THE ENTROPY DISSIPATION METHOD FOR SPATIALLY INHOMOGENEOUS REACTION–DIFFUSION TYPE SYSTEMS

MARCO DI FRANCESCO, KLEMENS FELLNER, AND PETER A. MARKOWICH

Abstract. We study the large–time asymptotics of reaction–diffusion type systems, which feature a monotone decaying entropy (Lyapunov, free energy) functional. We consider both bounded domains and confining potentials on the whole space for arbitrary space dimensions. Our aim is to derive quantitative expressions for (or estimates of) the rates of convergence towards an (entropy minimising) equilibrium state in terms of the constants of diffusion and reaction and with respect to conserved quantities. Our method, the so called entropy approach, seeks to quantify convergence to equilibrium by using functional inequalities, which relate quantitatively the entropy and its dissipation in time. The entropy approach is well suited to nonlinear problems and known to be quite robust with respect to model variations. It has already been widely applied to scalar diffusion–convection equations and the main goal of this paper is to study its generalisation to systems of partial differential equations, which contain diffusion and reaction terms and admit fewer conservation laws than the size of the system. In particular, we successfully apply the entropy approach to general linear systems and to a nonlinear example of reaction–diffusion–convection system arising in solid state physics as a paradigm for general nonlinear systems.

1. Introduction

Many physical, chemical and biological processes are driven on the one hand by diffusion, a random particle motion microscopically described by a Brownian stochastic process, and on the other hand, by reactions representing instantaneous interactions between particles. Typical examples where both mechanisms occur are chemical kinetics, population dynamics, flame propagation and combustion, movement of biological cells in plants and animals, and charge carrier transport in semiconductors.

To reiterate the classical mathematical modelling, let $U = U(x,t) \in \mathbb{R}^N$ denote the concentration vector describing N interacting species, where $x \in \mathbb{R}^n$ denotes the position variable and $t > 0$ time. Then, according to Fick's law the diffusion part of the motion is described by the linear parabolic evolution equation :

$$U_t = \operatorname{div}(D \nabla U),$$

where $D = D(x,t,U)$ is a positive definite, symmetric diffusion matrix, in general depending on position x, time t, and on the concentration vector U itself.

On the other hand given a reaction process in terms of a 'local' dynamical system of the form $U_t = F(x,t,U)$, then the interaction of both reaction and diffusion leads to the following reaction–diffusion (RD) systems:

$$U_t = \operatorname{div}(D(x,t,U) \nabla U) + F(x,t,U).$$

If the two processes occur in a spatially bounded domain Ω, suitable boundary conditions have to be imposed, for instance *Dirichlet conditions* $U = U_D$ on $\partial\Omega$ or, with n denoting the outer normal, *Neumann conditions* $(D \nabla U) \cdot n = U_N$ on $\partial\Omega$ and also linear combinations of either conditions (*Robin conditions*). Note that the total number of particles is conserved if zero outflow, i.e. $U_N = 0$, is prescribed through the boundary $\partial\Omega$.

2000 *Mathematics Subject Classification.* Primary 35K57, Secondary 35B40, 35Q80.

Key words and phrases. Reaction–diffusion systems, entropy–entropy dissipation approach, large–time asymptotics.

MDF was partially supported by the Italian research project 'Modelli iperbolici non lineari in fluido dinamica' (INdAM GNAMPA 2008).

This article has partly been supported by the KAUST Investigator Award 2008 and the Wolfson Research Merit Award (Royal Society) of Peter A. Markowich.

Also reaction–diffusion equations posed on the whole n–dimensional space occur in many important applications. In these cases, a confinement of particles occurs often by means of an exterior potential acting as a trap. A classical example of a confined diffusion process is the Fokker-Planck equation describing the interaction of Brownian diffusion with a harmonic (i.e. quadratic) potential as confinement.

Reaction–diffusion systems are nowadays considered a classical topic, going (at least) back to the pivotal works of e.g. Fisher [Fis37] and Kolmogorov et al. [KPP37]. The mathematical literature on reaction-diffusion equations is vast and we shall refer to classical textbooks such as [Smo83, Rot84, Ama85, Mur03] and the references therein for typical questions such as existence of solutions and their global boundedness, stability and asymptotics, travelling waves, geometry and topology of attracting sets and singular limits.

From very early on, the question of stability vs. instability of linear and nonlinear reaction–diffusion systems has been a key issue to mathematicians, physicists and biologists. In particular in 1952, A. M. Turing [Tur52] first pointed out diffusion–induced instability of stable homogeneous reaction systems in chemistry (well known as *Turing instability*). More general, the classical mathematical analysis of the long–time asymptotic behaviour involves linearised stability techniques, spectral theory, perturbation and invariant regions arguments, and Lyapunov stability (see e.g. [CHS78, FHM97]).

The approach presented in this paper is different and motivated by the recent great progress in the understanding of long–time asymptotics of scalar linear and nonlinear diffusion and diffusion–convection equations due to the so called *entropy approach*. The entropy method refers to the general idea of a *functional inequality relation* between an *entropy functional* of a system and its monotone change in time, usually called the *entropy dissipation*. Such an entropy–entropy dissipation inequality entails convergence to an entropy minimising equilibrium state, at first in entropy and further in L^1 using Cziszár-Kullback-Pinsker type inequalities (cf. [Ott01, DPD02, CJM+01, AMTU01]). The entropy approach is per se a nonlinear method avoiding any kind of linearisation and capable of providing explicitly computable convergence rates. Moreover, being based on functional inequalities rather than particular differential equations, it has the advantage of being quite robust with respect to model variations.

Related to the context of this paper, the entropy approach has already been applied to semiconductor drift–diffusion–Poisson systems [AMT00], or to drift–diffusion–reaction–Poisson systems on bounded domains in $\mathbb{R}^2$ [GGH96, GH97] but with the drawback of a proof based on an indirect contradiction argument without control on the rates and constants. We also mention the paper [DGJ97] dealing with general cross–diffusion systems. It was first in [DF06, DF08, DF] that explicit exponential convergence to equilibrium has been shown via entropy methods for nonlinear reaction–diffusion systems modelling reversible mass action kinetics of two, three, and four species. A general framework for reaction–diffusion systems however (long–time asymptotic convergence, convergence rates, etc.) is still lacking.

