Generalized Stirling Permutations and Iteration Theory

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Joint work with Wojciech Jabłoński

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Using generalized Stirling permutations, I will present a combinatorial proof that a certain family of power series suggested by Wojciech Jabłoński is indeed a two-parameter iteration group.

Iteration Groups

$\mathbb{K}$ be a field of characteristic 0,
$(G,*)$ a group,
$\mathbb{K}[[x]]$ the ring of formal powerseries $f(x)=\sum_{n\geq 0} a_n x^n$, $a_n \in \mathbb{K}$,
$\Gamma := \{g(x) \in \mathbb{K}[[x]] \mid g(x) = \sum_{n\geq 1} b_n x^n, \}$,
for $f \in \mathbb{K}[[x]]$, $g \in \Gamma$, the composition $(f \circ g) = \sum_{n\geq 0} a_n [g(x)]^n$.

An iteration group is a family $(F_g(x))_{g\in G}$ of elements in Γ such that

$$F_{g*g'}(x) = (F_g \circ F_{g'})(x), \text{ or } F(g*g',x) = F(g,F(g',x)), \quad g,g' \in G,$$

where $F(g,x) = F_g(x)$. Thus it is a solution of the translation equation.

E.g. $\mathbb{K} = \mathbb{C}$ and $G = (\mathbb{C},+)$ was studied by L. Reich.

Generalized Factorial

$\ell \geq 1$ a positive integer,
generalized factorial

$$(k\ell+1)!_\ell := \prod_{j=0}^{k} (j\ell+1) = 1 \cdot (\ell+1) \cdots (k\ell+1), \qquad k \in \mathbb{Z},$$

if $k < 0$, then $(k\ell+1)!_\ell = 1$, the empty product,
if $\ell = 1$, then $(k\ell+1)!_\ell = (k+1)!$.

A non-commutative two-parameter Iteration Group

W. Jabłoński studies iteration groups where G is different from $\mathbb{K}$.

$\ell \geq 1$ a positive integer,
$G := (\mathbb{K}^\times \times \mathbb{K}, \diamond)$,

$$(y_1, y_2) \diamond (z_1, z_2) = (y_1 z_1, y_1 z_2 + y_2 z_1^{\ell+1}), \qquad (y_1, y_2), (z_1, z_2) \in G$$

G is not commutative since $\ell + 1 \geq 2$.

W.J. proves in

> ***An explicit example of an iteration group in the ring of formal power series***, in Aequationes Mathematicae **98**, Nr. 3, 837–850, (2024)

the following theorem.

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Theorem 1

The family $\left(F^{(\ell)}_{(z_1,z_2)}(x)\right)_{(z_1,z_2)\in\mathbb{K}^\times\times\mathbb{K}}$ given by

$$F^{(\ell)}_{(z_1,z_2)}(x) = \sum_{n\geq 0}\left(\frac{((n-1)\ell+1)!_\ell}{n!}\cdot\frac{z_2^n}{z_1^{n-1}}\right)x^{nl+1}, \quad (z_1,z_2)\in\mathbb{K}^\times\times\mathbb{K}, \tag{1}$$

is a non-commutative, two-parameter iteration group in $\mathbb{K}[[x]]$ if and only if

$$\frac{((n-1)\ell+1)!_\ell}{(n-k)!((k-1)\ell+1)!_\ell} = \sum_{c=(c_1,\ldots,c_{k\ell+1})}\prod_{j=1}^{k\ell+1}\frac{((c_j-1)\ell+1)!_\ell}{c_j!} \tag{2}$$

holds true for all $n\in\mathbb{N}$ and $0\leq k\leq n$, where c is a composition of $n-k$, i.e. $c_1+\cdots+c_{k\ell+1}=n-k$ and all c_i are non-negative integers.

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$\ell=1$: equation (2) is true.

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$\ell=1$: equation (2) is true. $\ell>1$: direct computations show that (2) is true for any n and $k=0$ or $n-4\leq k\leq n$. (The proof for $k=n-5$ is similar to but more complicated than for $k=n-4$.)

Stirling Permutations

$n \in \{1, 2, 3, \ldots\}$

Permutation of order** n**: any sequence of length n which contains each element of $\{1, \ldots, n\}$ exactly once.

E.g. for $n = 3$: 123, 132, 213, 231, 312, 321

for $n = 0$: (), the empty word

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Stirling permutation of order n**:** any sequence $(v_1, \ldots, v_{2n})$ of length $2n$ which contains each element of $\{1, \ldots, n\}$ exactly twice, ***such that for all*** $1 \leq i \leq j \leq k \leq 2n$ ***if*** $v_i = v_k$ ***then*** $v_j \geq v_i$**.** Between two occurrences of the number $r \in \{1, \ldots, n\}$ only numbers $s > r$ may appear.

