Bayreuther Mathematische Schriften **74** (2005), 100–122

Enumeration of the Semilinear Isometry Classes of Linear Codes

Harald Fripertinger

Abstract

In joint papers with A. Kerber we showed how to enumerate the linear isometry classes of linear codes by certain substitutions into cycle index polynomials of the action of projective linear groups on projective spaces. In the present paper it is described how to generalize this approach to the enumeration of the semilinear isometry classes. Moreover, we motivate that with respect to error correcting properties semilinear isometry is the natural equivalence relation on the set of all linear codes.

1 Introduction

The methods and results presented in this paper are interesting in the framework of *classification of discrete structures under group actions* as described by A. Kerber in [10] or [11]. In the second section notions from functional equations are used.

Let $\mathbb{F}_q$ be a finite field of cardinality $q = p^r$, where the prime number p is its characteristic, and r is a positive integer. The Galois group Gal $:=$ $\mathrm{Gal}\,[\,\mathbb{F}_q : \mathbb{F}_p\,]$ is the group of all automorphisms of $\mathbb{F}_q$. It is a cyclic group of order r and it is generated by the Frobenius-automorphism $\tau\colon \mathbb{F}_q \to \mathbb{F}_q$, $\tau(\kappa) := \kappa^p$. In the sequel we indicate the Galois group just by Gal.

For $1 \leq k \leq n$ a linear (n,k)-code C over $\mathbb{F}_q$ is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$. It is usually described by a generator matrix, a $k \times n$-matrix over $\mathbb{F}_q$, the rows of which form a basis of C. In the present paper we write all vectors as row vectors. Therefore, an element v of $\mathbb{F}_q^n$ is

of the form $v = (v_0, v_1, \ldots, v_{n-1})$. It is also considered as the mapping v from the set $n := \{0, \ldots, n-1\}$ to $\mathbb{F}_q$ defined by $v(i) := v_i$.
In general a linear code possesses many different bases, i.e. many different generator matrices, and it is clear from linear algebra that the set of all generator matrices of a linear code with generator matrix Γ is given by

$$\{B \cdot \Gamma \mid B \in \mathrm{GL}_k(q)\},$$

where $\mathrm{GL}_k(q)$ is the group of all regular $k \times k$-matrices over $\mathbb{F}_q$.
The Hamming-distance on $\mathbb{F}_q^n$ is the mapping

$$d: \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{Z}_{\geq 0} \ : \ d(v, w) := |\{i \in n \mid v_i \neq w_i\}|.$$

Using a linear code C, the number of errors which can be detected or corrected by maximum-likelihood decoding is determined by the minimum distance

$$\mathrm{dist}(C) := \min\{d(c, c') \mid c, c' \in C, \ c \neq c'\} = \min\{d(c, 0) \mid c \in C \setminus \{0\}\}.$$

In fact, if $\mathrm{dist}(C) \geq 2t + 1$, then it is possible to correct up to t errors with this method.
Two linear codes having the same metrical structure have the same coding theoretic properties, thus they are considered to be equivalent. Therefore, two (n, k)-codes C and C' are of the same quality, if there exists a mapping

$$\iota: \mathbb{F}_q^n \to \mathbb{F}_q^n$$

with $\iota(C) = C'$ preserving the Hamming distance, i.e.

$$d(v, v') = d(\iota(v), \iota(v')) \ \text{ for all } \ v, v' \in \mathbb{F}_q^n.$$

Mappings with the last property are called *isometries.*

There exist several different equivalence relations on the set of linear (n, k)-codes. Here we just want to mention two of them.
Two linear (n, k)-codes C and C' are said to be *linearly isometric* if there exists a linear isometry of $\mathbb{F}_q^n$ mapping C onto C'. (In the literature, such codes are often simply called isometric, or monomially isometric.) Generalizing Slepian's approach [16], based on methods from [5] we enumerated the linear isometry classes of linear codes in [6], [3], and [4]. We described the group of linear isometries of $\mathbb{F}_q^n$ as the wreath product $\mathbb{F}_q^* \wr_n S_n$. It operates in a canonical way on the set of all functions from n to $\mathbb{F}_q^k \setminus \{0\}$. By an

application of Lehmann's Lemma (cf. [13] or [14]) we succeeded in replacing this action by an action of S_n on the set of functions from n to $\mathrm{PG}_{k-1}(q)$, the $(k-1)$-dimensional projective space over $\mathbb{F}_q$. Its elements, the so called points of $\mathrm{PG}_{k-1}(q)$, are the orbits of $\mathbb{F}_q^*$ on $\mathbb{F}_q^k \setminus \{0\}$.
A mapping $\sigma: \mathbb{F}_q^n \to \mathbb{F}_q^n$ is called *semilinear* if for all $u, v \in \mathbb{F}_q^n$, all $\kappa \in \mathbb{F}_q$ and a given automorphism α of $\mathbb{F}_q$ we have

$$\sigma(u+v) = \sigma(u) + \sigma(v), \qquad \sigma(\kappa u) = \alpha(\kappa)\sigma(u).$$

Two linear (n,k)-codes C and C' are said to be *semilinearly isometric* if there exists a semilinear isometry of $\mathbb{F}_q^n$ mapping C onto C'. In the next section we will motivate semilinear isometry as the natural equivalence relation on the set of all linear (n,k)-codes over $\mathbb{F}_q$. In the third section we describe the enumeration of the semilinear isometry classes of linear (n,k)-codes.
If $q = p$, then there are no differences between the linear and semilinear isometry classes since there is only the trivial automorphism of $\mathbb{F}_p$. Thus, we are mainly interested in the situation $q = p^r$ and $r > 1$.
We might have imposed a seemingly weaker condition by asking for the existence of a *local* (semi)linear isometry between two linear (n,k)-codes C and C' only, but it can be shown, see [9] and [1], that each such local isometry can be extended to a *global* isometry of $\mathbb{F}_q^n$.

2 Motivation of Semilinear Isometries

Here we describe the group of all isometries of $\mathbb{F}_q^n$ which map subspaces onto subspaces. To be more precise, the image of a subspace under an isometry is supposed to be a subspace of $\mathbb{F}_q^n$. Under these assumptions we derive for $n \geq 3$ that these mappings preserve the dimension, i.e. they map (n,k)-codes to (n,k)-codes, and that they are the *semilinear* isometries of $\mathbb{F}_q^n$. In order to prove this we need a more detailed analysis of isometries. At first we prove that it is enough to investigate isometries ι of $\mathbb{F}_q^n$ with $\iota(0) = 0$.

