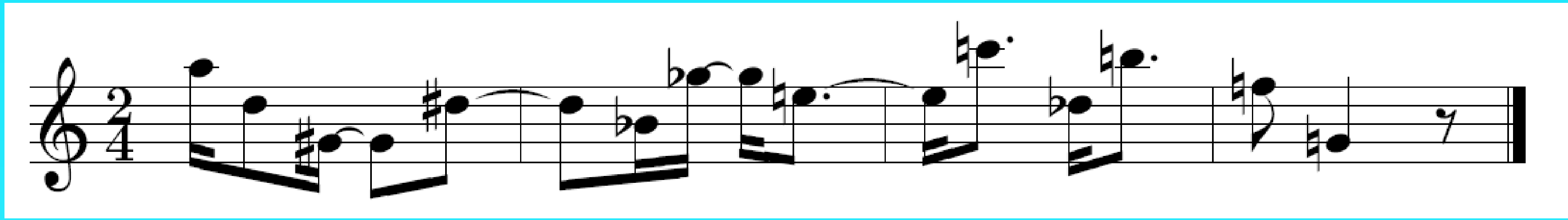

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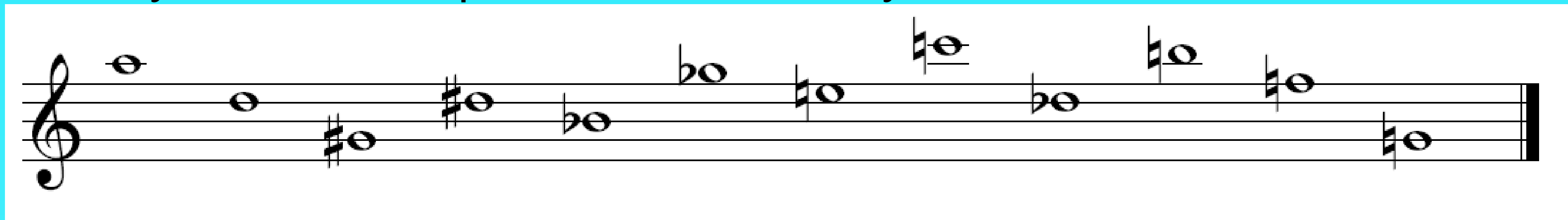

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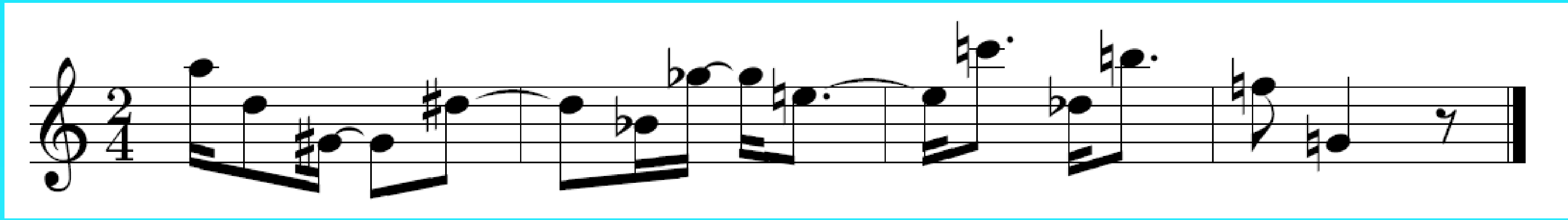

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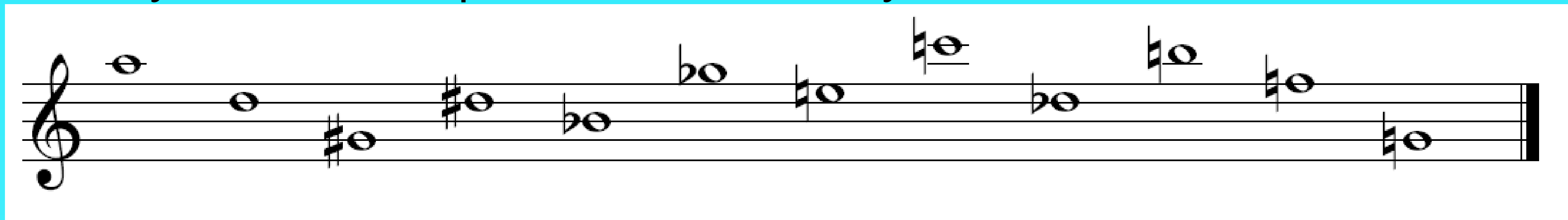
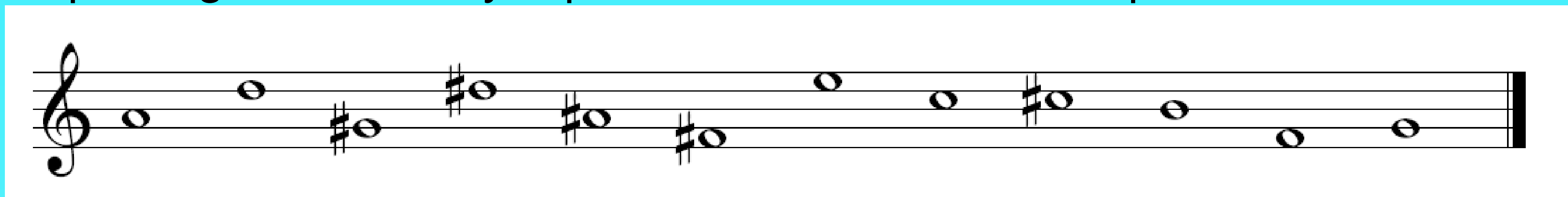


Replacing the tones by representatives of octave equivalence.



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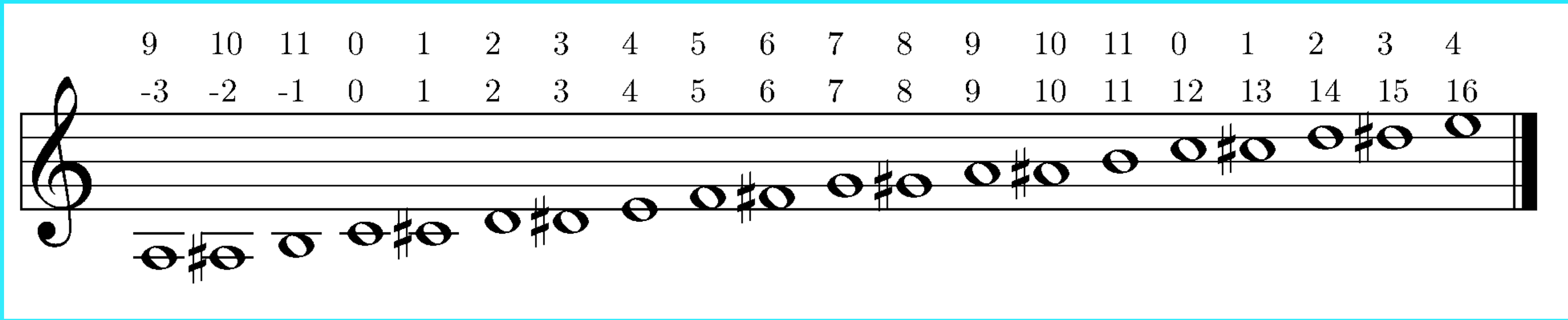
The chromatic scale



Here is a short part of the chromatic scale together with the labelling of the tones in $\mathbb{Z}$ and pitch classes in $\mathbb{Z}_{12}$.