E.g. for $n = 3$: 112233, 133122, 123321, 233211, ...
however 213312, 212133, 312213, 112331 are ***not*** Stirling permutations.

$\ell \geq 1$ a positive integer,

ℓ-Stirling permutation of order n: any sequence $(v_1, \ldots, v_{\ell n})$ of length ℓn which contains each element of $\{1, \ldots, n\}$ exactly ℓ times, ***such that for all*** $1 \leq i \leq j \leq k \leq \ell n$ ***if*** $v_i = v_k$ ***then*** $v_j \geq v_i$**.** Between two occurrences of the number $r \in \{1, \ldots, n\}$ only numbers $s \geq r$ may appear.

E.g. for $\ell = 3$, $n = 3$: 222333111, 133312221, 333112221, 112333221, …

The element n occurs always as a block of length ℓ.

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For $\ell = 1$ we obtain permutations of order n.

For $\ell = 2$ we obtain Stirling-permutations of order n.

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If $i + 1 < k$, $v_i = v_k$, and $v_i \neq v_j$ for all $j \in \{i+1, \ldots, k-1\}$, then $k - i = m\ell + 1$, where m is the number of different values occurring between v_i and v_k. If there exists $j \in \{i+1, \ldots, k-1\}$ such that $v_j = s > v_i$, then all ℓ occurrences of s lie between v_i and v_k.

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$n = 2$: Take the only ℓ-Stirling permutation of order 1 and insert the block 2^ℓ at the beginning, or in between any two 1s or at the end of the block 1^ℓ.

$$2^\ell 1^\ell,\ 12^\ell 1^{\ell-1},\ 112^\ell 1^{\ell-2},\ \dots,\ 1^{\ell-1}2^\ell 1,\ 1^\ell 2^\ell.$$

E.g for $\ell = 4$ and $n = 2$:
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E.g for $\ell = 4$ and $n = 2$:
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$n > 2$: Take any ℓ-Stirling permutation v of order $n-1$ and insert the block n^ℓ at the beginning, or in between any two entries of v or at the end of v. They all will be pairwise different and each ℓ-Stirling permutation v of order n can be obtained in this way.

Theorem 2
The number of ℓ-Stirling permutations of order n is

$$\prod_{j=0}^{n-1}(j\ell+1) = ((n-1)\ell+1)!_\ell.$$

For $n = 0, 1$ the formula is true. By the induction hypothesis the number of ℓ-Stirling permutations of order $n-1$ is $((n-2)\ell+1)!_\ell$. The block n^ℓ can be inserted in $(n-1)\ell+1$ ways.

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On any totally ordered set we can construct ℓ-Stirling permutations, e.g. on subsets A of $\{1,\dots,n\}$. Let $S^{(\ell)}(A)$ be the set of all ℓ-Stirling permutations with entries from A. Then

$$|S^{(\ell)}(A)|=((|A|-1)\ell+1)!_\ell.$$

Combinatorial Proof of (2)

(2) is equivalent to

$$((n-1)\ell+1)!_\ell = (n-k)!((k-1)\ell+1)!_\ell \sum_c \prod_{j=1}^{k\ell+1} \frac{((c_j-1)\ell+1)!_\ell}{c_j!}$$

$$= ((k-1)\ell+1)!_\ell \sum_c \binom{n-k}{c_1 \dots c_{k\ell+1}} \prod_{j=1}^{k\ell+1} ((c_j-1)\ell+1)!_\ell .$$

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Let

$$\mathscr{A} = \{(A_1,\dots,A_{k\ell+1}) \mid A_j \subseteq \{k+1,\dots,n\}, \dot\bigcup_{j=1}^{k\ell+1} A_j = \{k+1,\dots,n\}\}.$$

We show that

$$\left|S^{(\ell)}(\{1,\dots,n\})\right| = \left| \dot\bigcup_{(A_1,\dots,A_{k\ell+1})\in\mathscr{A}} S^{(\ell)}(\{1,\dots,k\}) \times \prod_{j=1}^{k\ell+1} S^{(\ell)}(A_j) \right|.$$

$$\Phi\colon S^{(\ell)}(\{1,\dots,n\}) \to \dot{\bigcup_{A=(A_1,\dots,A_{k\ell+1})\in\mathscr{A}}} S^{(\ell)}(\{1,\dots,k\}) \times \{A\} \times \prod_{j=1}^{k\ell+1} S^{(\ell)}(A_j)$$

$$\Phi(v) = (v\downarrow_{\{1,\dots,k\}}, (A_1,\dots,A_{k\ell+1}), (v\downarrow_{A_1},\dots,v\downarrow_{A_{k\ell+1}}).$$

The restriction $v' := v\downarrow_{\{1,\dots,k\}}$ belongs to $S^{(\ell)}(\{1,\dots,k\})$. It consists of all entries of v which are $\le k$.