Lemma 1. *If $\iota: \mathbb{F}_q^n \to \mathbb{F}_q^n$ is an isometry, then*

$$\iota': \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ \iota'(v) := \iota(v) - \iota(0), \qquad v \in \mathbb{F}_q^n,$$

is also an isometry of $\mathbb{F}_q^n$ and $\iota'(0) = 0$.
Conversely, if $\iota': \mathbb{F}_q^n \to \mathbb{F}_q^n$ is an isometry with $\iota'(0) = 0$, then for any $v_0 \in \mathbb{F}_q^n$ the mapping

$$\iota: \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ \iota(v) := \iota'(v) + v_0, \qquad v \in \mathbb{F}_q^n,$$

is an isometry with $\iota(0) = v_0$. □

Consequently, it is enough to consider isometries ι with $\iota(0) = 0$, which is, for example, the case when ι maps subspaces onto subspaces. If $\iota(0) = 0$, then ι also preserves the weight, since

$$\mathrm{wt}(\iota(v)) = d(\iota(v), 0) = d(\iota(v), \iota(0)) = d(v, 0) = \mathrm{wt}(v), \qquad v \in \mathbb{F}_q^n.$$

Lemma 2. *Each isometry* ι *on a finite vector space* $\mathbb{F}_q^n$ *is bijective. If it satisfies* $\iota(0) = 0$, *then it permutes the orbits*

$$\mathbb{F}_q^*(e^{(i)}) = \{\kappa e^{(i)} \mid \kappa \in \mathbb{F}_q^*\}$$

of the unit vectors $e^{(i)}$ *with respect to the action of* $\mathbb{F}_q^*$ *by left multiplication. In formal terms:*

$$\exists\ \pi \in S_n\ \forall\ i \in n\colon\ \iota\left(\mathbb{F}_q^*(e^{(i)})\right) = \mathbb{F}_q^*(e^{(\pi(i))}).$$

Proof. Since ι is an isometry it is injective and, therefore, bijective. For each $i \in n$ and $\lambda \in \mathbb{F}_q^*$, there exists $k \in n$ and $\mu \in \mathbb{F}_q^*$ such that

$$\iota(\lambda e^{(i)}) = \mu e^{(k)}.$$

This follows from $1 = \mathrm{wt}(\lambda e^{(i)}) = \mathrm{wt}(\iota(\lambda e^{(i)}))$. Moreover, this index k does not depend on λ: Suppose that for $\lambda = 1$ we have $\iota(e^{(i)}) = \nu e^{(j)}$. Then, for $\lambda \neq 1$ we get

$$1 = d(\lambda e^{(i)}, e^{(i)}) = d(\iota(\lambda e^{(i)}), \iota(e^{(i)})) = d(\mu e^{(k)}, \nu e^{(j)}),$$

and this implies $j = k$. Thus we obtain, for the index j defined by $\iota(e^{(i)}) = \nu e^{(j)}$,

$$\iota(\mathbb{F}_q^*(e^{(i)})) \subseteq \mathbb{F}_q^*(e^{(j)}).$$

The bijectivity of ι on the orbit $\mathbb{F}_q^*(e^{(i)})$ shows that $\iota(\mathbb{F}_q^*(e^{(i)}))$ is in fact *equal* to $\mathbb{F}_q^*(e^{(j)})$, and the bijectivity of ι on the whole vector space assures the existence of some $\pi \in S_n$ which satisfies

$$\iota\left(\mathbb{F}_q^*(e^{(i)})\right) = \mathbb{F}_q^*(e^{(\pi(i))}),$$

for all $i \in n$. □

Lemma 3. *Let* ι *be an isometry of* $\mathbb{F}_q^n$ *with* $\iota(0) = 0$. *For* $i \neq k$ *and* $\lambda, \mu \in \mathbb{F}_q^*$ *we have,*

$$\iota(\lambda e^{(i)} + \mu e^{(k)}) = \iota(\lambda e^{(i)}) + \iota(\mu e^{(k)}).$$

Proof. The assumptions imply $2 = \mathrm{wt}(\lambda e^{(i)} + \mu e^{(k)})$. Together with

$$\begin{aligned} 1 &= d(\lambda e^{(i)}, \lambda e^{(i)} + \mu e^{(k)}) = d(\mu e^{(k)}, \lambda e^{(i)} + \mu e^{(k)}) \\ &= d(\iota(\lambda e^{(i)}), \iota(\lambda e^{(i)} + \mu e^{(k)})) = d(\iota(\mu e^{(k)}), \iota(\lambda e^{(i)} + \mu e^{(k)})), \end{aligned}$$

this yields the statement. □

By induction on the number of nonzero summands it is possible to prove the following

Lemma 4. *Let ι be an isometry of $\mathbb{F}_q^n$ with $\iota(0) = 0$, then the image of $v \in \mathbb{F}_q^n$ under ι is*

$$\iota(v) = \iota\left(\sum_{i=0}^{n-1} v_i e^{(i)}\right) = \sum_{i=0}^{n-1} \iota(v_i e^{(i)}).$$

□

We are now in a position to describe the group of isometries ι which satisfy $\iota(0) = 0$ as a wreath product. Since

$$\iota(v_i e^{(i)}) \in \iota(\mathbb{F}_q^*(e^{(i)})) = \mathbb{F}_q^*(e^{(\pi(i))}),$$

we can obtain the scalar factor of $e^{(\pi(i))}$ in $\iota(v_i e^{(i)})$ (if $v_i \neq 0$, otherwise we can simply neglect this summand since $\iota(0) = 0$) by the application of a suitable permutation $\varphi(\pi(i))$ of the scalars that keeps 0 fixed,

$$\iota(v_i e^{(i)}) = \varphi(\pi(i))(v_i) e^{(\pi(i))}.$$

Or, in formal terms and since we have to take all the indices into account, there exists a mapping

$$\varphi : n \to S_{\mathbb{F}_q^*},$$

from n to the symmetric group

$$S_{\mathbb{F}_q^*} := \{\rho \mid \rho : \mathbb{F}_q \to \mathbb{F}_q \text{ bijective, and } \rho(0) = 0\}$$

on $\mathbb{F}_q^*$ (considered as the subgroup of the symmetric group $S_{\mathbb{F}_q}$ on $\mathbb{F}_q$ consisting of the permutations ρ of $\mathbb{F}_q$ that keep the zero element fixed: $\rho(0) = 0$), which satisfies

$$\iota(v_0, \ldots, v_{n-1}) = (\varphi(0)(v_{\pi^{-1}(0)}), \ldots, \varphi(n-1)(v_{\pi^{-1}(n-1)})).$$

This proves the following useful description of the group of isometries:

Theorem 5. *The group of isometries ι, with $\iota(0)=0$, on the finite vector space $\mathbb{F}_q^n$, is the wreath product*

$$S_{\mathbb{F}_q^*} \wr_n S_n$$

of the symmetric group $S_{\mathbb{F}_q^*}$ on $\mathbb{F}_q$ and the symmetric group S_n on n. The action is the following one:

$$S_{\mathbb{F}_q^*} \wr_n S_n \times \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ ((\varphi;\pi),v) \mapsto (\varphi(0)(v_{\pi^{-1}(0)}),\ldots,\varphi(n-1)(v_{\pi^{-1}(n-1)})).$$

□

Together with Lemma 1 we obtain

Theorem 6. *The group of all isometries ι on the finite vector space $\mathbb{F}_q^n$, is the wreath product*

$$S_{\mathbb{F}_q} \wr_n S_n$$

of the symmetric group $S_{\mathbb{F}_q}$ on $\mathbb{F}_q$ and the symmetric group S_n on n. The action is the following one:

$$S_{\mathbb{F}_q} \wr_n S_n \times \mathbb{F}_q^n \to \mathbb{F}_q^n \ : \ ((\varphi;\pi),v) \mapsto (\varphi(0)(v_{\pi^{-1}(0)}),\ldots,\varphi(n-1)(v_{\pi^{-1}(n-1)})).$$

□

Now we want to show that isometries which *map subspaces onto subspaces* are semilinear mappings, if $n \geq 3$:

Lemma 7. *If the isometry $\iota\colon \mathbb{F}_q^n \to \mathbb{F}_q^n$, $n \geq 3$, maps subspaces onto subspaces, then for each $u \in \mathbb{F}_q^n$ we have*

$$\iota(\mathbb{F}_q^*(u)) = \mathbb{F}_q^*(\iota(u)).$$

Moreover, there exists an automorphism α of $\mathbb{F}_q$ such that, for each $\kappa \in \mathbb{F}_q$,

$$\iota(\kappa u) = \alpha(\kappa)\iota(u).$$

Proof. Since ι maps subspaces onto subspaces, the space $\{0\}$ must be mapped onto itself, whence $\iota(0)=0$. Therefore, the assertion is obviously true for $u=0$.

