On n-associative formal power series

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n-associativity

$$F(x_1, x_2, \ldots, x_n) \in \mathbb{C}[[x_1, x_2, \ldots, x_n]],\ n \geq 3,$$

$F(0, 0, \ldots, 0) = 0$, so F can be substituted into any formal power series

$$F(x_1, x_2 \ldots, x_n) = \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{N}_0^n \\ i_1+\ldots+i_n\geq 1}} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

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$$F(F(x_1, x_2, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) \tag{1}$$

$$F(x_1, F(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}) \tag{2}$$

$$\dots$$

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F is n-associative if $(1) = (2) = \dots = (n)$. This is a system of $\binom{n}{2}$ equations in $\mathbb{C}[[x_1, x_2, \dots, x_{2n-1}]]$.

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Let $\varphi_i(x_i) = F(0,\ldots,0,x_i,0,\ldots,0)$, $i \in \{1,\ldots,n\}$, where x_i is in the i-th position.

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4. If $\varphi_1(x) = 0$ and $\varphi_n(x) = x$, then $\varphi_2 = \ldots = \varphi_{n-1} = 0$.
5. If $\varphi_1 = \varphi_n = 0$, then $\varphi_2 = \ldots = \varphi_{n-1} = 0$.

Trivial n-associative series

Theorem 1. Let F be an n-associative formal power series.

1. If $\varphi_1(x) = x$ and $\varphi_n(x) = 0$, then $F(x_1, \ldots, x_n) = x_1$.
2. If $\varphi_1(x) = 0$ and $\varphi_n(x) = x$, then $F(x_1, \ldots, x_n) = x_n$.

The situation $\varphi_1 = \varphi_n = 0$

Lemma 2. Let F be an n-associative formal power series with $\varphi_1 = \varphi_n = 0$, then $x_1 \cdots x_n$ divides $F(x_1, \ldots, x_n)$.

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Lemma 3. Let $F \neq 0$ be an n-associative formal power series, then $\mathrm{ord}_{x_1} F(x_1, \ldots, x_n), \mathrm{ord}_{x_n} F(x_1, \ldots, x_n) \in \{0, 1\}$.

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Lemma 4. Let $F(x_1, \ldots, x_n) = x_1 \cdots x_n F_1(x_1, \ldots, x_n)$ be an n-associative formal power series where $F_1(0, \ldots, 0) = a_{1,\ldots,1} = 0$, then $F = 0$.

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Outline of the proof: If $F \neq 0$, then there must be coefficients $a_{1,i_2,\ldots,i_n} \neq 0$.
Case 1: There exists some $\mu \neq \nu \in \{2, \ldots, n\}$ such that $i_\nu = 1$ and $i_\mu > 1$. This situation is impossible.
Case 2: $i_\nu > 1$ for all $\nu \in \{2, \ldots, n\}$. This is also impossible.
Is there a more elegant proof possible?

Lemma 5. Let $F(x_1,\dots,x_n) = x_1\cdots x_n F_1(x_1,\dots,x_n)$ be an n-associative formal power series where $F_1(0,\dots,0) = a_{1,\dots,1} \neq 0$, then $F_1(x_1,\dots,x_n) \in \mathbb{C}[[x_1,\dots,x_n]]^*$ and

$$F_1(x_1,\dots,x_n) = \frac{F_1(x_n,0,\dots,0)F_1(x_1,\dots,x_{n-1},0)}{F_1(F(x_1,\dots,x_n),0,\dots,0)}.$$

Theorem 2. Let $F(x_1,\ldots,x_n) = x_1\cdots x_n F_1(x_1,\ldots,x_n)$ be an n-associative formal power series where $F_1(x_1,\ldots,x_n) \in \mathbb{C}[[x_1,\ldots,x_n]]^*$. Consider some $\sigma \in \mathbb{C}$ where $\sigma^{n-1} = a_{1,\ldots,1}^{n-2} = F_1(0,\ldots,0)^{n-2}$ and define a formal series

$$f_\sigma(x) = \sigma^{-1} x F_1(x,0,\ldots,0),$$

then $f \in \Gamma$, i.e. f is invertible with respect to substitution, and

$$F(x_1,\ldots,x_n) = f_\sigma^{-1}(f_\sigma(x_1)\cdots f_\sigma(x_n)).$$

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Conversely, for each $f(x) = cx + \ldots \in \mathbb{C}[[x]]$, $c \neq 0$, the series $F(x_1,\dots,x_n) = f^{-1}(f(x_1)\cdots f(x_n))$ is n-associative.