Tone rows

In Music: a sequence of 12 tones so that different tones are not octave equivalent.

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Simple model:

$$f : \{1, \ldots, 12\} \to \mathbb{Z}_{12}, \quad \text{bijective.}$$

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(Actually the last pitch class is already determined.)

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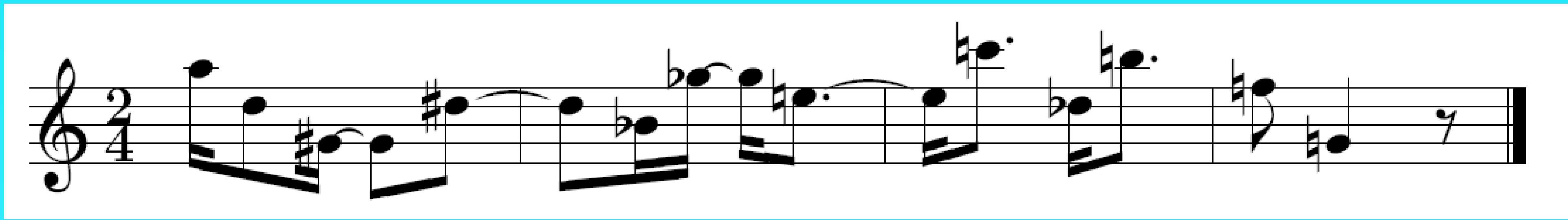
(Actually the last pitch class is already determined.)

This leads to a total of

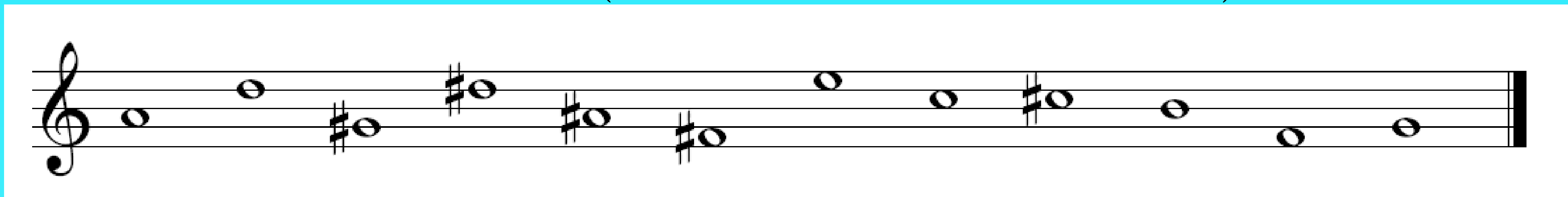
$$12 \cdot 11 \cdots 2 \cdot 1 = 479\,001\,600 = 12!$$

tone rows.

O. Messiaen: Le Merle Noir



Reduction to pitch classes $(9, 2, 8, 3, 10, 6, 4, 0, 1, 11, 5, 7)$.



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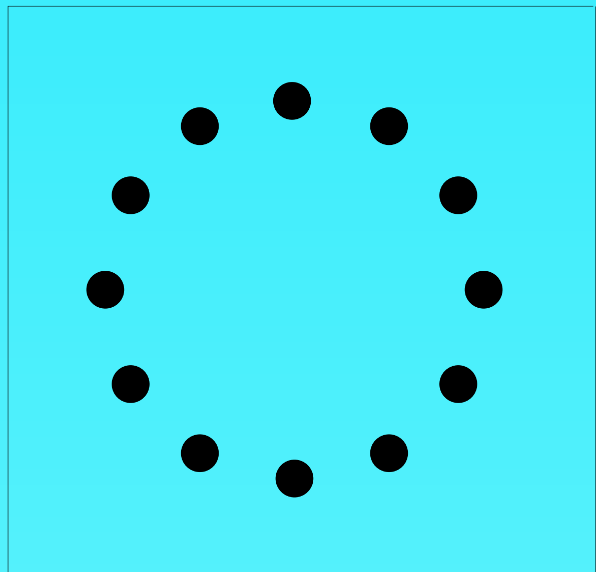

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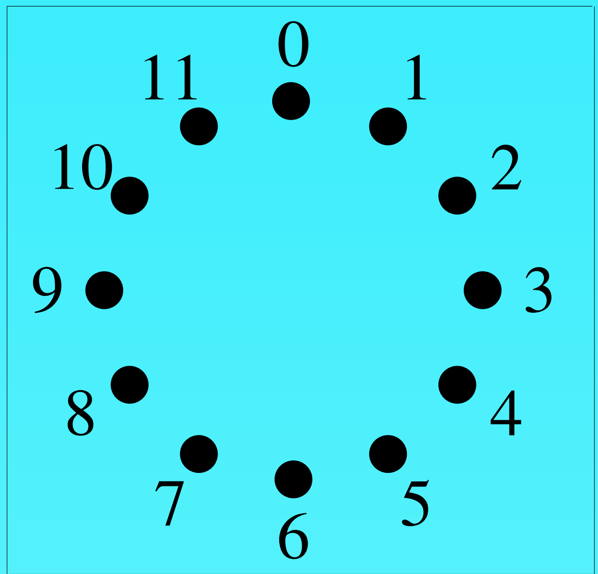

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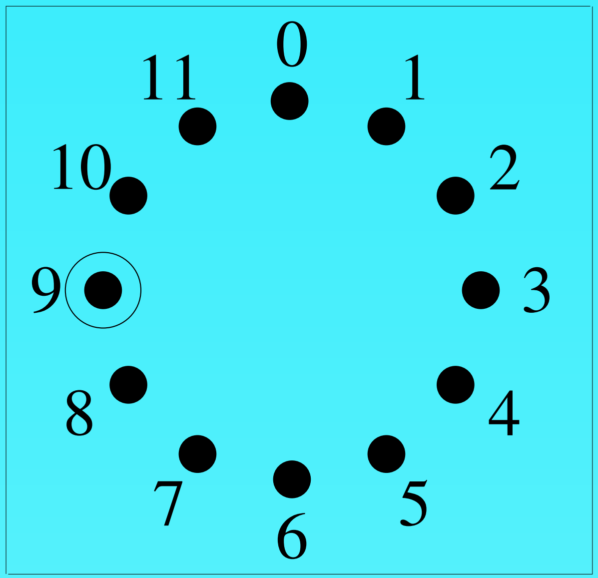