Assume that $u \neq 0$. Since ι is bijective and since it maps subspaces to subspaces, $\iota(\langle u\rangle)$ is a one-dimensional subspace, and so, using $\iota(u)\neq 0$, we obtain

$$\iota(\langle u\rangle) = \langle \iota(u)\rangle.$$

Moreover, as $\iota(0)=0$,

$$\iota(\mathbb{F}_q^*(u)) = \mathbb{F}_q^*(\iota(u)).$$

Hence, there is a permutation of the scalars

$$\Phi_u \in S_{\mathbb{F}_q^*} \leq S_{\mathbb{F}_q},$$

depending possibly on the vector u, which satisfies

$$\iota(\kappa u) = \Phi_u(\kappa)\iota(u).$$

We have to show that Φ_u is independent of u and that it is a field automorphism.
For the special case $e := \sum_i e^{(i)}$ we have

$$\iota(\kappa e) = \Phi_e(\kappa)\iota(e) = \Phi_e(\kappa)\sum_{i=0}^{n-1}\varphi(\pi(i))(1)e^{(\pi(i))}, \qquad \kappa \in \mathbb{F}_q^*,$$

as well as

$$\iota(\kappa e) = \sum_{i=0}^{n-1}\varphi(\pi(i))(\kappa)e^{(\pi(i))}, \qquad \kappa \in \mathbb{F}_q^*,$$

so that we obtain

$$\forall\ i \in n\ :\ \Phi_e(\kappa) = \frac{\varphi(\pi(i))(\kappa)}{\varphi(\pi(i))(1)}, \qquad \kappa \in \mathbb{F}_q^*. \tag{1}$$

Now we prove that $\Phi_e(\kappa\mu) = \Phi_e(\kappa)\Phi_e(\mu)$ for $\kappa,\mu \in \mathbb{F}_q$. The assertion is trivial for $\kappa = 0$ or $\mu = 0$. So it is possible to restrict attention to $\kappa,\mu \in \mathbb{F}_q^*$. To begin with, we consider another special case (recalling that $n > 2$, by assumption): Let

$$w := e^{(0)} + \mu e^{(i)},$$

for $i \neq 0$ and $\mu \in \mathbb{F}_q^*$. The corresponding equation

$$\iota(\kappa w) = \Phi_w(\kappa)\iota(w), \qquad \kappa \in \mathbb{F}_q^*,$$

implies that

$$\begin{aligned}&\varphi(\pi(0))(\kappa)e^{(\pi(0))} + \varphi(\pi(i))(\kappa\mu)e^{(\pi(i))}\\ &= \Phi_w(\kappa)\big(\varphi(\pi(0))(1)e^{(\pi(0))} + \varphi(\pi(i))(\mu)e^{(\pi(i))}\big).\end{aligned}$$

Comparing the coefficients of the basis vectors on both sides we obtain two useful identities. The coefficients of $e^{(\pi(0))}$ give

$$\varphi(\pi(0))(\kappa) = \Phi_w(\kappa)\varphi(\pi(0))(1),$$

so that we can deduce

$$\Phi_w(\kappa) = \frac{\varphi(\pi(0))(\kappa)}{\varphi(\pi(0))(1)} = \Phi_e(\kappa), \qquad \kappa \in \mathbb{F}_q^*,$$

and hence $\Phi_w = \Phi_e$ in this particular situation. The second identity, obtained by comparing the coefficients of $e^{(\pi(i))}$, is

$$\varphi(\pi(i))(\kappa\mu) = \Phi_w(\kappa)\varphi(\pi(i))(\mu).$$

Using $\Phi_w = \Phi_e$ and dividing both sides by $\varphi(\pi(i))(1)$ we derive that

$$\Phi_e(\kappa\mu) = \Phi_e(\kappa)\Phi_e(\mu), \qquad \kappa, \mu \in \mathbb{F}_q^*,$$

i.e. Φ_e is multiplicative.
We want to show that $\Phi_u = \Phi_e$, for all $u \neq 0$. According to Lemma 4 and (1), for $u = \sum_i u_i e^{(i)}$ we get

$$\begin{aligned} \iota(u) &= \sum_{i=0}^{n-1} \iota(u_i e^{(i)}) = \sum_{i=0}^{n-1} \varphi(\pi(i))(u_i) e^{(\pi(i))} \\ &= \sum_{i=0}^{n-1} \Phi_e(u_i)\varphi(\pi(i))(1) e^{(\pi(i))}. \end{aligned}$$

This together with the multiplicativity of Φ_e yields for $\kappa \in \mathbb{F}_q^*$ that

$$\begin{aligned} \iota(\kappa u) &= \sum_{i=0}^{n-1} \Phi_e(\kappa u_i)\varphi(\pi(i))(1) e^{(\pi(i))} \\ &= \Phi_e(\kappa) \sum_{i=0}^{n-1} \Phi_e(u_i)\varphi(\pi(i))(1) e^{(\pi(i))} \\ &= \Phi_e(\kappa)\iota(u), \end{aligned}$$

which can be compared with the identity

$$\iota(\kappa u) = \Phi_u(\kappa)\iota(u),$$

obtaining $\Phi_e(\kappa) = \Phi_u(\kappa)$ for all $\kappa \in \mathbb{F}_q^*$. Hence we have proved that in fact $\Phi_u = \Phi_e$, as stated.
It remains to show that Φ_e is additive, i.e.

$$\Phi_e(\kappa + \lambda) = \Phi_e(\kappa) + \Phi_e(\lambda), \qquad \kappa, \lambda \in \mathbb{F}_q.$$

Since by assumption $n \geq 3$, we can consider

$$u := e^{(0)} + e^{(1)}, \quad w := e^{(1)} + e^{(2)}$$

and the subspace $U := \langle\{u, w\}\rangle$ generated by these two vectors. Since, for $\lambda, \mu \in \mathbb{F}_q^*$, the vectors $\iota(\lambda u)$, $\iota(\mu w)$ and $\iota(\lambda u) + \iota(\mu w)$ are contained in the subspace $\iota(U)$, there exists some $z \in U$, for which $\iota(z) = \iota(\lambda u) + \iota(\mu w)$ is equal to

$$\Phi_e(\lambda)\varphi(\pi(0))(1)e^{(\pi(0))} + \Phi_e(\lambda)\varphi(\pi(1))(1)e^{(\pi(1))}$$
$$+\Phi_e(\mu)\varphi(\pi(1))(1)e^{(\pi(1))} + \Phi_e(\mu)\varphi(\pi(2))(1)e^{(\pi(2))}.$$

On the other hand, since

$$z = z_0 e^{(0)} + z_1 e^{(1)} + z_2 e^{(2)}$$

with $z_1 = z_0 + z_2$, we have the following expression for $\iota(z)$,

$$\Phi_e(z_0)\varphi(\pi(0))(1)e^{(\pi(0))} + \Phi_e(z_1)\varphi(\pi(1))(1)e^{(\pi(1))} + \Phi_e(z_2)\varphi(\pi(2))(1)e^{(\pi(2))}.$$

From these two representations of $\iota(z)$ we derive

$$\Phi_e(z_0) = \Phi_e(\lambda), \quad \Phi_e(z_1) = \Phi_e(\lambda) + \Phi_e(\mu), \quad \Phi_e(z_2) = \Phi_e(\mu).$$