In order to demonstrate the entropy approach we consider the homogeneous Neumann problem for the heat equation on a bounded domain $\Omega \in \mathbb{R}^n$:

$$\begin{cases} u_t = \Delta u, \quad u(t=0,x) = u_I(x), & x \in \Omega \\ n(x)\cdot\nabla u(x,t) = 0. & x \in \partial\Omega \end{cases}$$

Due to the homogeneous Neumann conditions all constants are stationary states and the equilibrium state is determined by the conservation of the initial mass:

$$\overline{u}(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx = \frac{1}{|\Omega)|}\int_\Omega u_I(x)\,dx =: \overline{u_I} \qquad \text{for all } t>0\,,$$

where $|\Omega|$ denotes the volume of the domain. Multiplying the heat equation by u and integrating by parts yields the entropy (free energy) dissipation equation

$$\frac{d}{dt}\int_\Omega (u-\overline{u})^2\,dx = -2\int_\Omega |\nabla u|^2\,dx \le -\frac{2}{D_\Omega^2}\int_\Omega (u-\overline{u})^2\,dx\,,$$

where we have used the $H^1(\Omega)$–Poincaré inequality with constant $1/D_\Omega^2$, which is the spectral gap of the homogeneous Neumann–Laplace operator. Hence, after integration in time we obtain the sharp decay estimate:

$$\int_\Omega (u(x,t)-\overline{u_I})^2\,dx \le \exp\left(-\frac{2}{D_\Omega^2}t\right)\int_\Omega (u_I-\overline{u_I})^2(x)\,dx.$$

To mimic the effect of a stable reaction term we add now a linear absorption term with a constant rate:

$$\begin{cases} u_t=\Delta u-\lambda u, \quad u(t=0,x)=u_I(x), & x\in\Omega \\ n(x)\cdot\nabla u(x,t)=0. & x\in\partial\Omega \end{cases}$$

Clearly, this shifts the spectrum and convergence to the unique equilibrium state $u_\infty=0$ ($\overline{u}$ is no more conserved) follows from the sharp estimate:

$$\int_\Omega u(x,t)^2\,dx \le \exp(-2\lambda t)\int_\Omega u_I(x)^2\,dx.$$

Thus, even in this most simplistic scalar example equation, diffusion and stable reaction do not 'cooperate' in the rate of decay to equilibrium (since constant states are not affected by diffusion). Nevertheless, full asymptotic cooperation between diffusion and stable reaction is observed for the intermediate asymptotic state $\overline{u}(t)$:

$$\int_\Omega \left(u(x,t)-\overline{u}(t)\right)^2\,dx \le \exp\left(-2\left(\lambda+\frac{1}{D_\Omega^2}\right)t\right)\int_\Omega \left(u_I(x)-\overline{u_I}\right)^2\,dx$$

Later on we shall see that linear systems combining 'purely diffusive' modes obeying conservation laws (like the Neumann–Laplace problem) with 'diffusive–reactive' modes, which are governed by both diffusion and reaction (similar to the Neumann–Laplace–absorption problem) are particular interesting as far as (non)cooperation of reaction and diffusion effects on the convergence rates are concerned. In the next section, an explicitly computable 2×2 system with constant coefficients will demonstrate the complicated system related interaction effects of diffusion and reaction. Such systems are typical in many physical and biological applications.

The aim of this paper is to start developing a framework for the quantitative analysis of the large–time asymptotics of stable reaction–diffusion–convection systems based on the entropy approach. In particular for drift–diffusion–reaction systems on the whole space with confinement, the present paper provides – up to our knowledge – the first attempt in this generality.

Outline: In Section 2, we outline the presented approach in the clearest possible way, namely for systems posed on bounded domains with constant equilibrium states. In Section 3, we shall turn to the analysis of whole-space systems with confinement in each component. Both sections 2 and 3 deal with linear systems. An example of a nonlinear 2×2 reaction–diffusion system with a quadratic reaction term and confinement will be considered in Section 4 to indicate a possible general strategy and the technical difficulties when dealing with nonlinear systems. However, a general theory for systems with nonlinear reaction is still be developed.

We summarise the main results contained in the paper:

- In Theorem 2.1 we prove exponential stability of constant equilibrium states for linear systems on bounded domains. The structural assumptions in this result prelude to those in (the more important) Theorem 3.1, which deals with inhomogeneous systems on the whole spaces and which is shortly resumed below. A significant aspect is that, unlike in Theorem 3.1, in Theorem 2.1 we can allow for degenerate diffusion.
- A general result on the exponential stability of inhomogeneous equilibria for linear systems on the whole space $\mathbb{R}^n$ with confinement is proven in Theorem 3.1, which – loosely speaking (without going into the technical assumptions here) – reads as: Let the reaction-diffusion system be linear and appropriately symmetrisable, assume that the reaction is 'stable', i.e. the eigenvalues of the reaction matrix are either zero (of position independent multiplicity) or uniformly negative on the bounded domain in $\mathbb{R}^n$, on which the system is posed, subject to homogeneous

Neumann boundary conditions. Note that each zero eigenvalues of the reaction matrix corresponds to a conservation law. Then, for each L^2 -initial state there is unique constant equilibrium state, to which the solution converges exponentially in L^2 as time tends to infinity. The rate of decay can be characterised in terms of the coefficient matrices of the system. This characterisation is typically more complex than a simple 'combination' of the diffusive and reactive rates.

- A refined convergence result for a linearised 2×2 drift–diffusion–recombination system with confining potentials is shown in Theorem 3.2. Again, without going into technical details, the result states the following: Let the linear reaction-diffusion system be posed on the whole space $\mathbb{R}^n$, with appropriate confinement in every component of the system. As above, let the reaction be stable, generating a certain number of conservation laws. Then a unique integrable equilibrium state exists for all initial data. The solution of the initial value problem converges exponentially to this equilibrium, again with a characterisable rate.
- For a nonlinear drift–diffusion–recombination example system we show exponential convergence towards the inhomogeneous equilibria in Theorem 4.1. We consider a nonlinear drift-diffusion system with a charge-carrier recombination-generation term (subject to linear confinement) arising in solid state physics and prove exponential convergence of the carrier densities towards the inhomogeneous equilibria (Theorem 4.1).

Notations: For the sake of clarity, we recall some basic notations and conventions on matrix calculus. The linear space of square $N \times N$ matrices is denoted by $\mathbb{R}^{N\times N}$. Given a function $\mathbb{R}^n \ni x \mapsto A(x) \in \mathbb{R}^{N\times N}$ with real coefficients, we denote its j–th row by $A^j(x)$ and its k–th column by $A_k(x)$. The components of a N dimensional column vector $U \in \mathbb{R}^N$ will be denoted by U_j, $j = 1, \dots, N$. Moreover, we adopt the following conventions:

- The (vector valued) divergence operator acts *row–wise* on matrices, i. e. $\mathrm{div}A(x) = (\mathrm{div}A^1(x), \dots, \mathrm{div}A^d(x))^T$.
- The Jacobian matrix of a vector map $\mathbb{R}^n \ni x \mapsto U(x) \in \mathbb{R}^N$ *has gradients on its rows*, i. e. $\nabla U(x) = [\nabla U_1(x), \dots, \nabla U_d(x)]^T$.