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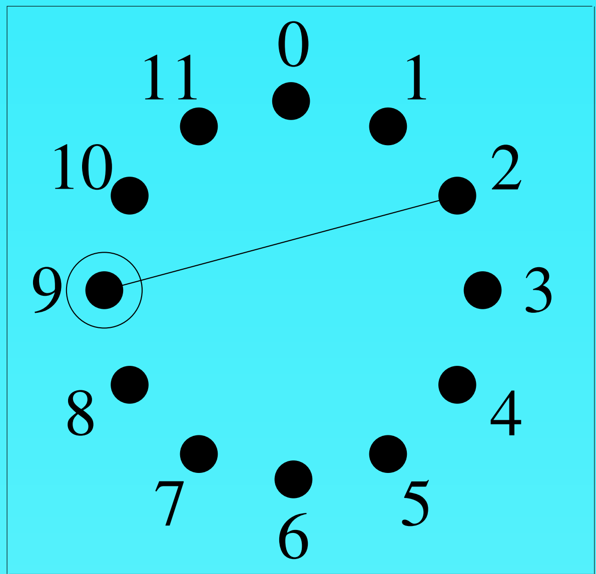

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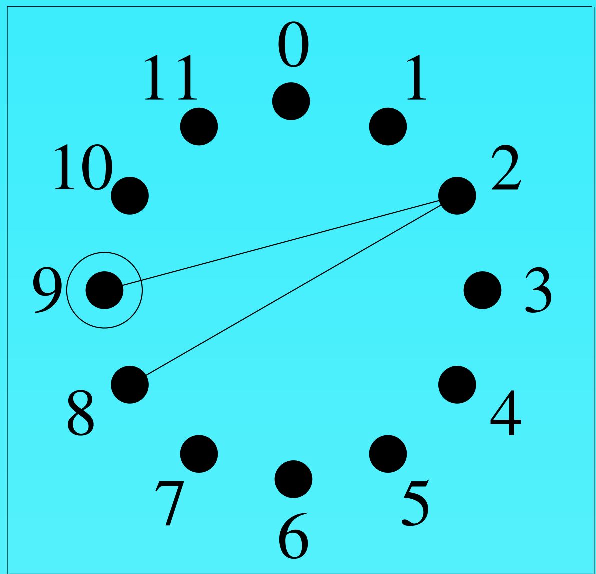

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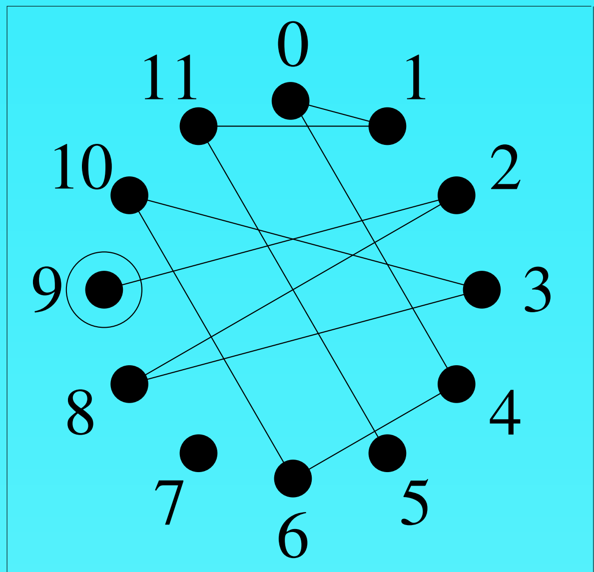

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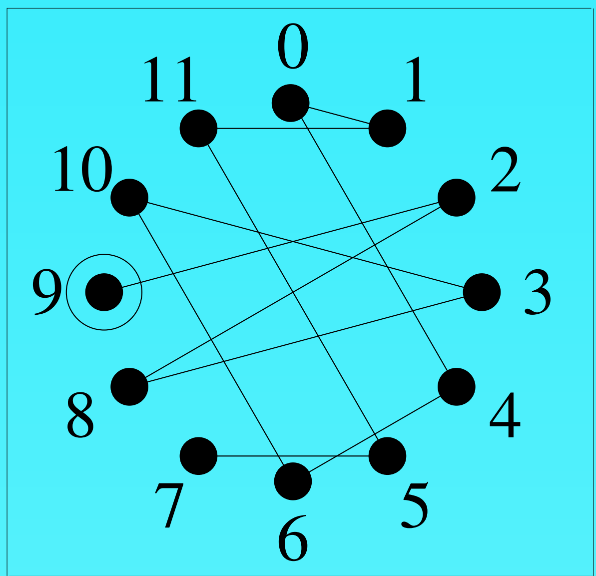

Classification of discrete structures

Discrete structures: subsets, unions, products of finite sets, mappings between finite sets, bijections or linear orders on finite sets, equivalence classes on finite sets, etc.

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Classification: more detailed information.

step 1: Determine the number of different objects.
step 2: Determine the number of objects with certain properties.
step 3: Determine a complete list of the elements of a discrete structure.

Equivalence classes of tone rows

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Reduction of the number of tone rows to the number of different similarity classes of tone rows.

Ilomäki, Tuukka. 2008. ***On the Similarity of Twelve-Tone Rows***. Vol. 30 of Studia Musica. Helsinki: Sibelius Academy.

Equivalence by Schönberg: Tone rows as equivalent whenever they can be constructed by transposing, inversion and/or retrograde from a single tone row.

Transposing

of the 12-scale: $\mathbb{Z} \longrightarrow \mathbb{Z}$, $k \mapsto k+1$, or $k \mapsto k+r$, $r \in \mathbb{Z}$.
of pitch classes: $T\colon \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$, $i \mapsto i+1$, or $T^r\colon \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$, $i \mapsto i+r$.
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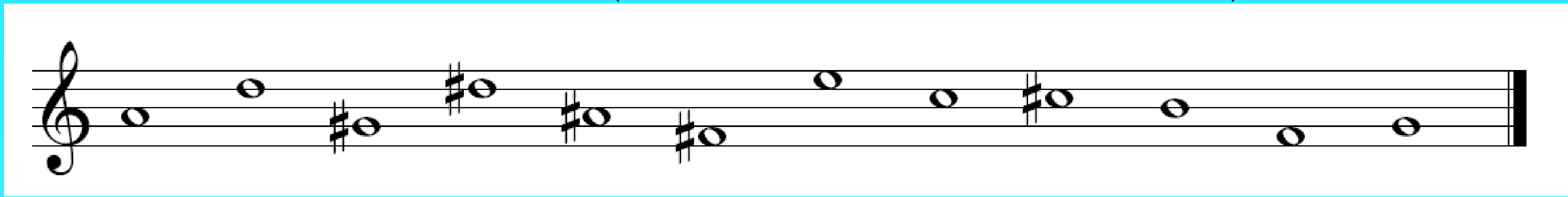
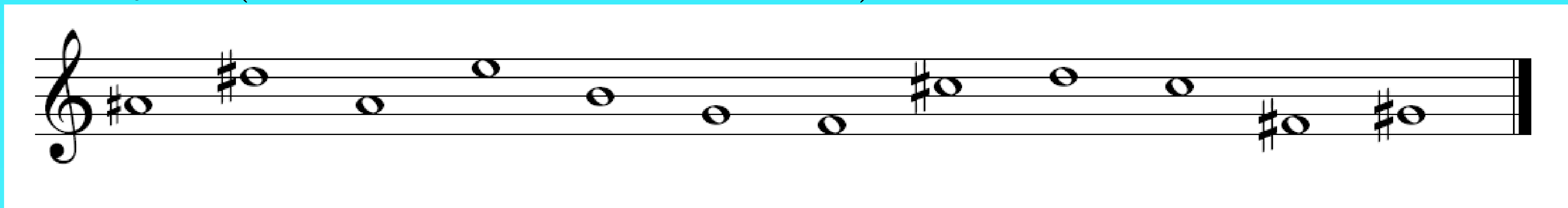
E.g. the transposed of $f = (9,2,8,3,10,6,4,0,1,11,5,7)$