Since Φ_e is a bijection on $\mathbb{F}_q$, we obtain $z_0 = \lambda$, $z_2 = \mu$. Moreover, since $z = z_0 u + z_2 w$, we get $z_1 = z_0 + z_2 = \lambda + \mu$ and, therefore,

$$\Phi_e(\lambda) + \Phi_e(\mu) = \Phi_e(z_1) = \Phi_e(\lambda + \mu),$$

which completes the proof of the additivity.
Hence, $\alpha := \Phi_e$ is in fact an automorphism of $\mathbb{F}_q$ which satisfies

$$\iota(\kappa u) = \alpha(\kappa)\iota(u) \qquad \kappa \in \mathbb{F}_q, \; u \in \mathbb{F}_q^n.$$

Finally

$$\iota(u + v) \quad = \quad \iota\Big(\sum_{i=0}^{n-1}(u_i + v_i)e^{(i)}\Big)$$

$$\begin{aligned}
&= \sum_{i=0}^{n-1} \iota\big((u_i + v_i)e^{(i)}\big) \\
&= \sum_{i=0}^{n-1} \alpha(u_i + v_i)\varphi(\pi(i))(1)e^{(\pi(i))} \\
&= \sum_{i=0}^{n-1} \alpha(u_i)\varphi(\pi(i))(1)e^{(\pi(i))} + \sum_{i=0}^{n-1} \alpha(v_i)\varphi(\pi(i))(1)e^{(\pi(i))} \\
&= \iota(u) + \iota(v),
\end{aligned}$$

which completes the proof. □

Summarizing, an isometry of $\mathbb{F}_q^n$ for $n \geq 3$ which maps subspaces onto subspaces is semilinear and is described by three mappings

$$\varphi\colon n \to S_{\mathbb{F}_q^*}, \quad \alpha \in \mathrm{Aut}(\mathbb{F}_q), \quad \pi \in S_n.$$

It acts on a vector $v \in \mathbb{F}_q^n$ by

$$\iota(v_0, \ldots, v_{n-1}) = (\alpha(v_{\pi^{-1}(0)})\varphi(0)(1), \ldots, \alpha(v_{\pi^{-1}(n-1)})\varphi(n-1)(1)).$$

The permutations $\varphi(i)$ are contained in $S_{\mathbb{F}_q^*}$, and so each factor $\varphi(i)(1)$ is contained in $\mathbb{F}_q^*$. Since we only need to know the values $\varphi(i)(1)$, $i \in n$, we can replace the mapping φ by the mapping

$$\psi\colon n \to \mathbb{F}_q^* \ : \ \psi(i) := \varphi(i)(1), \qquad i \in n.$$

Therefore, we can write ι as the triple (α, ψ, π), where α is an automorphism of $\mathbb{F}_q$, ψ is a mapping $n \to \mathbb{F}_q^*$ and π is a permutation of n. This allows the slightly simpler expression for $\iota(v)$ given by

$$(\alpha, \psi, \pi)(v_0, \ldots, v_{n-1}) = (\psi(0)\alpha(v_{\pi^{-1}(0)}), \ldots, \psi(n-1)\alpha(v_{\pi^{-1}(n-1)})).$$

We collect these results in the following

Theorem 8. *For $n = 1$, the isometries mapping subspaces onto subspaces are exactly the isometries of $\mathbb{F}_q$ which map 0 onto 0. According to Theorem 5 these are the elements of $S_{\mathbb{F}_q^*}$.*
For $n = 2$, the isometries mapping subspaces onto subspaces are exactly the mappings of the form (α, ψ, π), where α is a group automorphism of the multiplicative group $\mathbb{F}_q^*$, ψ is a mapping $n \to \mathbb{F}_q^*$ and π is a permutation of n.

For $n \geq 3$, the isometries mapping subspaces onto subspaces are exactly the semilinear mappings of the form (α, ψ, π), where α is a field automorphism of $\mathbb{F}_q$, ψ is a mapping $n \to \mathbb{F}_q^*$ and π is a permutation of n. These mappings form a group, the group of semilinear isometries. □

3 Semilinear Isometries

Our first aim is to show that the group of semilinear isometries is a *generalized wreath product*. Therefore, we prefer to write the semilinear isometry ι in the form $(\psi; (\alpha, \pi))$. Applying the two semilinear isometries $\iota_2 = (\psi_2; (\alpha_2, \pi_2))$ and $\iota_1 = (\psi_1; (\alpha_1, \pi_1))$ to the vector $v = (v_0, \ldots, v_{n-1}) \in \mathbb{F}_q^n$ and indicating $\iota_1(v)$ by $v' = (v'_0, \ldots, v'_{n-1})$ we obtain

$$\iota_2(\iota_1(v)) = \iota_2(v') = \big(\psi_2(0)\alpha_2(v'_{\pi_2^{-1}(0)}), \ldots, \psi_2(n-1)\alpha_2(v'_{\pi_2^{-1}(n-1)})\big) =$$

$$\Big(\ldots, \psi_2(i)\alpha_2\big(\psi_1(\pi_2^{-1}(i))\alpha_1(v_{\pi_1^{-1}(\pi_2^{-1}(i))})\big), \ldots\Big) =$$

$$\Big(\ldots, \psi_2(i)\alpha_2(\psi_1(\pi_2^{-1}(i)))(\alpha_2 \circ \alpha_1)(v_{(\pi_2 \circ \pi_1)^{-1}(i)}), \ldots,\Big).$$

This formula motivates the following

Lemma 9. *The group of all semilinear isometries of $\mathbb{F}_q^n$ is the semidirect product*

$$(\mathbb{F}_q^*)^n \rtimes (\mathrm{Gal} \times S_n)$$

with the multiplication

$$(\psi_2; (\alpha_2, \pi_2)) \cdot (\psi_1; (\alpha_1, \pi_1)) := (\psi_2 \psi_{1(\alpha_2, \pi_2)}; (\alpha_2\alpha_1, \pi_2 \circ \pi_1)),$$

where

$$\psi_{1(\alpha_2, \pi_2)}(i) := \alpha_2(\psi_1(\pi_2^{-1}(i))), \qquad i \in n,$$

and

$$\psi_2\psi_1(i) := \psi_2(i)\psi_1(i), \qquad i \in n.$$

□

Therefore, the identity element is $(1; (\mathrm{id}_{\mathbb{F}q}, \mathrm{id}_n))$, where 1 is the mapping $i \mapsto 1$, $i \in n$, and $\mathrm{id}_{\mathbb{F}q}$ respectively id_n are the identity elements in Gal respectively S_n. The inverse of $(\psi; (\alpha, \pi))$ is $(\psi^{-1}_{(\alpha^{-1}, \pi^{-1})}; (\alpha^{-1}, \pi^{-1}))$ where $\psi^{-1}(i) := (\psi(i))^{-1}$, $i \in n$, and $\psi^{-1}_{(\alpha,\pi)} := (\psi_{(\alpha,\pi)})^{-1} = (\psi^{-1})_{(\alpha,\pi)}$.

Representing the product of two semilinear isometries in this way, it is easy to realize certain similarities with ordinary wreath products.