A_1 is the set of elements of v occurring in front of the first element of v'.

A_2 is the set of elements of v occurring between the first and second element of v'.

$A_{k\ell+1}$ is the set of elements of v occurring after the last element of v'.

The subsequence of v occurring in front of the first element of v' is an ℓ-Stirling permutation, thus it belongs to $S^{(\ell)}(A_1)$.

The subsequence of v occurring between the first and second element of v' is an element of $S^{(\ell)}(A_2)$.

The subsequence of v occurring after the last element of v' is an element of $S^{(\ell)}(A_{k\ell+1})$.

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For example $\ell = 3$, $n = 5$, $k = 2$.

$$v = 555222113344431.$$

Then $v' = v\downarrow_{\{1,2\}}$=222111.

$A_1 = \{5\}$, $A_2 = A_3 = A_4 = A_5 = \emptyset$, $A_6 = \{3,4\}$, $A_7 = \emptyset$.

The A_i are pairwise disjoint, and their union gives $\{3,4,5\}$.

$v\downarrow_{A_1} = 555$, $v\downarrow_{A_6} = 334443$, $v\downarrow_{A_j} = ()$ for $j \in \{2,3,4,5,7\}$.

Thus $\Phi(v) =$

$$\Big(222111, \big(\{5\}, \emptyset, \emptyset, \emptyset, \emptyset, \{3,4\}, \emptyset\big), \big(555, (), (), (), (), 334443, ()\big)\Big).$$

We show that Φ is bijective by proving that Ψ is its inverse.

$$\Psi\colon \dot{\bigcup_{A=(A_1,\dots,A_{k\ell+1})\in\mathscr{A}}} S^{(\ell)}(\{1,\dots,k\}) \times \{A\} \times \prod_{j=1}^{k\ell+1} S^{(\ell)}(A_j) \to S^{(\ell)}(\{1,\dots,n\})$$

$$\Psi\big(w_1\dots w_{k\ell},(A_1,\dots,A_{k\ell+1}),(z_1,\dots,z_{k\ell+1})\big)$$

is the concatenation of

$$z_1\|w_1\|z_2\|w_2\|\dots\|z_{k\ell}\|w_{k\ell}\|z_{k\ell+1}. \tag{4}$$

By construction each element of $\{1,\dots,n\}$ occurs exactly ℓ times in (4).
$r>k$: there exists exactly one j so that $r\in A_j$. Between two occurrences of r in z_j only numbers $\geq r$ appear, since $z_j\in S^{(\ell)}(A_j)$.
$r\leq k$: If $s<r$, then all occurrences of r and s lie in $w_1\dots w_{k\ell}$, hence no s occurs between two occurrences of r.
Ψ is well defined.
$\Phi\circ\Psi=\mathrm{id}$ and $\Psi\circ\Phi=\mathrm{id}$.

$$
\begin{aligned}
((n-1)\ell+1)!_\ell &= \left|S^{(\ell)}(\{1,\dots,n\})\right| \\
&= \left| \dot{\bigcup_{(A_1,\dots,A_{k\ell+1})\in\mathscr{A}}} S^{(\ell)}(\{1,\dots,k\}) \times \prod_{j=1}^{k\ell+1} S^{(\ell)}(A_j) \right| \\
&= \sum_{(A_1,\dots,A_{k\ell+1})\in\mathscr{A}} \left|S^{(\ell)}(\{1,\dots,k\})\right| \prod_{j=1}^{k\ell+1} \left|S^{(\ell)}(A_j)\right| \\
&= ((k-1)\ell+1)!_\ell \sum_c \binom{n-k}{c_1\dots c_{k\ell+1}} \prod_{j=1}^{k\ell+1} ((c_j-1)\ell+1)!_\ell
\end{aligned}
$$

where the sum is taken over all compositions $c=(c_1,\dots,c_{k\ell+1})$ of $n-k$ into $k\ell+1$ summands, since

$$
|\{(A_1,\dots,A_{k\ell})\in\mathscr{A} : |A_j|=c_j,\ j=1,\dots,k\ell+1\}| = \binom{n-k}{c_1\dots c_{k\ell+1}}
$$

and (3) is established.

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