is $T \circ f = (10,3,9,4,11,7,5,1,2,0,6,8)$



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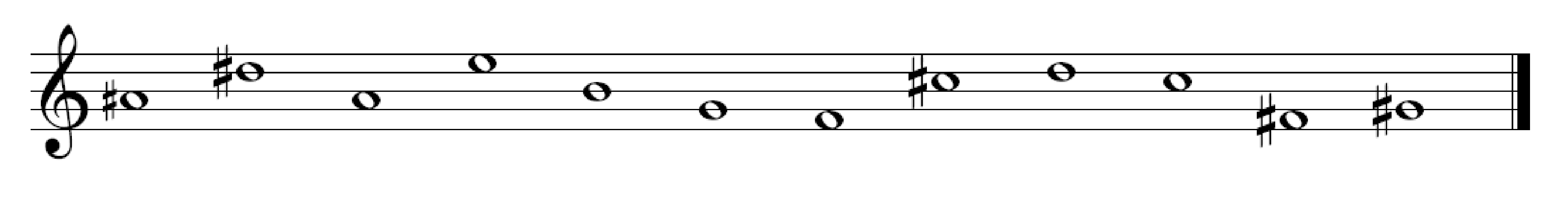
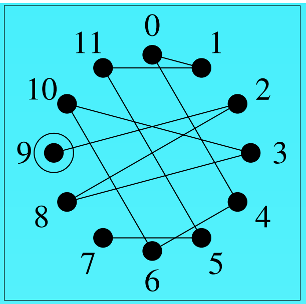
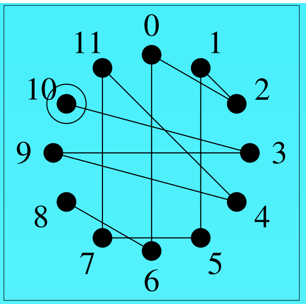


Rotation of the circular representation.





Inversion

of the 12-scale: $\mathbb{Z} \longrightarrow \mathbb{Z}$, $k \mapsto 2t_0 - k$, or $k \mapsto 2t_0 + 1 - k$, $t_0 \in \mathbb{Z}$.

of pitch classes: $I\colon \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$, $i \mapsto -i$, or $T^r \circ I\colon \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ $i \mapsto -i + r$.

of tone rows: $f \mapsto (T^r \circ I) \circ f$.

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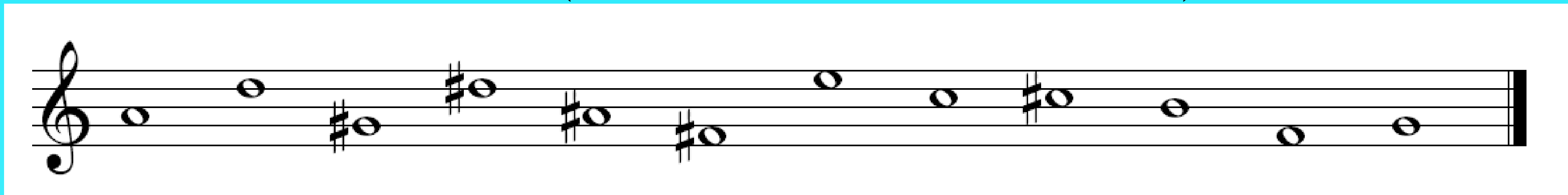

is $I \circ f = (3, 10, 4, 9, 2, 6, 8, 0, 11, 1, 7, 5)$

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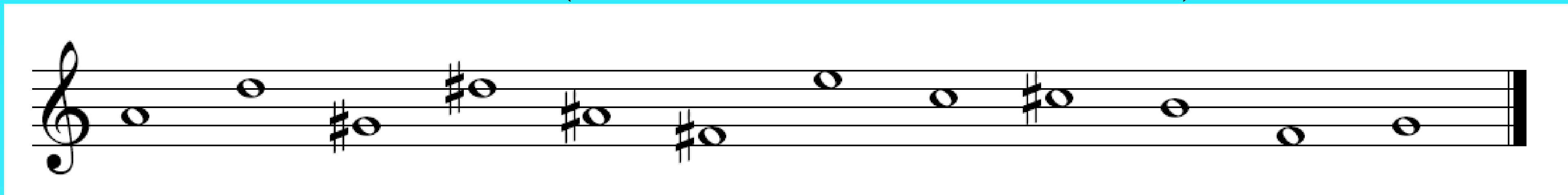

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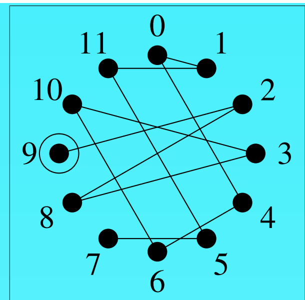

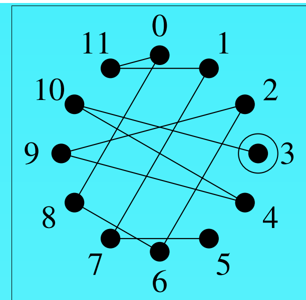

Mirror image of the circular representation.

Retrograde and cyclic shift of a tone row

Two permutations of $\{1,2,\dots,12\}$:
$R=(1,12)(2,11)\cdots(6,7)$
$S=(1,2,3,\dots,12)$.

Retrograde of f***:*** $f\circ R$
Cyclic shift of f***:*** $f\circ S$, or $f\circ S^r$.

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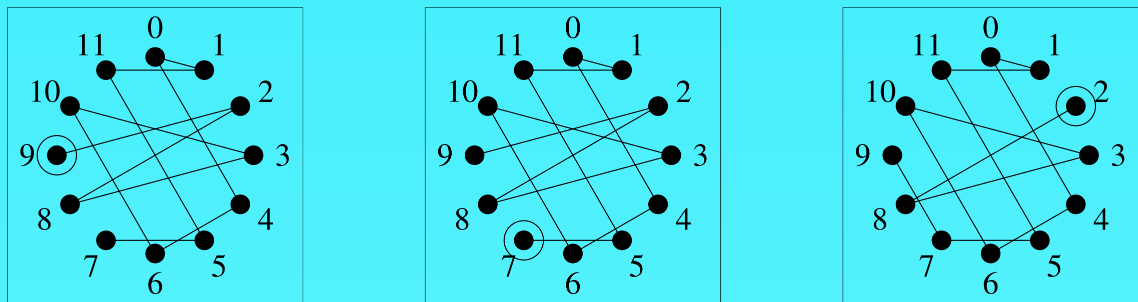
$f=(9,2,8,3,10,6,4,0,1,11,5,7)$,
$f\circ R=(7,5,11,1,0,4,6,10,3,8,2,9)$, and
$f\circ S=(2,8,3,10,6,4,0,1,11,5,7,9)$.