The situation $\varphi_1(x) = \varphi_n(x) = x$

This is a generalization of the formal group laws of dimension one.

By Lemma 1 we have $\varphi_2^{n-1}(x) = x$ and $\varphi_j = \varphi_2^{j-1}$, $j \in \{3, \ldots, n\}$.

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Iterative roots of unity are of the form $T^{-1}(\rho T(x))$ for some $\rho \in \mathbb{C}$ with $\rho^{n-1} = 1$ and $T(x) = x + \ldots \in \Gamma$.

Thus $\varphi_j(x) = T^{-1}(\rho^{j-1}T(x))$, $j \in \{2,\dots,n-2\}$, for some $\rho \in \mathbb{C}$, $\rho^{n-1} = 1$, and $T(x) = x + \ldots \in \Gamma$.

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If F is an n-associativ series with $\varphi_1(x) = \varphi_n(x) = x$, then
$a_{1,0,\dots,0} = a_{0,\dots,0,1} = 1$,
$a_{\nu,0,\dots,0} = a_{0,\dots,0,\nu} = 0$, $\nu \geq 2$, and
$a_{0,1,0\dots,0} = \rho$, $a_{0,0,1,0\dots,0} = \rho^2$, $\dots$, $a_{0,\dots,0,1,0} = \rho^{n-2}$.

Multiplicative powers of F.

Lemma 6. Let $F(x_1, x_2 \ldots, x_n) = \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{N}_0^n \\ i_1+\ldots+i_n\geq 1}} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$ be a formal power series of order 1. For $\nu \in \mathbb{N}$ we have

$$F(x_1, x_2 \ldots, x_n)^\nu = \sum_{\substack{(i_1,\ldots,i_n)\in\mathbb{N}_0^n \\ i_1+\ldots+i_n\geq \nu}} A^{(\nu)}_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where

$$A^{(\nu)}_{i_1,\ldots,i_n} = \sum_{(i_1,\ldots,i_n)}^{\nu}{}^{*} \prod_{s=1}^{\nu} a_{i_{1,s},\ldots,i_{n,s}}$$

where we are summing over all matrices $(i_{r,s})_{\substack{1\leq r\leq n \\ 1\leq s\leq \nu}}$ with entries in $\mathbb{N}_0$, so that

$\sum_{s=1}^{\nu} i_{r,s} = i_r$, for all $r \in \{1, \ldots, n\}$, and

$\sum_{r=1}^{n} i_{r,s} > 0$ for all $s \in \{1, \ldots, \nu\}$.

Let F be an n-associative formal power series: Comparison of coefficients of $x_1^{j_1} \cdots x_n^{j_n} x_{n+1}$ with $j_1 + \ldots + j_n > 0$ in (1) $= \ldots =$ (n) we obtain a formula to express the coefficients $a_{\mathbf{i}}$ of F as polynomials in other coefficients $a_{\mathbf{j}}$ of F where the indices $\mathbf{j}$ are strictly smaller than $\mathbf{i}$ with respect to the colexicographical order.

$(\ell_1, \ldots, \ell_n)$ is smaller than $(i_1, \ldots, i_n)$ if there exists some $r \in \{1, \ldots, n\}$ so that $\ell_r < i_r$ and $\ell_s = i_s$ for all $r < s \leq n$.

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Consider some $k\in\{2,\ldots,n\}$ and assume $j_{k+1}=\ldots=j_n=0$, then

$$a_{j_1,\ldots,j_{k-1},j_k+1,0,\ldots,0}=\frac{\rho^{k-2}}{j_k+1}\left(\sum_{i_1=1}^{j_1+\ldots+j_k}a_{i_1,1,0,\ldots,0}\sum_{(j_1,\ldots,j_k,0,\ldots,0)}^{i_1}{}^{*}\prod_{s=1}^{i_1}a_{j_{1,s},\ldots,j_{k,s},0\ldots,0}\right.$$

$$\left.-\sum_{i_k=1}^{j_k}a_{j_1,\ldots,j_{k-1},i_k,0,\ldots,0}\,i_k\,\rho^{n-k}\,a_{j_k-i_k+1,1,0,\ldots,0}\right).$$