The standard scalar product in $\mathbb{R}^N$ will be denoted by $a \cdot b$. We also recall the definition of scalar product of two matrices as $A : B = \mathrm{Tr}(AB^T) = \sum_{i=1}^N A^i \cdot B^i$. For future use, we recall the formula $S\ \mathrm{div}A(x) = \mathrm{div}(SA(x))$ with $A(x), S \in \mathbb{R}^{N\times N}$.

2. Linear systems on bounded domains with constant equilibria

This section studies linear reaction–diffusion systems on bounded domains $\Omega \in \mathbb{R}^n$ with zero outflow condition, which feature unique, spatial homogeneous equilibrium solutions. However, we emphasise that this section is not intended to be merely an exercise on linear techniques. Our goal here is to capture (in the simplest possible setting) the system related structural difficulties in determining the large time asymptotics of linear(ised) reaction-diffusion systems by applying the entropy approach (as outlined in the introduction), which promises to be rather robustly extendable to nonlinear problems.

It is well known that reaction-diffusion systems may feature Turing instabilities, a phenomenon where stable steady states of a spatial homogeneous reaction system are rendered unstable when diffusion, sufficiently different in the components of the system, is added. However, as we shall see, Turing instabilities can not occur in systems featuring an entropy functional, which dissipates due to both reaction and diffusion until a unique entropy minimising equilibrium state is reached.

Nevertheless, even for those 'entropically stabilised' systems, it remains a non-trivial question how convergence to equilibrium can be quantified in terms of reaction rates and diffusion constants, in particular as in general no maximum-principle holds. Let us discuss an explicitly computable *example in 1D*:

We consider reversible mass-action type reactions $\alpha\mathcal{A} \longleftrightarrow \beta\mathcal{B}$ with $\alpha, \beta \in \mathbb{N}$ for two species $\mathcal{A}$ and $\mathcal{B}$ described by the concentrations a and b on the interval $\Omega = [0, 1]$ with homogeneous Neumann boundary conditions :

$$\begin{pmatrix} \partial_t a - d_a\, \partial_{xx} a \\ \partial_t b - d_b\, \partial_{xx} b \end{pmatrix} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \cdot \begin{pmatrix} l\, a^\alpha \\ k\, b^\beta \end{pmatrix},$$

where d_a and d_b are positive diffusion constants and l and k are reaction rates. The above system has a unique positive, constant equilibrium state (a_∞, b_∞), which balances the reaction (i.e. $l\, a_\infty^\alpha = k\, b_\infty^\beta$) and satisfies the conservation of mass $\beta a_\infty + \alpha b_\infty = \int_{\mathbb{R}^N} (\beta a_{in} + \alpha b_{in})\, dx$ for given nonnegative initial data.

Linearising around the equilibrium states and denoting $u = a - a_\infty$ and $v = b - b_\infty$ leads to the linear system

$$\left(\begin{array}{c} \partial_t u - d_a\, \partial_{xx} u \\ \partial_t v - d_b\, \partial_{xx} v \end{array} \right) = F \left(\begin{array}{cc} -\alpha^2 & \alpha\beta \\ \alpha\beta & -\beta^2 \end{array} \right) \cdot \left(\begin{array}{cc} \frac{1}{a_\infty} & 0 \\ 0 & \frac{1}{b_\infty} \end{array} \right) \cdot \left(\begin{array}{c} u \\ v \end{array} \right), \tag{1}$$

where $F = F(\alpha_\infty, \beta_\infty) = l a_\infty^\alpha = k b_\infty^\beta$. By rescaling we are able to set, without loss of generality, $\alpha_\infty = \beta_\infty = 1 = F$, i.e.

$$\left(\begin{array}{c} \partial_t u - d_a\, \partial_{xx} u \\ \partial_t v - d_b\, \partial_{xx} v \end{array} \right) = \left(\begin{array}{cc} -\alpha^2 & \alpha\beta \\ \alpha\beta & -\beta^2 \end{array} \right) \cdot \left(\begin{array}{c} u \\ v \end{array} \right),$$

which can be solved explicitly in terms of Fourier series: Denoting $\tilde\varphi : [-1,1] \to \mathbb{R}$ the even mirrored extension of a function $\varphi : [0,1] \to \mathbb{R}$, the eigenproblem $\varphi = \lambda_k \varphi$ on $[0,1]$ with homogeneous Neumann boundary conditions transforms into the eigenproblem $\tilde\varphi = \lambda_k \tilde\varphi$ on $[-1,1]$ with periodic boundary conditions. The resulting othonormed eigensystem ($\lambda_0 = 0$ with eigenfunction $\tilde\varphi_0 = \frac{1}{\sqrt{2}}$ and $\lambda_k = -(k\pi)^2$, $\tilde\varphi_k = \cos(k\pi x)$ for all $k \geq 1$) Fourier-decomposes the mirrored concentrations $\tilde u = \sum_{k=0}^\infty u^k(t) \tilde\varphi_k$ and $\tilde v = \sum_{k=0}^\infty v^k(t) \tilde\varphi_k$, i.e.

$$\partial_t \left(\begin{array}{c} u^k \\ v^k \end{array} \right) = \left(\begin{array}{cc} d_a \lambda_k - \alpha^2 & \alpha\beta \\ \alpha\beta & d_b \lambda_k - \beta^2 \end{array} \right) \cdot \left(\begin{array}{c} u^k \\ v^k \end{array} \right).$$

The individual Fourier-modes yield the eigenvalues $\mu_{1/2}(k) = \tau/2 \pm \sqrt{\tau^2/4 - \Delta}$ with the negative trace $\tau = (d_a + d_b)\lambda_k - (\alpha^2 + \beta^2) < 0$ and the nonnegative determinant $\Delta = d_a d_b \lambda_k^2 - \lambda_k (d_a \beta^2 + d_b \alpha) \geq 0$. In particular we have $\Delta > 0$ iff $k \geq 1$ and then $\mu_2(k) < \tau/2 < \mu_1(k) < 0$. Moreover we have $\mu_1(k+1) < \mu_1(k)$, and the convergence to equilibrium is determined by the eigenvalues

$$\mu_1(0) = 0\,, \qquad \mu_2(0) = -(\alpha^2 + \beta^2) < 0\,,$$
$$\mu_1(1) = -\tfrac{d_a + d_b}{2}\pi^2 - \tfrac{\alpha^2+\beta^2}{2} + \sqrt{(\pi^2(d_a - d_b) + \alpha^2 - \beta^2)^2/4 + \alpha^2\beta^2} < 0\,,$$