Let us recall some basic facts about the wreath product. Consider an action ${}_GX$ and a group H. The *wreath product* of H with G, with respect to ${}_GX$, consists of the underlying set

$$H \wr_X G := H^X \times G = \{(\varphi; g) \mid \varphi : X \to H,\ g \in G\},$$

and the multiplication

$$(\varphi; g)(\varphi'; g') := (\varphi\varphi'_g; gg'),$$

where $(\varphi\varphi'_g)(x) := \varphi(x) \cdot \varphi'_g(x)$ and $\varphi'_g(x) := \varphi'(g^{-1}x)$ for $x \in X$. Its identity element is

$$1_{H \wr_X G} = (1; 1_G),$$

where $1 \in H^X$ is the constant mapping $x \mapsto 1_H$, and 1_G, 1_H denote the identity elements of G and H, respectively. The inverse of $(\varphi; g) \in H \wr_X G$ is

$$(\varphi; g)^{-1} = (\varphi^{-1}_{g^{-1}}; g^{-1}),$$

where

$$\varphi^{-1}(x) := \varphi(x)^{-1} \quad \text{and} \quad \varphi^{-1}_{g^{-1}} := (\varphi_{g^{-1}})^{-1} = (\varphi^{-1})_{g^{-1}}.$$

If moreover H acts on Y, then the canonical action of $H \wr_X G$ on Y^X is described by

$$H \wr_X G \times Y^X \to Y^X : ((\varphi; g), f) \mapsto \widetilde{f}, \text{ where } \widetilde{f}(x) := \varphi(x) f(g^{-1}x). \tag{2}$$

Now we come back to the semilinear isometry group. For defining the multiplication in $H \wr_X G$ we used the canonically induced action of G on H^X. Here, in the situation of the group of semilinear isometries, we have $X = n$ and $H = \mathbb{F}_q^*$. The group $\mathrm{Gal} \times S_n$ does not act on n, but on $(\mathbb{F}_q^*)^n$. Therefore, it is not necessary to consider an induced action on $(\mathbb{F}_q^*)^n$, and we say that the group of semilinear isometries is the generalized wreath product of $\mathbb{F}_q^*$ and $\mathrm{Gal} \times S_n$ which we indicate by

$$\mathbb{F}_q^* \wr\wr_n (\mathrm{Gal} \times S_n).$$

Its order is equal to $(q-1)^n \cdot r \cdot n!$. The generalization of the natural action of a wreath product, given by (2), to this generalized wreath product is

$$(\psi;(\alpha,\pi))(v) = \big(\psi(0)\alpha(v_{\pi^{-1}(0)}),\dots,\psi(n-1)\alpha(v_{\pi^{-1}(n-1)})\big)$$

which is the action of the semilinear isometry $(\psi;(\alpha,\pi))$ on $\mathbb{F}_q^n$.

We describe codes by their generator matrices. The set of generator matrices of linear (n,k)-codes over $\mathbb{F}_q$, is indicated by $\mathbb{F}_q^{k\times n,k}$. This is the set of all $k\times n$-matrices over $\mathbb{F}_q$ of rank k. Then the set of semilinear isometry classes of (n,k)-codes is equal to the set of orbits

$$\mathbb{F}_q^* \wr_n (\mathrm{Gal}\times S_n)\backslash\!\backslash \left(\mathrm{GL}_k(q)\backslash\!\backslash \mathbb{F}_q^{k\times n,k}\right),$$

where the operation of $(\psi;(\alpha,\pi)) \in \mathbb{F}_q^* \wr_n (\mathrm{Gal}\times S_n)$ on the orbit $\mathrm{GL}_k(q)(\Gamma)$ is given by

$$\big((\psi;(\alpha,\pi)),\mathrm{GL}_k(q)(\Gamma)\big) \mapsto \mathrm{GL}_k(q)(\hat\Gamma) \quad\text{where}\quad \hat\Gamma(i) = \psi(i)\alpha(\Gamma(\pi^{-1}(i))).$$

Here again we identify the matrix Γ with the function $\Gamma\colon n \to \mathbb{F}_q^k$ where the transposed vector $\Gamma(i)$ is the i-th column of Γ.
We want to prove that this operation of the generalized wreath product on the set of generator matrices is well defined. For $A \in \mathrm{GL}_k(q)$ and $\tilde\Gamma$ given by $\tilde\Gamma(i) := \psi(i)\alpha((A\cdot\Gamma)(\pi^{-1}(i)))$ we have $\mathrm{GL}_k(q)(\tilde\Gamma) = \mathrm{GL}_k(q)(\hat\Gamma)$, since $\tilde\Gamma(i) = \psi(i)\alpha(A)\cdot\alpha(\Gamma(\pi^{-1}(i)))$ and $\alpha(A) \in \mathrm{GL}_k(q)$. (We just remind the reader that $\alpha \in \mathrm{Gal}$ induces a group automorphism of $\mathrm{GL}_k(q)$ when α is applied to each component of the matrices in $\mathrm{GL}_k(q)$.)
In the situation of linear isometries the actions of the isometry group and of the linear group were commuting and we obtained an action of the direct product of these two groups on $\mathbb{F}_q^{k\times n,k}$. In general, the action of the semilinear isometry group does not commute with the action of $\mathrm{GL}_k(q)$. For $A \in \mathrm{GL}_k(q)$, $(\psi;(\alpha,\pi)) \in \mathbb{F}_q^* \wr_n (\mathrm{Gal}\times S_n)$ and $\Gamma \in \mathbb{F}_q^{k\times n,k}$ we have

$$A\cdot(\psi;(\alpha,\pi))\Gamma =$$
$$\big(\psi(0)A\cdot\alpha(\Gamma(\pi^{-1}(0))),\dots,\psi(n-1)A\cdot\alpha(\Gamma(\pi^{-1}(n-1)))\big)$$

and

$$(\psi;(\alpha,\pi))A\cdot\Gamma =$$
$$\big(\psi(0)\alpha(A)\cdot\alpha(\Gamma(\pi^{-1}(0))),\dots,\psi(n-1)\alpha(A)\cdot\alpha(\Gamma(\pi^{-1}(n-1)))\big).$$

Therefore, in general we do not get an action of the direct product.

Again, similarly as in [6] we eliminate the rank condition on the $k \times n$-matrices and consider the set of *all* $k \times n$-matrices over $\mathbb{F}_q$ which do not contain zero columns. They can be expressed as functions $n \to \mathbb{F}_q^k \setminus \{0\}$. Thus our task is to determine the cardinality of

$$\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n) \backslash\!\backslash \left(\mathrm{GL}_k(q) \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})^n\right).$$

For that reason we describe a generalization of Lehmann's Lemma (cf. [13] or [14]). We generalize it in two ways, since on the one hand we are dealing with an action of the generalized wreath product, and on the other hand this wreath product operates on $\mathrm{GL}_k(q)$-orbits of functions and not just on a set of functions. However we do not formulate it for arbitrary group actions but for the situation of the present problem.

Generalization of Lehmann's Lemma *If the mapping*

$$\varphi\colon \mathrm{GL}_k(q) \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})^n \to \mathrm{GL}_k(q) \backslash\!\backslash \left(\mathbb{F}_q^* \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})\right)^n$$

is given by

$$\mathrm{GL}_k(q)(\Gamma) \mapsto \mathrm{GL}_k(q)(\bar{\Gamma}) \quad \textit{where} \quad \bar{\Gamma}(i) = \mathbb{F}_q^*(\Gamma(i)),$$

then the mapping

$$\Phi\colon \left(\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)\right) \backslash\!\backslash \left(\mathrm{GL}_k(q) \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})^n\right) \to$$

$$(\mathrm{Gal} \times S_n) \backslash\!\backslash \left(\mathrm{GL}_k(q) \backslash\!\backslash \left(\mathbb{F}_q^* \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})\right)^n\right)$$

defined by

$$\left(\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)\right)(\mathrm{GL}_k(q)(\Gamma)) \mapsto (\mathrm{Gal} \times S_n)(\varphi(\mathrm{GL}_k(q)(\Gamma)))$$

is a bijection. On the right hand side we have an operation of $(\mathrm{Gal} \times S_n)$ *on the set of orbits* $\mathrm{GL}_k(q) \backslash\!\backslash \left(\mathbb{F}_q^* \backslash\!\backslash (\mathbb{F}_q^k \setminus \{0\})\right)^n$ *of the form*

$$(\alpha, \pi)\, \mathrm{GL}_k(q)(\bar{\Gamma}) = \mathrm{GL}_k(q)(\hat{\Gamma})$$

where $\hat{\Gamma}(i) = \alpha(\bar{\Gamma}(\pi^{-1}(i))) = \alpha(\mathbb{F}_q^*(\Gamma(\pi^{-1}(i)))) = \mathbb{F}_q^*(\alpha(\Gamma(\pi^{-1}(i))))$, $i \in n$.