Theorem 3. Let F be an n-associative formal power series with $\varphi_1(x) = \varphi_n(x) = x$. For each $(j_1, \ldots, j_n) \in \mathbb{N}_0^n$ with $\sum_{i=1}^n j_i > 0$ there exists a universal polynomial $P_{j_1+\ldots+j_n}(z_0, z_1, \ldots, z_{j_1+\ldots+j_n})$ over $\mathbb{Q}$, so that

$$a_{j_1,\ldots,j_n} = P_{j_1+\ldots+j_n}(a_{0,1,0,\ldots,0}, a_{1,1,0,\ldots,0}, \ldots, a_{j_1+\ldots+j_n,1,0,\ldots,0}).$$

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Corollary. Let F be an n-associative formal power series with $\varphi_1(x) = \varphi_n(x) = x$. Then all the coefficients of F are uniquely determined by the sequence $(a_{\nu,1,0,\ldots,0})_{\nu \geq 0}$.

Lemma 7. Consider a sequence $(\alpha_\nu)_{\nu\geq 0}$ of complex numbers where $\alpha_0 = \tilde\rho$ is an $n-1$-th root of unity. Define a formal power series $f(x) = x + \sum_{k\geq 2} f_k x^k \in \Gamma$, $f_1 = 1$, recursively by

$$f_k = \frac{1}{k\tilde\rho}\sum_{\nu=1}^{k-1} \nu f_\nu \alpha_{k-\nu}, \qquad k \geq 2.$$

Then $F(x_1,\ldots,x_n) = f^{-1}(f(x_1) + \tilde\rho f(x_2) + \tilde\rho^2 f(x_3) + \ldots + f(x_n))$ is an n-associative formal power series and its coefficients satisfy $a_{\nu,1,0,\ldots,0} = \alpha_\nu$, $\nu \geq 0$.

Theorem 4. Let F be an n-associative formal power series with $\varphi_1(x) = \varphi_n(x) = x$. Then $\rho = a_{0,1,0,\dots,0}$ is an $n-1$-th root of unity. Determine a formal power series $f(x) = x + \sum_{k \geq 2} f_k x^k \in \Gamma$, $f_1 = 1$, recursively by

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Then $F(x_1,\ldots,x_n) = f^{-1}(f(x_1) + \rho f(x_2) + \rho^2 f(x_3) + \ldots + f(x_n))$.

Conversely, each formal power series $f(x) = x + \ldots \in \Gamma$ and each root ρ of unity of order $n-1$, determines an n-associative series $F(x_1,\ldots,x_n) = f^{-1}(f(x_1) + \rho f(x_2) + \rho^2 f(x_3) + \ldots + f(x_n))$, where ρ is the coefficient of x_2 in F.

Commutative or symmetric n-associative series

The series $F(x_1,\ldots,x_n)$ is symmetric, if
$F(x_1,\ldots,x_n) = F(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all π in the symmetric group S_n.

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n-associativity implies $k(n-1)+1$-associativity

Lemma 8. If $F(x_1,\ldots,x_n)$ is n-associative, then $F(F(x_1,\ldots,x_n),x_{n+1},\ldots,x_{2n-1})$ is $(2n-1)$-associative.

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Lemma 8. If $F(x_1,\dots,x_n)$ is n-associative, then $F(F(x_1,\dots,x_n),x_{n+1},\dots,x_{2n-1})$ is $(2n-1)$-associative.

From an n-associative power series F it is possible to construct a $k(n-1)+1$-associative series for all $k \in \mathbb{N}$.

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Let $F(x_1,\ldots,x_N)=\sum_{i=1}^{N}\varphi_i(x_i)+\tilde{F}(x_1,\ldots,x_N)$, $N\geq 3$, be an N-associative series. From Theorems 1, 2 and 4 we deduce:

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If $\varphi_1=0$ or $\varphi_N=0$, then each N-associative series can be constructed from a suitabel 2-associative series, or from a suitable n-associative series where $n-1$ is a divisor of $N-1$.

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If ρ is a primitive root of order d (where d is a divisor of $N-1$), then F can be constructed from a suitable $\ell d+1$-associative series, $\ell\in\mathbb{N}$, where ℓd is a divisor of $N-1$.

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If ρ is a primitive root of order $N-1$, then F cannot be constructed from an n-associative series with $n<N$.

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