Further operations

Quart-circle or ***multiplication*** $Q\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$, $i \mapsto 5i$.
Five-step: Similar operation on $\{1,\ldots,12\}$.

Quart-circle of f: $Q \circ f$.
Quint-circle of f: $(I \circ Q) \circ f$.
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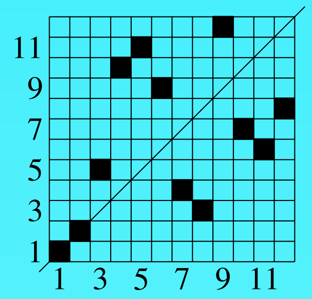
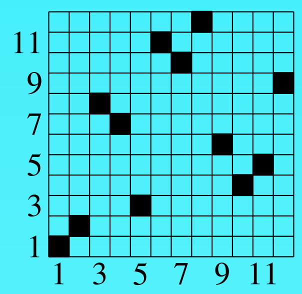
Five-step of f: $f \circ F$.

Parameter exchange: Consider f as a bijection between $\{1, \ldots, 12\}$ and $\{1, \ldots, 12\}$, i.e. $f \in S_{12}$, then f^{-1} is also a tone row.

Parameter exchange of f: $P(f) = f^{-1}$.

$M_f =$ 

$M_{f^{-1}} =$ 

Permutation groups

$\langle T \rangle$ and $\langle S \rangle$ are ***cyclic groups*** of order 12 isomorphic to C_{12}.

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Theorem. *Let π be a permutation of $\mathbb{Z}_{12}$, then $\pi(i+1) = \pi(i) + 1$ for all $i \in \mathbb{Z}_{12}$, or $\pi(i+1) = \pi(i) - 1$ for all $i \in \mathbb{Z}_{12}$, if and only if $\pi \in D_{12}$.*

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$\langle T \rangle$ and $\langle S \rangle$ are ***cyclic groups*** of order 12 isomorphic to C_{12}.

$\langle T, I \rangle$ and $\langle S, R \rangle$ are permutation groups, both isomorphic to the ***dihedral group*** D_{12} consisting of 24 elements.

Theorem. *Let π be a permutation of $\mathbb{Z}_{12}$, then $\pi(i+1) = \pi(i) + 1$ for all $i \in \mathbb{Z}_{12}$, or $\pi(i+1) = \pi(i) - 1$ for all $i \in \mathbb{Z}_{12}$, if and only if $\pi \in D_{12}$.*

$\langle T, I, Q \rangle$ and $\langle S, R, F \rangle$ are isomorphic to the group of all ***affine mappings*** on $\mathbb{Z}_{12}$ abbreviated by $\mathrm{Aff}_1(\mathbb{Z}_{12})$. It is the set of all mappings $f(i) = ai + b$, where $a, b \in \mathbb{Z}$, $\gcd(a, 12) = 1$. Thus $a \in \{1, 5, 7, 11\}$ and $b \in \{0, \ldots, 11\}$.

UNI GRAZ

Equivalent tone rows

Standard setting: f' is ***equivalent*** to f if f' can be constructed from f by any combination of transposing, inversion, cyclic shift and retrograde.

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Let $\mathscr{R}$ be the set of all tone rows, i.e. bijective mappings from $\{1,\ldots,12\}$ to $\mathbb{Z}_{12}$. We consider the following mapping

$$(\langle T,I\rangle \times \langle S,R\rangle) \times \mathscr{R} \to \mathscr{R}$$

$$((\varphi,\pi),f) \mapsto \varphi \circ f \circ \pi^{-1}.$$

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Circular representation of all equivalent rows:

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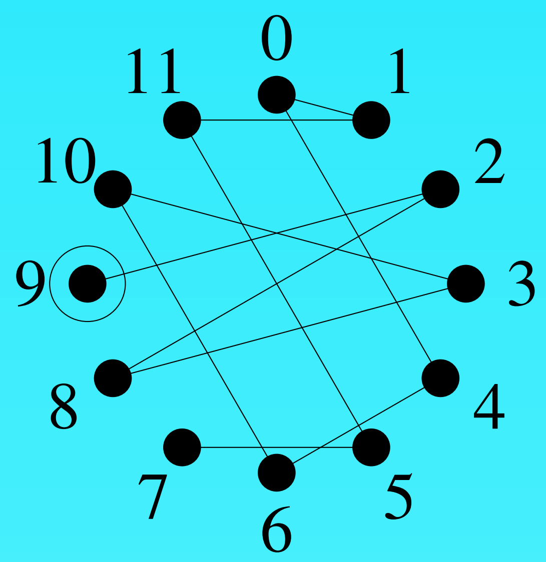
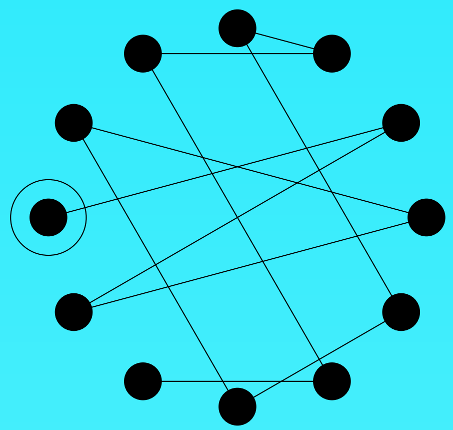
We start with f.



Circular representation of all equivalent rows:

We start with f.

Because of transposing we delete the labels.



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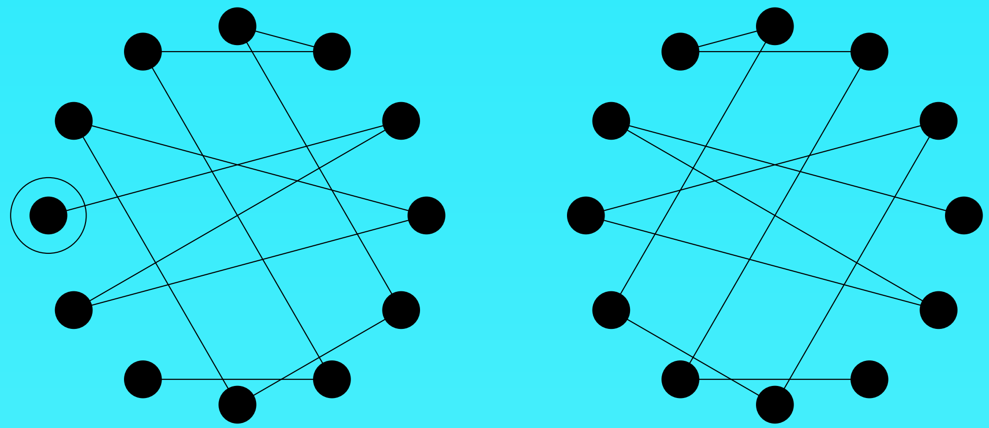
Inversion of f is the mirror of the given graph.