where $\mu_1(0)$ corresponds to the mass–conservation law and the rate of convergence is given by the minimum of $|\mu_2(0)|$ and $|\mu_1(1)|$. One can check that

$$|\mu_1(1)| > |\mu_2(0)| \;\Leftrightarrow\; \pi^2 > \frac{\alpha^2}{d_b} + \frac{\beta^2}{d_a}\,, \quad \left(\Leftrightarrow\; F\pi^2 > \frac{\alpha^2}{d_b} + \frac{\beta^2}{d_a} \text{ unrescaled.} \right)$$

and the rate of convergence depends in a non-trivial fashion on the reaction parameters, the diffusion constants and a geometric parameter of the domain. Note that the value of $|\mu_1(1)|$ involves both reaction and diffusion parameters, except in case of equal diffusion constants $d_a = d_b = d$, when $|\mu_1(1)| = d\pi^2$ reflects only the diffusion. In the degenerate cases of one vanishing diffusivity $d_a = 0$ or $d_b = 0$ we have exponential decay with a rate involving the other positive diffusivity and the reaction.

2.1. **Statement of the problem and remarks.** We consider linear systems in the symmetrised form

$$\partial_t (SU) = \operatorname{div}(\widetilde{D}(x) \nabla U) + \widetilde{R}(x) U, \tag{2}$$

with $U = U(x,t) \in \mathbb{R}^N$, $N \geq 1$ denoting the unknowns depending on time $t \geq 0$ and position $x \in \Omega \subset \mathbb{R}^n$ within a bounded domain Ω with sufficiently smooth (e.g. Lipschitz) boundary. We further prescribe the initial datum

$$U(x,0) = U_0(x), \tag{3}$$

and assume zero-flux boundary conditions

$$(\widetilde{D}(x) \nabla U(x)) \cdot \nu(x) = 0 \qquad \text{on } \partial\Omega, \tag{4}$$

where $\partial\Omega \ni x \mapsto \nu(x) \in \mathbb{R}^n$ denotes the outer normal unit vector at the boundary $\partial\Omega$. The structural assumptions on system (2) are the following:

(A1) *Non-degenerate symmetrizer matrix S:* The matrix $S \in \mathbb{R}^{N\times N}$ is constant, symmetric and strictly positive definite. More precisely, there exists a constant $\underline{s} > 0$ such that $S\xi \cdot \xi \geq \underline{s}|\xi|^2$ for all $\xi \in \mathbb{R}^N$. For future use we also define $\overline{s}$ as the maximum eigenvalue of S.

(A2) *Positive semi-definite diffusion matrix $\widetilde{D}(x)$:* The matrix $\widetilde{D}(x)$ is symmetric with eigenvalues $(0,\dots,0,\mu_{d+1}(x),\dots,\mu_N(x))$ for an integer $d < N$ and $0 < \underline{\mu} \leq \mu_j(x)$ for all $j \in \{d+1,\dots,N\}$. Moreover, $\widetilde{D}(x)$ has a constant eigenvector othonormed matrix F, i. e.

$$F\widetilde{D}(x)F^T = \mathrm{diag}(\underbrace{0,\dots,0}_{d},\mu_{d+1}(x),\dots,\mu_N(x)) \tag{5}$$

(A3) *Negative semi-definite reaction matrix $\widetilde{R}(x)$:* The matrix $\widetilde{R}(x)$ is symmetric with eigenvalues $(0,\dots,0,\lambda_{c+1}(x),\dots,\lambda_N(x))$ for an integer $1 \leq c < N$ and $\lambda_j(x) \leq -\underline{r} < 0$ for all $j \in \{c+1,\dots,N\}$. Moreover, $\widetilde{R}(x)$ admits a constant orthogonal eigenvector matrix E, i. e.

$$E\widetilde{R}(x)E^T = \Lambda(x), \qquad \Lambda(x) := \mathrm{diag}(\underbrace{0,\dots,0}_{c},\lambda_{c+1}(x),\dots,\lambda_N(x)) \tag{6}$$

(A4) *Compatibility of reaction and diffusion matrices:* The following condition holds

$$\mathrm{Ker}\widetilde{D}(x) \cap \mathrm{Ker}\widetilde{R}(x) = \{0\} \qquad \text{for all } \ x \in \Omega. \tag{7}$$

Remark 2.1 (Simultaneously symmetrisable systems). A major subclass of systems (2) which we have in mind are linear reaction–diffusion systems of the form

$$\partial_t U = \mathrm{div}\,(D(x)\nabla U) + R(x)U, \tag{8}$$

where the matrices $D(x)$ and $R(x)$ are, although not necessarily symmetric, *simultaneously symmetrisable*, i. e. there exists a non degenerate, symmetric, positive definite, constant matrix $S \in \mathbb{R}^{N\times N}$ such that $SD(x)$ and $SR(x)$ are symmetric. Multiplication of system (8) by the symmetrizer matrix S yields the symmetrised system (2) with $\widetilde{D} = SD$ and $\widetilde{R} = SR$ (see, for instance, (1) in the above example with the symmetrizer matrix $S = \mathrm{diag}(a_\infty^{-1}, b_\infty^{-1})$). We remark that the structural assumptions and the theorems presented in this section can be stated more straightforwardly in the symmetrised version (2).

Remark 2.2 (Weaker assumptions: symmetry of $\widetilde{D}$ and $\widetilde{R}$ is not necessary). The set of assumptions (A1)-(A4) could be relaxed by dropping the symmetry assumptions on $\widetilde{D}(x)$ and $\widetilde{R}(x)$ and by instead requiring that the symmetric parts $\widetilde{D}(x) + \widetilde{D}(x)^T$ and $\widetilde{R}(x) + \widetilde{R}(x)^T$ satisfy all other hypotheses set up above for $\widetilde{D}(x)$ and $\widetilde{R}(x)$ respectively. In terms of the original formulation (8) of a reaction diffusion system, this means that one can avoid requiring that $D(x)$ and $R(x)$ are simultaneously symmetrisable as stated in Remark 2.1 by instead supposing that

$$\begin{cases} \widehat{D} := SD + DS, & \text{satisfys the assumptions on } \widetilde{D}(x) \text{ in (A2)} \\ \widehat{R} := SR + R^T S, & \text{satisfys the assumptions on } \widetilde{R}(x) \text{ in (A3).} \end{cases} \tag{9}$$

The exponential convergence statement in Theorem 2.1 holds still valid under assumptions (9), with the only problem of a more involved notation. Since the existence of a simultaneous symmetrizer S is guaranteed in all the applied models motivating our theory, we shall simplify the treatment by requiring (A1)-(A4) instead of (9). We shall prove in Proposition 2.2 that (9) are the minimal assumptions needed to have a reasonable energy structure for (8).