Proof. As in the proof of the original lemma we see that for $f_1, f_2 \in Y^X$ the following facts are equivalent:

$$\Phi(\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)(f_1)) = \Phi(\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)(f_2))$$

$$(\mathrm{Gal} \times S_n)(\varphi(f_1)) = (\mathrm{Gal} \times S_n)(\varphi(f_2))$$

$$\varphi(f_2) \in (\mathrm{Gal} \times S_n)(\varphi(f_1))$$

$$\varphi(f_2) = \alpha \circ \varphi(f_1) \circ \pi \quad \text{for some } \alpha \in \mathrm{Gal} \text{ and some } \pi \in S_n$$

$$\varphi(f_2)(x) = \alpha\big(\varphi(f_1)(\pi(x))\big) \quad \text{for some } \alpha \in \mathrm{Gal},\ \pi \in S_n, \text{ and all } x \in X$$

$$\varphi(f_2)(x) = \varphi(\alpha \circ f_1)(\pi(x)) \quad \text{for some } \alpha \in \mathrm{Gal},\ \pi \in S_n, \text{ and all } x \in X$$

$$\mathbb{F}_q^*(f_2(x)) = \mathbb{F}_q^*((\alpha \circ f_1)(\pi(x))) \quad \text{for some } \alpha \in \mathrm{Gal},\ \pi \in S_n, \text{ and all } x \in X$$

$$f_2(x) \in \mathbb{F}_q^*((\alpha \circ f_1)(\pi(x))) \quad \text{for some } \alpha \in \mathrm{Gal},\ \pi \in S_n, \text{ and all } x \in X$$

$$f_2 = (\psi; (\alpha, \pi)) f_1 \quad \text{for some } (\psi; (\alpha, \pi)) \in \mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)$$

$$f_2 \in \mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)(f_1)$$

$$\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)(f_2) = \mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)(f_1).$$

Reading these implications from bottom to top we deduce that Φ is well defined. From top to bottom it follows that Φ is injective. In order to prove that Φ is surjective, we realize that φ is surjective. □

As an immediate consequence and using the fact that $\mathbb{F}_q^* \backslash\backslash (\mathbb{F}_q^k \setminus \{0\})$ is the projective space $\mathrm{PG}_{k-1}(q)$ we obtain that

$$\Big|\big(\mathbb{F}_q^* \wr_n (\mathrm{Gal} \times S_n)\big) \backslash\backslash \big(\mathrm{GL}_k(q) \backslash\backslash (\mathbb{F}_q^k \setminus \{0\})^n\big)\Big| =$$
$$\big|(\mathrm{Gal} \times S_n) \backslash\backslash \big(\mathrm{GL}_k(q) \backslash\backslash \mathrm{PG}_{k-1}(q)^n\big)\big|.$$

It is still possible to find a simpler expression for

$$(\mathrm{Gal} \times S_n) \backslash\backslash \big(\mathrm{GL}_k(q) \backslash\backslash \mathrm{PG}_{k-1}(q)^n\big).$$

It is well known that the action of a direct product can be split into two actions. Hence we obtain

$$\mathrm{Gal} \backslash\backslash \big(S_n \backslash\backslash (\mathrm{GL}_k(q) \backslash\backslash \mathrm{PG}_{k-1}(q)^n)\big)$$

which is the same as

$$\mathrm{Gal} \backslash\backslash \big((\mathrm{GL}_k(q) \times S_n) \backslash\backslash \mathrm{PG}_{k-1}(q)^n\big) \tag{3}$$

since the actions of $\mathrm{GL}_k(q)$ and S_n commute. An element $A \in \mathrm{GL}_k(q)$ leaves each point of the projective space $\mathrm{PG}_{k-1}(q)$ fixed if and only if A belongs to the center

$$\mathcal{Z}_k = \left\{ \kappa \cdot I_k \;\middle|\; \kappa \in \mathbb{F}_q^* \right\}$$

of $\mathrm{GL}_k(q)$. Hence, multiplications by elements of the center do not matter, they can be factored out. For that reason, we introduce the *projective linear group* as the factor group $\mathrm{PGL}_k(q) := \mathrm{GL}_k(q)/\,\mathcal{Z}_k$. The general linear group acts as the projective linear group on the projective space.

Applying the automorphism $\alpha \in \mathrm{Gal}$ to all components of a vector induces a permutation representation of Gal on $\mathbb{F}_q^k$. This induced mapping $\alpha\colon \mathbb{F}_q^k \to \mathbb{F}_q^k$ is semilinear. The group of all semilinear bijections of $\mathbb{F}_q^k$ is the *general semilinear group* $\Gamma\mathrm{L}_k(q)$. It is the smallest permutation group of $\mathbb{F}_q^k$ containing $\mathrm{GL}_k(q)$ and Gal. To be more precise, it is the semidirect product $\mathrm{GL}_k(q) \rtimes \mathrm{Gal}$. The center of $\Gamma\mathrm{L}_k(q)$ is again $\mathcal{Z}_k$ and the factor group $\mathrm{P}\Gamma\mathrm{L}_k(q) := \Gamma\mathrm{L}_k(q)/\,\mathcal{Z}_k$ is the *projective semilinear group.*

Coming back to (3), we have to consider the action of Gal on the orbits $(\mathrm{GL}_k(q) \times S_n) \backslash\!\backslash \mathrm{PG}_{k-1}(q)^n$. An application of the automorphism α to the orbit $(\mathrm{GL}_k(q) \times S_n)(\bar{\Gamma})$ yields the orbit $(\mathrm{GL}_k(q) \times S_n)(\hat{\Gamma})$ where $\hat{\Gamma}(i) = \alpha(\bar{\Gamma}(i)) = \mathbb{F}_q^*(\alpha(\Gamma(i)))$. The orbits of (3) can be represented as the elements of

$$(\mathrm{P}\Gamma\mathrm{L}_k(q) \times S_n) \backslash\!\backslash \mathrm{PG}_{k-1}(q)^n, \tag{4}$$

since $\mathrm{P}\Gamma\mathrm{L}_k(q) = (\mathrm{GL}_k(q) \rtimes \mathrm{Gal})/\,\mathcal{Z}_k$.

From (4) it is easy to deduce that linear codes could also be represented by multisets of columns of generator matrices or multisets of points of the projective space $\mathrm{PG}_{k-1}(q)$ (cf. [12]). Using this approach (4) can be replaced by the action of $\mathrm{P}\Gamma\mathrm{L}_k(q)$ on the set of multisets of size n with elements from $\mathrm{PG}_{k-1}(q)$.