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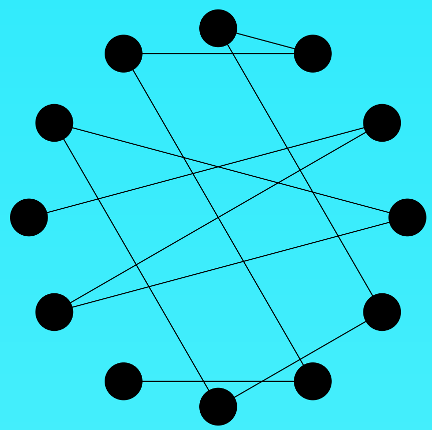
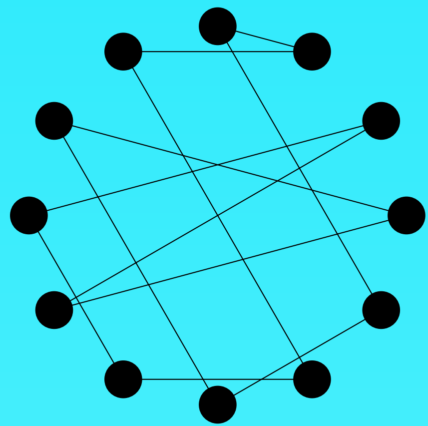
Because of retrograde we don't show the first tone.



Circular representation of all equivalent rows:

We start with f.
Because of transposing we delete the labels.
Inversion of f is the mirror of the given graph.
Because of retrograde we don't show the first tone.
Because of cyclic shifts we insert the missing edge.



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General setting:

G a permutation group on $\{1,2\ldots,12\}$,
H a permutation group on $\mathbb{Z}_{12}$.

$$(H \times G) \times \mathscr{R} \to \mathscr{R}$$

$$((\varphi,\pi),f) \mapsto \varphi \circ f \circ \pi^{-1}. \qquad (*)$$

f' is equivalent to f if and only if $f' = \varphi \circ f \circ \pi^{-1}$ for some (φ,π) in $H \times G$.

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Even more general: $\mathscr{R} = S_{12}$, $\mathfrak{G}$ a group acting on S_{12}.

$$\mathfrak{G} \times \mathscr{R} \to \mathscr{R}$$

$$(g, f) \mapsto g * f.$$

f' is equivalent to f if and only if f' is obtained from f by application of some $g \in \mathfrak{G}$.

Group Actions

A multiplicative group G with neutral element 1 acts on a set X if there exists a mapping

$$*\colon G \times X \to X, \qquad *(g,x) \mapsto g * x (= gx),$$

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Notation: We usually write gx instead of $g * x$.

Orbit of $x \in X$: $G(x) = \{gx \mid g \in G\}$.

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$$|G\backslash\backslash X| = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

Classification of tone rows 1st step

How many non-equivalent tone rows?

Classification of tone rows 1st step

How many non-equivalent tone rows? How many orbits are there?

Classification of tone rows 1st step

How many non-equivalent tone rows? How many orbits are there?

	acting group	# of orbits
1.	$\langle T\rangle \times \langle R\rangle$	19 960 320
2.	$\langle T,I\rangle \times \langle R\rangle$	9 985 920
3.	$\langle T\rangle \times \langle S\rangle$	3 326 788
4.	$\langle T,I\rangle \times \langle S\rangle$	1 664 354
5.	$\langle T,I\rangle \times \langle S,R\rangle$	836 017
6.	$\langle T,I,Q\rangle \times \langle S,R\rangle$	419 413
7.	$\langle T,I\rangle \times \langle S,R,F\rangle$	419 413
8.	$\langle T,I,Q\rangle \times \langle S,R,F\rangle$	211 012
9.	$\mathfrak{D}_{12}$	420 948
10.	$\mathfrak{A}_{12}$	106 986

$\mathfrak{D}_{12}$ generated by $D_{12} \times D_{12}$ and P.
$\mathfrak{A}_{12}$ generated by $\mathrm{Aff}_1(\mathbb{Z}_{12}) \times \mathrm{Aff}_1(\mathbb{Z}_{12})$ and P.

Arnold Schönberg considered 2.

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5. is the standard settings for our classification.

Also used by:

Josef Matthias Hauer

Ronald C. Read. Combinatorial problems in the theory of music. *Discrete Mathematics*, 167-168(1-3):543–551, 1997.

Solomon Wolf Golomb and ***Lloyd Richard Welch***. On the enumeration of polygons. *American Mathematical Monthly*, 87:349–353, 1960.

David J. Hunter and ***Paul T. von Hippel***. How rare is symmetry in musical 12-tone rows? *American Mathematical Monthly*, 110(2):124–132, 2003.

The orbit of a tone row

In connection with the orbit of f we solve the following problems:

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- Determine the ***standard representative*** or the ***normal form*** of the orbit $G(f)$ as the smallest element in $G(f)$ with respect to the lexicographical order
- Given two tone rows f_1 and f_2 belonging to the same orbit, determine an element $g \in G$ so that $f_2 = g f_1$.

Database: The equivalence class of (9,2,8,3,10,6,4,0,1,11,5,7)
Database: The normal form (9,2,8,3,10,6,4,0,1,11,5,7)

Properties of tone rows

- Normalform

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- Interval structure

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- Invariance of the orbit under P

Interval structure

Interval from a to b in $\mathbb{Z}_{12}$: the difference $b - a \in \mathbb{Z}_{12}$.

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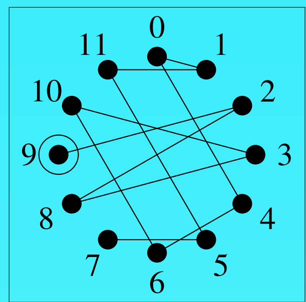
Interval structure

Interval from a to b in $\mathbb{Z}_{12}$: the difference $b-a \in \mathbb{Z}_{12}$.
Sequence of twelve intervals:

$$g = (f(2)-f(1), f(3)-f(2), \ldots, f(12)-f(11), f(1)-f(12))$$

Its orbit under $D_{12} \times D_{12}$ is the ***interval structure***.

The interval structure of $f = (9,2,8,3,10,6,4,0,1,11,5,7)$



is the $D_{12} \times D_{12}$-orbit of $(5,6,7,7,8,10,8,1,10,6,2,2)$ represented by $(1,8,10,8,7,7,6,5,2,2,6,10)$.
Database: Search for tone rows of given interval structure

The stabilizer type of a tone row

f a tone row, G a group describing the equivalence classes of tone rows.

$G_f = \{g \in G \mid g * f = f\}$ the stabilizer.

$G(f) = \{g * f \mid g \in G\}$ the orbit. Its size is $|G| / |G_f|$.

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Stabilizer type of the orbit $G(f)$ is the conjugacy class $\tilde{G}_f = \{gG_f g^{-1} \mid g \in G\}$.