Remark 2.3 (Analogy with systems of conservation laws). The set of condition (A1)–(A4) and its generalisation (9) share similarities with the structural condition required in the study of nonlinear systems of conservation laws, cf. for instance [Fri54, KL89, SK85]. In particular, condition (A4) is reminiscent of the so called Kawashima condition, which is well known to ensure, for instance, convergence to equilibrium for hyperbolic–parabolic systems by ensuring that the normal modes corresponding to the nonlinear convection part do not 'meet' the degeneracy in the parabolic part.

Remark 2.4 (Unique equilibria in case of semidefinite diffusion and reaction matrices). In case of non zero integers d and c in the assumptions (A2) and (A3), the diffusion part and the reaction part separately are not able to drive the system toward a unique homogeneous equilibrium state. In particular, the degeneracy of the reaction matrix implies that a certain number of integral quantities is conserved with respect to time (see Lemma 2.1). Therefore, the system needs a partial amount of diffusion to be driven toward a unique homogeneous equilibrium. More precisely, the diffusion operator should be non degenerate along the 0-eigenvectors of the reaction matrix. This is assured by the orthogonality condition (A4).

Remark 2.5. We remark that the assumption of a constant eigenvector matrix E in (A3) may be satisfied even in the case of a matrix $\widetilde{R}(x)$ with variable coefficients. Consider as an example (with the notation introduced in (A1)–(A4)) the following 3×3 case

$$\widetilde{D}(x) \equiv \mathbb{I}_3, \qquad \widetilde{R}(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a(x) & b(x) \\ 0 & b(x) & a(x) \end{bmatrix} \qquad E = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \tag{10}$$

with $a(x) - b(x)$ and $a(x) + b(x)$ both uniformly negative.

Remark 2.6. The hypotheses of a constant eigenvector matrix F in (A2) is not needed in case the diffusion matrix $D(x)$ is uniformly non degenerate.

Before stating the theorems we shall present two important examples of reaction–diffusion systems satisfying the set of assumptions (A1)-(A4). Both examples will be stated in the original form (8).

Example 2.1 (Semiconductor device with trapped state)**.** Let us consider the system

$$\begin{cases} \partial_t n - \Delta n = C_c n_{tr} - C_a n (N_{tr} - n_{tr}) =: R_n \\ \partial_t p - \Delta p = C_d (N_{tr} - n_{tr}) - C_b\, p\, n_{tr} =: R_p \\ \partial_t n_{tr} = R_p - R_n \end{cases} \tag{11}$$

modelling transport and diffusion of charged particles in a semiconductor device combined with a recombination–generation mechanism called *band–trap capture and emission* (due to the presence of impurities, see [MRS90, Chapter 2.6] for further explanations). Here n denotes the density of electrons, p denotes the density of holes and n_{tr} is the densities of occupied traps. The parameter N_{tr} represents the density of traps, therefore $n_{tr} < N_{tr}$. For simplicity, we have neglected the effect of the self–consistent potential, therefore coupling occurs only due to the recombination–generation terms. The equilibrium vectors $(n^\infty, p^\infty, n_{tr}^\infty)$ satisfy

$$n^\infty p^\infty = \frac{C_c C_d}{C_a C_b}, \qquad n_{tr}^\infty = N_{tr} \frac{C_a n^\infty + C_d}{C_a n^\infty + C_c + C_b p^\infty + C_d} < N_{tr},$$

whereas the uniqueness of the equilibrium state is achieved by imposing

$$|\Omega|(n^\infty - p^\infty + n_{tr}^\infty) = M.$$

The linearisation of system (11) around the unique equilibrium state gives the linear system

$$U_t - \Delta(DU) = R \tag{12}$$

where

$$D = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} C_a(n_{tr}^\infty - N_{tr}) & 0 & C_c + C_a n^\infty \\ 0 & -C_b n_{tr}^\infty & -C_d - C_b p^\infty \\ -C_a(n_{tr}^\infty - N_{tr}) & -C_b n_{tr}^\infty & -C_d - C_b p^\infty - C_c - C_a n^\infty \end{bmatrix}.$$

A symmetrizer for R is the diagonal matrix $S = \operatorname{diag}\left(\frac{1}{n^\infty},\ \frac{1}{p^\infty},\ \frac{1}{n^\infty}\left(1 + \frac{C_d}{p^\infty C_b}\right)\right)$ and the symmetrised reaction matrix reads

$$\widetilde{R} = SR = \begin{bmatrix} \frac{C_a}{n^\infty}(n_{tr}^\infty - N_{tr}) & 0 & \frac{C_c}{n^\infty} + C_a \\ 0 & -\frac{C_b n_{tr}^\infty}{p^\infty} & -C_b - \frac{C_d}{p^\infty} \\ \frac{C_c}{n^\infty} + C_a & -C_b - \frac{C_d}{p^\infty} & -N_{tr}(C_a n^\infty + C_d)\frac{C_d + C_b p^\infty}{C_b p^\infty} \end{bmatrix}.$$

The matrix $\widetilde{R}$ has the only 0–eigenvector $E^1 := (n^\infty, p^\infty, \frac{C_c C_d}{C_a(p^\infty C_b + C_d)})$, which implies that the assumption (A5) is satisfied. An elementary but tedious calculations shows that the two remaining eigenvalues λ_2 and λ_3 of $\widetilde{R}$ are both strictly nonnegative. Therefore, the symmetrised form of the linear system (12) satisfies all assumptions (A1)–(A4).

Example 2.2 (Reaction–diffusion system with four species)**.** We consider a diffusive and reversible chemical reaction of the type $\mathcal{A}_1 + \mathcal{A}_3 \rightleftharpoons \mathcal{A}_2 + \mathcal{A}_4$ on a bounded domain $\Omega \subset \mathbb{R}^n$ (cf. [DF08, DF]). Denote by $a_i(x,t)$, $i = 1, \dots, 4$ the concentration of the four reacting species $\mathcal{A}_1, \dots, \mathcal{A}_4$ and by $d_i > 0$, $i = 1, \dots, 4$ their respective diffusivity constants. Assuming mass action kinetics for the reactions, we obtain the system

$$\partial_t a_i = d_i \Delta a_i + (-1)^i (a_1 a_3 - a_2 a_4),$$

where we have rescaled – without loss of generality – the reaction rates to one. The stationary states $(a_1^\infty, \dots, a_4^\infty)$ consists of the unique set of positive constants, which balance the reaction, i.e. $a_1^\infty a_3^\infty = a_2^\infty a_4^\infty$, and satisfy the conservation laws of the systems. Linearisation around those states produces the linearised reaction matrix R and the associated diagonal symmetrizer matrix S

$$R := \begin{bmatrix} -a_3^\infty & a_4^\infty & -a_1^\infty & a_2^\infty \\ a_3^\infty & -a_4^\infty & a_1^\infty & -a_2^\infty \\ -a_3^\infty & a_4^\infty & -a_1^\infty & a_2^\infty \\ a_3^\infty & -a_4^\infty & a_1^\infty & -a_2^\infty \end{bmatrix}, \qquad S := \operatorname{diag}\left(\frac{1}{a_1^\infty}, \dots, \frac{1}{a_4^\infty} \right).$$

It is easy to check that all assumptions (A1)–(A4) are satisfied in this case.