There are several properties of codes which are common to all members of a semilinear isometry class. A code C is called *nonredundant* if its generator matrices Γ do not contain columns of zeros. A nonredundant code C is called *projective* if the columns of any generator matrix Γ of C are pairwise linearly independent. Generalizing this definition, an arbitrary (n,k)-code C is called *injective* if the mapping

$$\Gamma\colon n \to \mathrm{PG}_{k-1}(q) \cup \{0\},$$

corresponding to the columns of an arbitrary generator matrix Γ of C, is injective. We call a code C *decomposable,* if it is linearly isometric to a code with a generator matrix in the form of a block diagonal matrix

$$\Gamma = \left(\begin{array}{c|c} \Gamma_1 & 0 \\ \hline 0 & \Gamma_2 \end{array} \right),$$

consisting of two generator matrices Γ_i of linear (n_i, k_i)-codes with $1 \leq k_i \leq n_i$ for $i = 1, 2$. In other words, it is linearly isometric to the direct outer sum of at least two codes. If C is not decomposable then it is *indecomposable*.
It is easy to check that together with one code all codes in its semilinear isometry class have (some of) these properties.
Analogously as we did with the linear isometry classes we introduce the notions

$$t_{nkq} := |(\mathrm{P\Gamma L}_k(q) \times S_n) \backslash\backslash \, \mathrm{PG}_{k-1}(q)^n|,$$
$$\bar{t}_{nkq} := \left|(\mathrm{P\Gamma L}_k(q) \times S_n) \backslash\backslash \, \mathrm{PG}_{k-1}(q)^n_{\mathrm{inj}}\right|,$$

where Y^X_{inj} denotes the set of all injective functions from X to Y. Moreover, let v_{nkq} denote the number of semilinear isometry classes of nonredundant (n, k)-codes over $\mathbb{F}_q$ and $\bar{v}_{nkq}$ the number of semilinear isometry classes of projective (n, k)-codes over $\mathbb{F}_q$. The symbols u_{nkq} and $\bar{u}_{nkq}$ are used to indicate the number of semilinear isometry classes of all, respectively injective, (n, k)-codes which may contain columns of zeros. The number of semilinear isometry classes of nonredundant indecomposable (n, k)-codes over $\mathbb{F}_q$ is denoted by r_{nkq} and of projective indecomposable (n, k)-codes over $\mathbb{F}_q$ by $\bar{r}_{nkq}$. These numbers satisfy certain relations similar to those of the corresponding numbers of linear isometry classes (cf. [6] or [16]).

Corollary 10.

- t_{nkq} *is the number of semilinear isometry classes of linear codes of length* n *and dimension at most* k*. If* $k > 1$*, then* $t_{n,k-1,q}$ *is also the number of* $\mathrm{P\Gamma L}_k(q) \times S_n$*-orbits of mappings* $f \in \mathrm{PG}_{k-1}(q)^n$ *corresponding to matrices of rank not greater than* $k - 1$.
- $\bar{t}_{nkq}$ *is the number of semilinear isometry classes of injective linear codes of length* n *and dimension at most* k.
- $u_{nkq} = \sum_{i=k}^{n} v_{ikq}$*, for* $n \geq k$*,*
  $\bar{u}_{kkq} = \bar{v}_{kkq}$*,* $\bar{u}_{nkq} = \bar{v}_{n-1,k,q} + \bar{v}_{nkq}$ *for* $n > k$.
- $v_{nkq} = t_{nkq} - t_{n,k-1,q}$*,* $\bar{v}_{nkq} = \bar{t}_{nkq} - \bar{t}_{n,k-1,q}$ *for* $1 < k \leq n$.
  The initial values for these recursions are $v_{n1q} = 1$ *for* $n \geq 1$*,* $\bar{v}_{11q} = 1$ *and* $\bar{v}_{n1q} = 0$ *for* $n > 1$.

- *For $n \geq 2$ we have*

$$r_{nkq} = v_{nkq} - \sum_a \sum_b \prod_{\substack{j=1 \\ a_j \neq 0}}^{n-1} \left(\sum_c U(c) \right),$$

where

$$U(c) = \prod_{i=1}^{j} C(S_{\nu(i,c)}, \nu(i,c))\big|_{z_\ell = r_{jiq}}$$

is a product computed from substitutions into the cycle index of symmetric groups of degree $\nu(i,c)$ given by

$$\nu(i,c) = |\{1 \leq \ell \leq a_j \mid c_\ell = i\}|.$$

The first sum runs through the cycle types $a = (a_1, \ldots, a_{n-1})$ of n with at least two summands, i.e. $a_i \in \mathbb{N}$ and $\sum i a_i = n$, and with the additional property $\sum a_i \leq k$, while the second sum is taken over the $(n-1)$-tuples $b = (b_1, \ldots, b_{n-1}) \in \mathbb{N}^{n-1}$, for which $a_i \leq b_i \leq i a_i$, and $\sum b_i = k$. The third sum runs through all the a_j-tuples $c = (c_1, \ldots, c_{a_j}) \in \mathbb{N}^{a_j}$ with the properties $j \geq c_1 \geq \ldots \geq c_{a_j} \geq 1$ and $\sum c_i = b_j$.

Analogously, $\bar{r}_{nkq}$ can be recursively evaluated from $\bar{v}_{nkq}$ and $\bar{r}_{jiq}$ with $j < n$.

The initial values for these recursions are $r_{11q} = 1 = \bar{r}_{11q}$. □

This way we have expressed all these numbers in terms of t_{nkq} and $\bar{t}_{nkq}$. The remaining problem is the evaluation of t_{nkq} and $\bar{t}_{nkq}$. In (4) we have the canonical action of a direct product on a set of functions. Since the group acting on the domain is the symmetric group, it is possible to apply a particular version of Pólya's theorem (cf. [15, equation (1, 27)], [2, Example 5.24]) in order to compute the generating function for the cardinalities of these orbit sets. We obtain the following

Corollary 11. *The generating functions for the numbers t_{nkq} and $\bar{t}_{nkq}$ can be obtained from the cycle index of the natural action of the projective semilinear group on the projective space in the following way:*

$$\sum_{n \in \mathbb{N}} t_{nkq} x^n = C(\mathrm{P\Gamma L}_k(q), \mathrm{PG}_{k-1}(q))\big|_{z_i := \sum_{j=0}^{\infty} x^{i \cdot j}},$$

and

$$\sum_{n\in\mathbb{N}} \bar{t}_{nkq}x^n = C(\mathrm{P\Gamma L}_k(q), \mathrm{PG}_{k-1}(q))\big|_{z_i:=1+x^i}.$$

□

Finally it remains to determine this cycle index of the natural action of the projective semilinear group on the projective space. In order to obtain numerical results we used the computer algebra system GAP [7] together with a particular extension for projective spaces [8]. We determined a complete system of representatives of the conjugacy classes of elements of $\mathrm{P\Gamma L}_k(q)$. Next we derived the permutation representation on $\mathrm{PG}_{k-1}(q)$ of each representative. Conjugate elements have the same cycle type, therefore we obtained a complete list of cycle types of elements of $\mathrm{P\Gamma L}_k(q)$. Finally we computed the cardinality of each conjugacy class. Multiplying each cycle type with the cardinality of the corresponding conjugacy class, summing over all conjugacy classes and dividing by the order of $\mathrm{P\Gamma L}_k(q)$ we derived the cycle index polynomial.

The smallest fields where differences occur between linear and semilinear isometries are $\mathbb{F}_4$ and $\mathbb{F}_8$. For $q = 4$ and $q = 8$ we have computed tables of t_{nkq}, v_{nkq}, r_{nkq}, u_{nkq}, $\bar{t}_{nkq}$, $\bar{v}_{nkq}$, and $\bar{r}_{nkq}$. These tables together with the cycle index polynomials of $\mathrm{P\Gamma L}_k(q)$ can be found online under

http://linearcodes.uni-bayreuth.de/CD/tables_semilinear.html

and

http://linearcodes.uni-bayreuth.de/CD/zykelind_pgglkq.html.

Replacing the action of the semilinear isometry group by the action of the linear isometry group we obtain corresponding numbers of linear isometry classes. We denote the numbers of linear isometry classes with the uppercase versions of the symbols introduced for the semilinear isometry classes. For instance V_{nkq} is the number of linear isometry classes of nonredundant linear (n, k)-codes. (In our previous papers and online we use S_{nkq}, $\bar{S}_{nkq}$ and W_{nkq} instead of V_{nkq}, $\bar{V}_{nkq}$ and U_{nkq}.)