In our standard situation we have ***17 different stabilizer types***.

name	generators	$\lvert U \rvert$	$\lvert \tilde{U} \rvert$	$\lvert \tilde{U}_i \backslash\backslash \mathscr{R} \rvert$
$\tilde{U}_1$	identity	1	1	827 282
$\tilde{U}_2$	(TI, S^6)	2	6	912
$\tilde{U}_3$	(T^6, R)	2	6	912
$\tilde{U}_4$	(T^6, S^6)	2	1	130
$\tilde{U}_5$	(I, SR)	2	36	942
$\tilde{U}_6$	(TI, R)	2	36	5 649
$\tilde{U}_7$	(T^4, S^4)	3	2	11
$\tilde{U}_8$	(T^3, S^3)	4	2	2
$\tilde{U}_9$	$(TI, S^6), (T^6, R)$	4	36	96
$\tilde{U}_{10}$	$(I, SR), (T^6, S^6)$	4	18	12
$\tilde{U}_{11}$	$(TI, R), (T^6, S^6)$	4	18	42
$\tilde{U}_{12}$	$(I, SR), (T^4, S^4)$	6	24	2
$\tilde{U}_{13}$	$(TI, R), (T^4, S^4)$	6	24	15
$\tilde{U}_{14}$	$(I, SR), (T^3, S^3)$	8	36	6
$\tilde{U}_{15}$	$(TI, R), (T^2, S^2)$	12	12	2
$\tilde{U}_{16}$	$(I, SR), (T, S)$	24	12	1
$\tilde{U}_{17}$	$(I, SR), (T, S^5)$	24	12	1

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Examples:

- Consider $f = (0, 1, 10, 8, 9, 11, 5, 3, 2, 4, 7, 6)$, then $T^6 f = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0)$ and $fR = (6, 7, 4, 2, 3, 5, 11, 9, 8, 10, 1, 0)$, thus $(T^6, R) * f = f$.

Examples:

- Consider $f = (0,1,10,8,9,11,5,3,2,4,7,6)$,
  then $T^6 f = (6,7,4,2,3,5,11,9,8,10,1,0)$
  and $fR = (6,7,4,2,3,5,11,9,8,10,1,0)$,
  thus $(T^6,R) * f = f$.

- Consider $f = (5,4,0,7,3,2,8,9,1,6,10,11)$,
  then $TIf = (8,9,1,6,10,11,5,4,0,7,3,2) = fS^6$
  and $T^6 f = (11,10,6,1,9,8,2,3,7,0,4,5) = fR$.
  The stabilizer of f is $\{\mathrm{id}, (TI,S^6), (T^6,R), (T^7I,S^6R)\}$.

Examples:

- Consider $f = (0,1,10,8,9,11,5,3,2,4,7,6)$,
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- The chromatic scale has a stabilizer of order 24.

  Database: The stabilizer of the chromatic scale

Examples:

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- The chromatic scale has a stabilizer of order 24.

  Database: The stabilizer of the chromatic scale

- 99.93% of all orbits of tone rows have the trivial stabilizer type.

Database: Search for tone rows of given stabilizer type

Hexachords and tropes

Hexachord: a 6-subset of $\mathbb{Z}_{12}$.

Trope: a 2-set $\{A, \mathbb{Z}_{12} \setminus A\}$, A a hexachord.

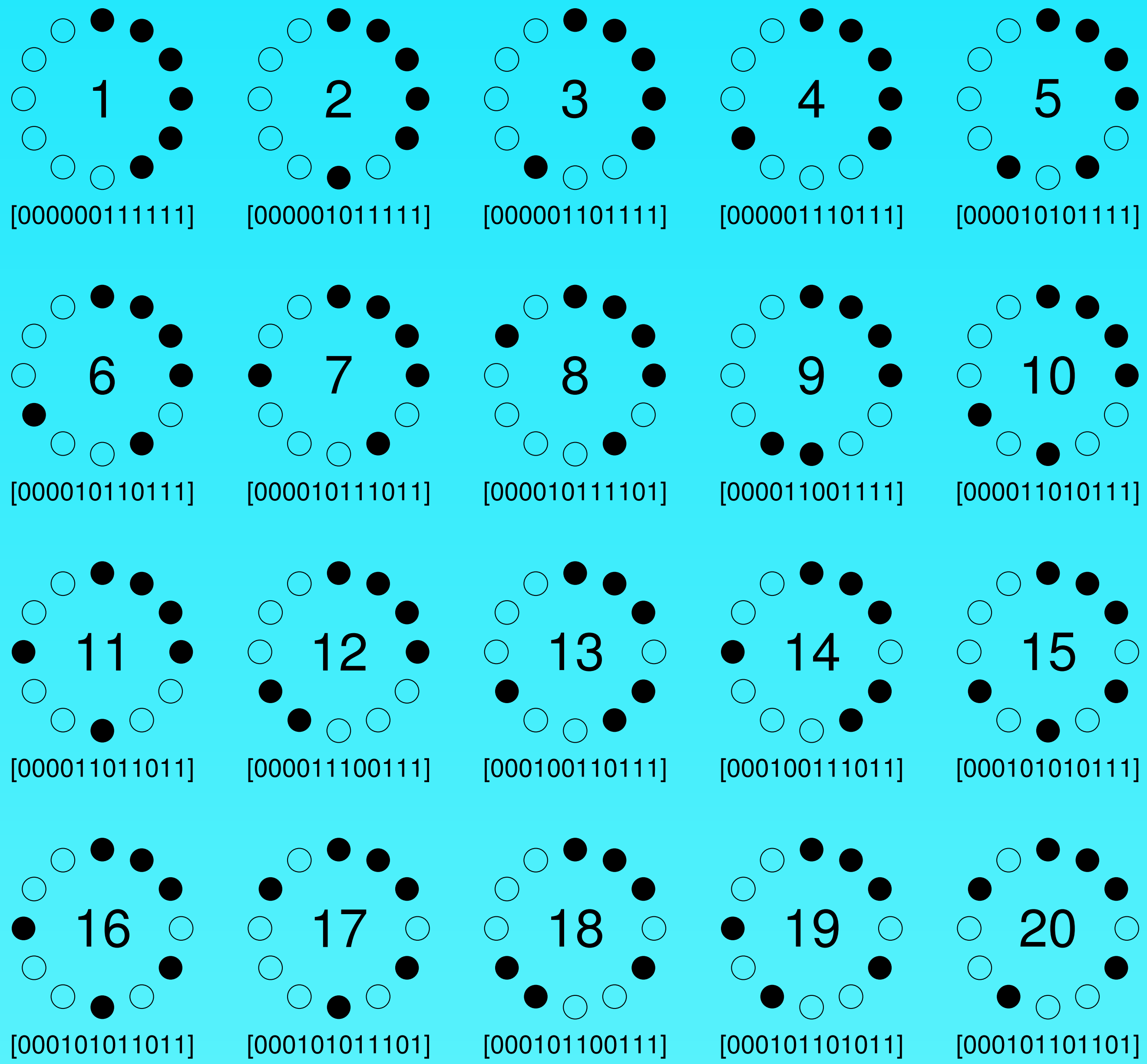
If a group G acts on $\mathbb{Z}_{12}$, then it also acts on the set of all hexachords $\mathscr{H}$, and on the set of all tropes $\mathscr{T}$.