2.2. **Stationary states and their exponential stability.** In this subsection we prove exponential convergence to constant equilibrium states for linear systems on bounded domains. We shall use the symmetrised form (2) and work always in the framework set by the assumptions (A1)–(A4). Note that global existence of classical solutions for linear systems follows readily by standard fix-point and Gronwall arguments (See e.g. [Rot84, Ama85]). A first natural problem is to detect and characterise stationary solutions to (2). In addressing these issues, we identify in the following Lemma the eigenvectors with eigenvalue zero of the symmetrised reaction matrix $\widetilde{R}$ as *conservation laws* of the system.

Lemma 2.1 (Conserved quantities)**.** *Let $U(x,t)$ be a classical solution to system (2). Then, for every $j \in \{1, \dots, c\}$, the quantity*

$$l_j := \int_\Omega E^j S U(x,t) dx \tag{13}$$

is conserved for all times $t \geq 0$.

Proof. We compute the time derivative of l_j, for $j \in \{1, \dots, c\}$. Due to (2) and (6), we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega E^j S U(x,t) dx &= \int_\Omega E^j (S U(x,t))_t dx = \int_\Omega E^j \operatorname{div}(\widetilde{D} \nabla U) dx + \int_\Omega E^j \widetilde{R}(x) U dx \\ &= \int_{\partial\Omega} E^j S D(x) \nabla U \nu(x) d\sigma(x) = 0 \end{aligned}$$

where we have used Green's theorem and the boundary condition (4). □

Proposition 2.1 (Stationary solutions)**.** *For every fixed set of quantities $l_1, \dots, l_c$, there exists a unique stationary solution U^∞ to (2) such that*

$$l_j := |\Omega| E^j S U^\infty \tag{14}$$

for all $j \in \{1, \dots, c\}$. Moreover, U^∞ is constant and it satisfies $U^\infty = E^T V^\infty$ with $V_{c+1}^\infty = \dots = V_N^\infty = 0$.

Proof. Let us multiply the stationary form of system (2) by $U^\infty(x)$ and integrate over Ω. After integration by parts and in view of the boundary conditions, we obtain

$$\int_\Omega \nabla U^\infty(x) : \widetilde{D}(x) \nabla U^\infty(x) dx = \int_\Omega (U^\infty(x))^T \widetilde{R} U^\infty(x) dx \leq 0.$$

But the term on the left hand side is also nonnegative due to assumption (A2)

$$\int_\Omega \nabla U^\infty(x) : \widetilde{D}(x)\nabla U^\infty(x)dx = \sum_{j=1}^N \int_\Omega \frac{\partial U^\infty}{\partial x_j}^T \widetilde{D}(x)\frac{\partial U^\infty}{\partial x_j}dx \geq 0.$$

Consequently, both above terms equal to zero and we have $U^\infty(x) \in \mathrm{Ker}\widetilde{R}(x)$. Let us then define

$$V^\infty := EU^\infty. \tag{15}$$

We have

$$0 = \widetilde{R}(x)U^\infty = \widetilde{R}(x)E^TV^\infty = E^T\Lambda(x)V^\infty.$$

Since E is nondegenerate, this implies $\Lambda(x)V^\infty = 0$. Therefore the set of possible stationary states is the d-dimensional linear subspace of all U^∞ with $\Lambda(x)V^\infty = 0$ which is equivalent to require

$$V^\infty_{c+1} = \ldots = V^\infty_N = 0. \tag{16}$$

On the other hand, condition (14) can be rewritten with (15) as

$$\frac{1}{|\Omega|}\begin{pmatrix} l_1 \\ \vdots \\ l_c \end{pmatrix} = \begin{bmatrix} E^1 \\ \vdots \\ E^c \end{bmatrix} SE^TV^\infty = \underbrace{\begin{bmatrix} E^1 \\ \vdots \\ E^c \end{bmatrix} S \begin{bmatrix} (E^1)^T & \ldots & (E^c)^T \end{bmatrix}}_{:=G} \begin{pmatrix} V_1^\infty \\ \vdots \\ V_c^\infty \end{pmatrix}, \tag{17}$$

and, as the matrix G is constant and nondegenerate, there exists a unique c–dimensional constant vector $(V_1^\infty, \ldots, V_c^\infty)^T$ satisfying (17). This completes the proof. □

For future use we introduce the, according to (15), the *normal modes* vector variable

$$V(x,t) := EU(x,t) \tag{18}$$

and the following *energy functional*

$$\mathcal{E}[U] := \frac{1}{2}\int_\Omega U(x)^T S\ U(x)dx. \tag{19}$$

In the following Proposition we will see that (19) is indeed a Lyapunov functional for (2), which will be used later on to prove convergence to equilibrium for solutions to (2). Before that let us detail further Remark 2.2. Going back to the original form (8) of a linear reaction diffusion system, we prove in the following proposition that a slightly weaker assumptions than simultaneous symmetrisation is necessary to have a quadratic Lyapunov functional for (8). More precisely we shall prove (under the assumption that $\widetilde{D}$ and $\widetilde{R}$ are constant, in order to avoid technicality) that conditions (9) are necessary for $\mathcal{E}$ defined in (19) to be a a Lyapunov functional for (8).