In Table 1 we compare the values of V_{nk4} and v_{nk4}. There are no differences for $k \le 7$ and $n \le 6$. For $k \ge 8$ the first line contains the numbers of linear isometry classes and the second line the numbers of semilinear isometry classes. Differences are marked by boldface numbers in the second row. Further values of V_{nk4} can be found online under

http://www.mathe2.uni-bayreuth.de/frib/codes/tables_5.html.

In Table 2 we compare the values of T_{nk8} and t_{nk8}. There are no differences for $k \le 5$ and $n \le 4$. For $k \ge 6$ the first line contains the numbers of linear isometry classes and the second line the numbers of semilinear isometry classes. Differences are marked by boldface numbers in the second row. Further values of linear and semilinear isometry classes over $\mathbb{F}_8$ can be computed or found online under

http://linearcodes.uni-bayreuth.de/CD/index.html.

Table 1: Comparison of V_{nk4} and v_{nk4}

$n\backslash k$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	1	0	0	0
4	1	4	3	1	0	0
5	1	6	9	4	1	0
6	1	9	24	17	5	1
7	1	12	55	70	28	6
8	1	17	131	323	189	44
8	1	17	**126**	**301**	**184**	44
9	1	22	318	1 784	1 976	490
9	1	22	**286**	**1 419**	**1 594**	**453**
10	1	30	772	12 094	36 477	13 752
10	1	**29**	**640**	**7 970**	**22 405**	**9 278**
11	1	37	1 881	89 437	923 978	948 361
11	1	**36**	**1 431**	**51 456**	**490 138**	**504 573**
12	1	48	4 568	668 922	25 124 571	91 149 571
12	1	**46**	**3 204**	**357 222**	**12 746 664**	**45 963 661**
13	1	59	10 857	4 843 901	665 246 650	9 163 203 790
13	1	**56**	**7 099**	**2 496 031**	**333 787 936**	**4 586 461 981**
14	1	74	25 276	33 456 545	16 677 221 922	887 802 519 854
14	1	**69**	**15 595**	**16 961 133**	**8 345 700 799**	**443 959 979 727**

Table 2: Comparison of T_{nk8} and t_{nk8}

$n\backslash k$	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	3	4	4
4	1	5	8	9
5	1	7	16	20
6	1	14	57	78
6	1	**12**	**43**	**62**
7	1	21	273	555
7	1	**17**	**143**	**289**
8	1	39	2 034	13 931
8	1	**27**	**792**	**4 979**
9	1	64	16 668	714 573
9	1	**40**	**5 806**	**239 355**
10	1	109	132 237	40 746 243
10	1	**61**	**44 619**	**13 586 393**
11	1	173	986 453	2 188 928 772
11	1	**89**	**329 959**	**729 659 322**
12	1	286	6 876 180	108 587 171 103
12	1	**136**	**2 294 446**	**36 195 786 755**
13	1	439	44 880 936	4 985 542 976 595
13	1	**197**	**14 965 218**	**1 661 847 901 869**
14	1	686	275 497 786	212 944 610 369 565
14	1	**292**	**91 842 474**	**70 981 537 714 473**
15	1	1 028	1 597 385 468	8 503 511 406 384 359
15	1	**420**	**532 481 348**	**2 834 503 805 580 423**
16	1	1 534	8 784 375 366	318 881 061 522 362 625
16	1	**606**	**2 928 163 108**	**106 293 687 186 817 717**
17	1	2 222	45 985 791 002	11 273 378 553 997 847 510
17	1	**854**	**15 328 669 468**	**3 757 792 851 378 389 530**
18	1	3 208	229 921 910 074	377 031 845 513 665 669 846
18	1	**1 206**	**76 640 772 664**	**125 677 281 838 050 117 086**
19	1	4 511	1 101 188 614 914	11 965 799 870 995 403 023 108
19	1	**1 665**	**367 063 122 976**	**3 988 599 956 999 028 257 910**
20	1	6 310	5 065 430 785 605	361 367 154 074 790 367 805 313
20	1	**2 294**	**1 688 477 385 597**	**120 455 718 024 932 032 208 289**

References

[1] A. Betten, M. Braun, H. Fripertinger, A. Kerber, A. Kohnert, and A. Wassermann. *Error–Correcting Linear Codes — Classification by Isometry and Applications*. Springer, Berlin, Heidelberg, New York, 2005 (in preparation).

[2] N.G. de Bruijn. Pólya's theory of counting. In E.F. Beckenbach, editor, *Applied Combinatorial Mathematics*, chapter 5, pages 144 – 184. Wiley, New York, 1964.

[3] H. Fripertinger. Enumeration of isometry classes of linear (n, k)-codes over $GF(q)$ in SYMMETRICA. *Bayreuther Mathematische Schriften*, 49:215 – 223, 1995. ISSN 0172-1062.

[4] H. Fripertinger. Zyklenzeiger linearer Gruppen und Abzählung linearer Codes. *Séminaire Lotharingien de Combinatoire*, Actes 33:1 – 10, 1995. ISSN 0755-3390.

[5] H. Fripertinger. Cycle Indices of Linear, Affine and Projective Groups. *Linear Algebra and Its Applications*, 263:133 – 156, 1997.

[6] H. Fripertinger and A. Kerber. Isometry Classes of Indecomposable Linear Codes. In G. Cohen, M. Giusti, and T. Mora, editors, *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, 11th International Symposium, AAECC-11, Paris, France, July 1995*, volume 948 of *Lecture Notes in Computer Science*, pages 194–204. Springer, 1995.

[7] The GAP Group, Aachen, Germany and St. Andrews, Scotland. *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2004.

[8] P. Govaerts and J. De Beule. pg, Projective Geometries, a share package for GAP 4. *http://cage.ugent.be/~jdebeule/pg/*.

[9] W. Heise and P. Quattrocchi. *Informations- und Codierungstheorie*. Springer Verlag, Berlin, Heidelberg, New York, Paris, Tokio, 2nd edition, 1989.

[10] A. Kerber. *Algebraic Combinatorics via Finite Group Actions*. B.I. Wissenschaftsverlag, Mannheim, Wien, Zürich, 1991. ISBN 3-411-14521-8.

[11] A. Kerber. *Applied Finite Group Actions*, volume 19 of *Algorithms and Combinatorics*. Springer, Berlin, Heidelberg, New York, 1999. ISBN 3-540-65941-2.

[12] I.N. Landjev. Linear codes over finite fields and finite projective geometries. *Discrete Math.*, 213(1-3):211–244, 2000. Selected topics in discrete mathematics (Warsaw, 1996).

[13] W. Lehmann. Das Abzähltheorem der Exponentialgruppe in gewichteter Form. *Mitteilungen aus dem Mathem. Seminar Giessen*, 112:19 – 33, 1974.

[14] W. Lehmann. *Ein vereinheitlichender Ansatz für die REDFIELD – PÓLYA – de BRUIJNSCHE Abzähltheorie*. PhD thesis, Universität Giessen, 1976.

[15] G. Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Mathematica*, 68:145 – 254, 1937.

[16] D. Slepian. Some Further Theory of Group Codes. *The Bell System Technical Journal*, 39:1219 – 1252, 1960.

Harald Fripertinger
Institut für Mathematik
Karl-Franzens Universität Graz
Heinrichstr. 36/4
A–8010 Graz, AUSTRIA
harald.fripertinger@uni-graz.at
http://www.uni-graz.at/harald.fripertinger/