G	$\lvert G\backslash\backslash\mathscr{H}\rvert$
$\langle 1 \rangle$	924
C_{12}	80
D_{12}	50
$\mathrm{Aff}_1(\mathbb{Z}_{12})$	34

G	$\lvert G\backslash\backslash\mathscr{T}\rvert$
$\langle 1 \rangle$	462
C_{12}	44
D_{12}	35
$\mathrm{Aff}_1(\mathbb{Z}_{12})$	26

Database: List all k-chords   List all tropes

List of all 35 orbits of tropes



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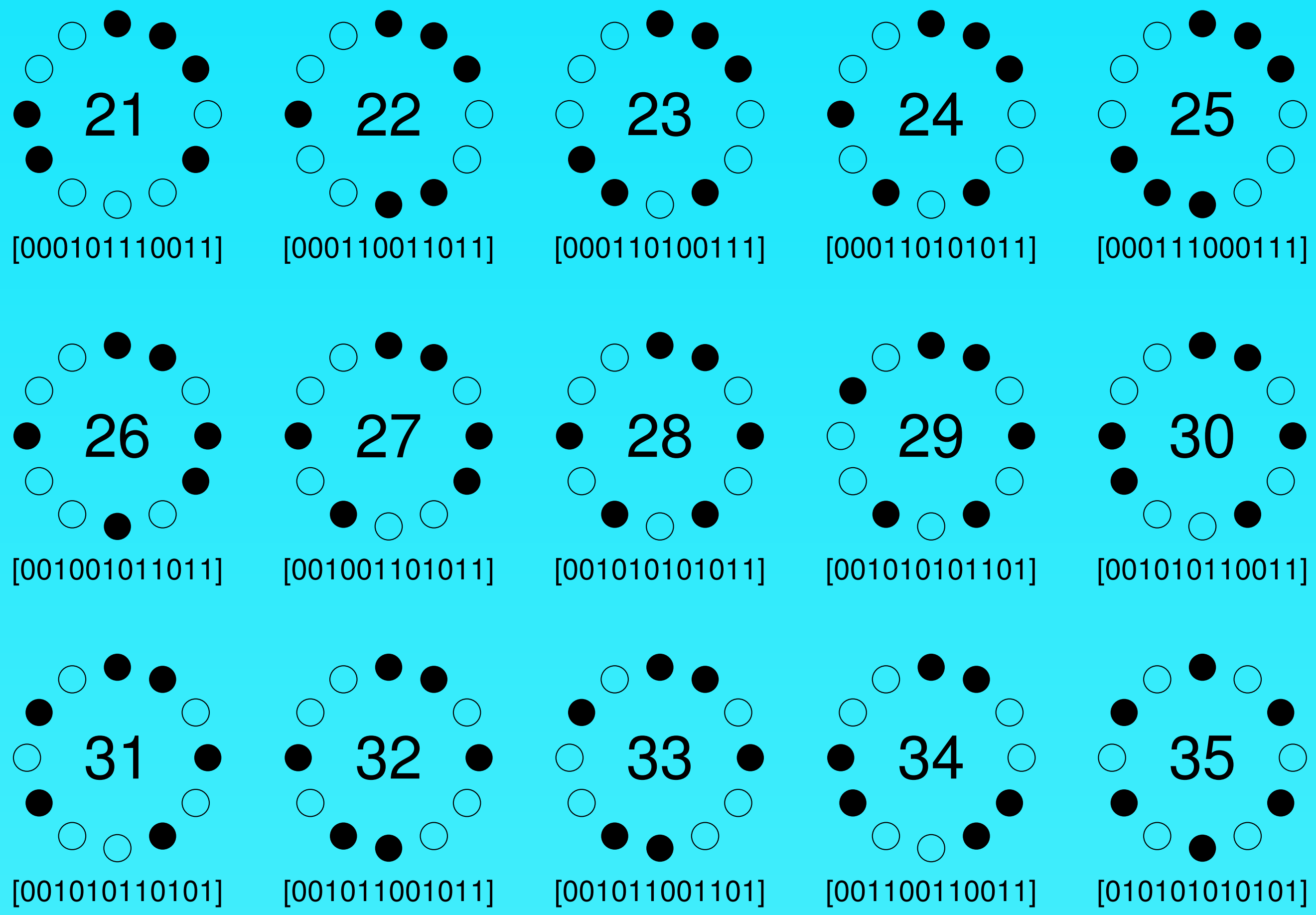

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Trope structure of a tone row

Consider a tone row f. We obtain in a natural way six 2-sets of hexachords defined by f:

$$
\begin{aligned}
\tau_1 &:= \{\{f(1),f(2),f(3),f(4),f(5),f(6)\},\{f(7),f(8),f(9),f(10),f(11),f(12)\}\}\\
\tau_2 &:= \{\{f(2),f(3),f(4),f(5),f(6),f(7)\},\{f(8),f(9),f(10),f(11),f(12),f(1)\}\}\\
\tau_3 &:= \{\{f(3),f(4),f(5),f(6),f(7),f(8)\},\{f(9),f(10),f(11),f(12),f(1),f(2)\}\}\\
\tau_4 &:= \{\{f(4),f(5),f(6),f(7),f(8),f(9)\},\{f(10),f(11),f(12),f(1),f(2),f(3)\}\}\\
\tau_5 &:= \{\{f(5),f(6),f(7),f(8),f(9),f(10)\},\{f(11),f(12),f(1),f(2),f(3),f(4)\}\}\\
\tau_6 &:= \{\{f(6),f(7),f(8),f(9),f(10),f(11)\},\{f(12),f(1),f(2),f(3),f(4),f(5)\}\}
\end{aligned}
$$

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Trope sequence: $(\tau_1, \tau_2, \ldots, \tau_6)$

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Database: 538 139 different trope structures
(1,1,1,1,1,1): exactly one tone row
(10,18,22,14,22,18) and (10,18,22,14,22,27): 48 different orbits of tone rows

Database: Compute the trope structure or Search for the trope structure

Which sequences of trope numbers are the trope number sequence of a tone row?

Further results

Close connection between the stabilizer type of a tone row and its trope structure.

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Tone rows and tropes by H.F. and Peter Lackner will appear in a special volume of the ***Journal of Mathematics and Music*** in 2015.

The database

Available from http://www.uni-graz.at/~fripert/db/

Data were computed with SYMMETRICA [2] and GAP [1].

Search routines using perl. Graphics uses javascript.

Musical information on tone rows appearing in works of various composers. Hence it is also possible to search for musical information on a given tone row. This opens the door for new research: Since we have normal forms of tone rows, it is easy to check, whether similar tone rows appeared in different compositions. Or knowing certain properties of tone rows it is interesting to study whether we can deduce from the composition that the composer was aware of these properties.

As a matter of fact, at the moment we have more than 1200 entries of musical information in our database. Of course this is not enough for doing some statistical analysis or to suggest trends in the usage of certain types of tone rows. Therefore, we try to collect further tone rows and data.

There are more than 500 entries with tone rows by J. M. Hauer. All tone rows from the Second Viennese School and a selection of compositions until today are input.

References

[1] M. Schönert et al. *GAP – Groups, Algorithms, and Programming*. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995.

[2] SYMMETRICA. A program system devoted to representation theory, invariant theory and combinatorics of finite symmetric groups and related classes of groups. Copyright by "Lehrstuhl II für Mathematik, Universität Bayreuth, 95440 Bayreuth".

`http://www.algorithm.uni-bayreuth.de/en/research/SYMMETRICA/`.

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