Proposition 2.2 (Lyapunov functionals and symmetrisation)**.** *Let R and D be constant matrices. Suppose the reaction matrix R has at least one left eigenvector V. Let S be a symmetric $N \times N$ matrix with constant coefficients. Then, the functional defined in (19) is a Lyapunov functional for (8) if and only if $SD + DS$ and $-(SR + R^TS)$ are nonnegative definite.*

Proof. We first prove the 'only if' statement by contradiction. For a classical solution $U(t)$ to (8) let us compute the evolution of the energy:

$$\frac{d}{dt}\mathcal{E}[U(t)] = -\frac{1}{2}\int_\Omega \nabla U : (SD + DS)\nabla U dx + \frac{1}{2}\int_\Omega U^T(SR + R^TS)U dx.$$

Case 1. Suppose first that $SD + DS$ is not nonnegative definite. Then there exists a vector $v \in \mathbb{R}^N$ such that $v^T(SD + DS)v =: -\alpha < 0$. Let us define the vector function

$U_0(x) := f(x)v$ for a scalar smooth function f. Then, we have

$$\int_\Omega \int_\Omega \nabla U_0 : (SD + DS)\nabla U_0 dx = \sum_{k=1}^N \int_\Omega \frac{\partial U_0}{\partial x_k}^T (SD + DS)\frac{\partial U_0}{\partial x_k} dx$$

$$\sum_{k=1}^N \int_\Omega v^T (SD + DS)v \left(\frac{\partial f}{\partial x_k}\right)^2 dx = -\alpha \int_\Omega |\nabla f(x)|^2 \, dx.$$

We then set $\beta := v^T(SR + R^T S)v$. We consider the solution to (8) with initial datum U_0. At time $t = 0$ we have

$$\frac{d}{dt}\mathcal{E}[U(t)] \geq \alpha \int_\Omega |\nabla f(x)|^2 \, dx - |\beta| \int_\Omega f(x)^2 dx.$$

Since the functional $\|\nabla f\|_{L^2}$ is unbounded on the set $\{f \in H^1(\Omega) : \ \|f\|_{L^2} = 1\}$, it is always possible to find a function $f \in H^1(\Omega)$ such that the right hand side above is strictly positive. Therefore, $\mathcal{E}[U(t)]$ is increasing at $t = 0$ and this proves the assertion.

Case 2. Suppose that $SR + R^T S$ is not nonpositive. Let $w \in \mathbb{R}^N$ such that $w^T(SR + R^T S)w =: \gamma > 0$. Let us consider the constant initial datum $U_0(x) \equiv w$. Trivially, one has $\frac{d}{dt}\mathcal{E}[U(t)] = \gamma > 0$ at time $t = 0$, which implies that the energy is increasing.

Vice versa, it is straightforward to see that $(SD+DS)$ and $(SR+R^T S)$ being nonnegative definite is a sufficient condition for $\mathcal{E}[U(t)]$ to be nonincreasing in time. □

Example 2.3 (Lyapunov functionals and stability)**.** Note that symmetrisation is not the only way to show stability for a reaction–diffusion system. Consider, for instance, the two dimensional system

$$U_t = d\Delta U + RU,$$

where $U = (u, v)^T \in \mathbb{R}^2$, $d > 0$ and

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

Suppose that the eigenvalues r_{11} and r_{22} of R are real, distinct and negative, and let E be the eigenvector matrix of R, such that $ERE^T = \mathrm{diag}(r_{11}, r_{22})$. Let us also suppose that $r_{12}^2 > 4r_{11}r_{22}$ ensuring that the symmetric part of $\frac{1}{2}(R + R^T)$ of R has two eigenvalues of distinct sign (easy to check). In this example where the diffusion matrix is a multiple of the identity, a diagonalisation method is more efficient than symmetrisation. Indeed, the new variable $V := EU$ satisfies the uncoupled system

$$V_t = d\Delta V + \mathrm{diag}(r_{11}, r_{22})V$$

which implies that V decays to zero exponentially. The symmetrisation method in the alternative sense (9) would also lead to the same result (by a more involved computation) by using the symmetrising matrix $S = \mathrm{diag}(s_1, \ s_2)$ where s_1 and s_2 are positive constants satisfying $4s_1r_{11}r_{22} > s_2r_{12}^2$, which is possible if and only if $s_2 < s_1$ as $r_{12}^2 > 4r_{11}r_{22}$.

Example 2.4 (Absence of symmetry and Turing instability)**.** For the sake of completeness we recall a well known example of a linear 2×2 reaction diffusion system featuring *Turing instability* (cf. [Gri91]), i. e. a system of the form

$$\partial_t U = \Delta DU + RU$$

with $U = (u, v)$, $D = \mathrm{diag}(d_1, d_2)$, d_1 and d_2 positive and $R = (r_{ij})_{i,j=1}^2$ having two distinct nonnegative real eigenvalues. We shall point out that Turing instabilities can occur only if R is not symmetric and $d_1 \neq d_2$, which partly motivates the symmetrisation setting (A1)-(A3) as a reasonable structural framework (besides the fact that such a framework fits certain target applications). Applying the Fourier's transform gives the following system for the vector $\widehat{U}(\xi, t)$

$$\widehat{U}_t = B(\xi)\widehat{U}, \qquad B(\xi) := \begin{bmatrix} -d_1|\xi|^2 + r_{11} & r_{12} \\ r_{21} & -d_2|\xi|^2 + r_{22} \end{bmatrix}.$$

The eigenvalues of R being negative implies in particular $r_{11} + r_{22} < 0$. By computing explicitly the eigenvalues of $B(\xi)$, one easily realizes that exponential growth of one of the

two normal modes on a closed non empty ξ–interval may occur only if $(r_{11}d_2+r_{22}d_1)>0$, and this condition can be satisfied only if $d_1 \neq d_2$ because of the original condition $r_{11}+r_{22}<0$ on the trace of R. Imposing R to be symmetric implies $\det R = r_{11}r_{22}-r_{12}^2 > 0$ and therefore r_{11} and r_{22} both have negative sign, which is in contradiction with the above condition $(r_{11}d_2+r_{22}d_1)>0$.

We now state the main result of this section.

Theorem 2.1 (Exponential convergence towards constant equilibrium states)**.** *Let $U(x,t)$ be a classical solution to (2) with initial datum $U_0(x) \in L^2(\Omega)$ satisfying*

$$l_j := \int_\Omega E^j S U_0(x) dx \tag{20}$$

for some given $l_1,\dots,l_c \in \mathbb{R}$. Then, for all times $t \geq 0$ the following estimate holds

$$\mathcal{E}[U(t)-U^\infty] \leq \mathcal{E}[U(0)-U^\infty] \exp\left(-\left(\frac{K\underline{r}}{\overline{s}(K+L)}-\delta\right)t\right), \tag{21}$$

where $K := \frac{\underline{\mu}}{\overline{C_P(\Omega)}}\ C_F$ and $C_P(\Omega)$ is the optimal constant in Poincaré inequality on the domain Ω, L and C_F are constants depending on the matrices E and F respectively, the constant $\overline{s}$, $\underline{\mu}$ and $\underline{r}$ are defined in the assumptions (A1)–(A3) and δ is an arbitrary positive small constant, $0<\delta \ll 1$. In particular, $U(t) \to U^\infty$ exponentially fast in L^2 as $t \to +\infty$.

Remark 2.7. In fact exponential convergence in all H^s norms holds by interpolation between the above L^2 decay and H^s a-priori bounds, which are propagated by the linear system.

Proof. For a solution $U(t)$ to (2) we compute the time derivative of $\mathcal{E}[U(t)-U^\infty]$. We shall implicitly use the boundary conditions (4) in the integration by parts. Let us then compute

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)-U^\infty] &= -\int_\Omega \nabla U : \widetilde{D}(x)\nabla U dx + \int_\Omega (U-U^\infty)^T \widetilde{R}(x)(U-U^\infty)dx \\ &= -\sum_{k=1}^N \int_\Omega \frac{\partial U}{\partial x_k}^T \widetilde{D}(x)\frac{\partial U}{\partial x_k}dx + \int_\Omega (U-U^\infty)^T \widetilde{R}(x)(U-U^\infty)\,dx. \end{aligned} \tag{22}$$

Now, let us define P_1 and P_2 to be the orthogonal projection operators on $\mathrm{Ker}\widetilde{D}$ and on $[\mathrm{Ker}\widetilde{D}]^\perp$ respectively. Then, it is clear that, for each $k=1,\dots,N$,

$$\frac{\partial U}{\partial x_k}^T \widetilde{D}(x)\frac{\partial U}{\partial x_k} = P_2\left(\frac{\partial U}{\partial x_k}\right)^T \widetilde{D} P_2\left(\frac{\partial U}{\partial x_k}\right) \geq \underline{\mu}\ \left|P_2\left(\frac{\partial U}{\partial x_k}\right)\right|^2.$$

We use now, in a crucial way, the assumption (A4), which ensures that the operator P_2 restricted to the linear space $\mathrm{Ker}\widetilde{R}$ is invertible. Indeed, we recall that, for every vector $v \in \mathbb{R}^N$, $P_2 v$ is defined as follow:

$$P_2 v = (0,\dots,0,F^{d+1}\cdot v,\dots,F^N\cdot v)^T,$$

where the F^j's are the rows of the matrix F defined in Assumption (A2). On the other hand, if $v \in \mathrm{Ker}\,\widetilde{R}$, due to the assumption (A4), there exists a $j \in \{d+1,\dots,N\}$ such that $v\cdot F^j \neq 0$, which implies $\mathrm{Ker}\, P_{2\llcorner_{\mathrm{Ker}\widetilde{R}}} = \{0\}$. Since the matrix F is constant, this implies that the inverse of $P_{2\llcorner_{\mathrm{Ker}\widetilde{R}}}$ is a bounded operator. Therefore, there exists a constant C_F depending on the matrix F such that

$$\left|P_2\left(\frac{\partial U}{\partial x_k}\right)\right|^2 \geq C_F \left|P_3\left(\frac{\partial U}{\partial x_k}\right)\right|^2$$

where P_3 denotes the orthogonal projection operator on the linear space $\mathrm{Ker}\widetilde{R}$. Recalling the notations in the assumption (A3), we can represent P_3 as follows: for every vector $v \in \mathbb{R}^N$, we have

$$P_3 v = (E^1\cdot v,\dots,E^c\cdot v,0,\dots,0)^T. \tag{23}$$

We then combine the above estimates with (22) to get

$$\frac{d}{dt}\mathcal{E}[U(t)-U^\infty] \le -\underline{\mu}\, C_F \sum_{k=1}^N \left\| P_3\left(\frac{\partial U}{\partial x_k}\right)\right\|^2_{L^2(\Omega)} + \int_\Omega (U-U^\infty)^T \widetilde{R}(x)(U-U^\infty)dx.$$

Since P_3 is a linear operator with constant coefficient, we can interchange it with the partial derivative operator. Moreover we use the Poincaré inequality

$$\|f-\bar f\|_{L^2(\Omega)} \le C_P(\Omega)\|\nabla f\|_{L^2(\Omega)}, \tag{24}$$

which applies to all scalar functions f with squared integrable gradient in Ω; here $\bar f$ denotes the average of f on the set Ω, namely $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f dx$. The result reads

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)-U^\infty] \le &-\frac{\underline{\mu}\ C_F}{C_P(\Omega)}\sum_{j=1}^N \|(P_3U)_j - \overline{(P_3U)_j}\|^2_{L^2(\Omega)} \\ &+ \int_\Omega (U-U^\infty)^T\widetilde{R}(x)(U-U^\infty)dx.\end{aligned} \tag{25}$$

Now we want to rephrase the above estimate (25) in terms of the normal mode variable V, by using the orthogonality of E together with the assumption (A3). As a trivial (but crucial) consequence of E having constant coefficients we shall also use the invariance of the averages after passing to normal modes, namely

$$\overline{V} = \overline{EU} = E\overline{U},$$

where $\overline{U}$ denotes the averaged vector $(\overline{U_1},\dots,\overline{U_N})^T$. Moreover, due to (23), we can write P_3U in terms of V as follows:

$$P_3U = ((E^1)^TE^TV,\dots,(E^c)^TE^TV,0,\dots,0)^T = (V_1,\dots,V_c,0,\dots,0).$$

The estimate (25) becomes

$$\frac{d}{dt}\mathcal{E}[U(t)-U^\infty] = -\frac{\underline{\mu}\ C_F}{C_P(\Omega)}\sum_{j=1}^c \|V_j-\overline{V_j}\|^2_{L^2(\Omega)} + \sum_{j=c+1}^N\int_\Omega \lambda_j(x)|V_j-V_j^\infty|^2dx,$$

where we have used (6). Therefore, (16) and the assumption (A3) give

$$\frac{d}{dt}\mathcal{E}[U(t)-U^\infty] \le -K\sum_{j=1}^c \|V_j-\overline{V_j}\|^2_{L^2(\Omega)} - \underline{r}\sum_{j=c+1}^N\int_\Omega V_j^2dx =: -D_D - D_R, \tag{26}$$

with $K := \frac{\underline{\mu}\ C_F}{C_P(\Omega)}$. The estimate (26) shows that both the diffusion part and the reaction part of system (8) produce dissipation of the energy functional $\mathcal{E}$. However, intuitively the dissipation term D_D due to the diffusion part drives V_j towards its average $\overline{V_j}$ for $j=1,\dots,d$, whereas the reaction dissipation term D_R drives V towards the linear space $V_{d+1}=\dots=V_N=0$, which contains the equilibrium vector V^∞. Since D_R is not enough to select V^∞ as the equilibrium state, we have to use part of the term D_D in order to achieve a term proportional to $-\mathcal{E}[V-V^\infty]$ in the right hand side of (26). We proceed as follows:

$$\begin{aligned}\sum_{j=1}^c\int_\Omega |V_j-\overline{V_j}|^2dx &\ge \alpha\sum_{j=1}^c\int_\Omega |V_j-\overline{V_j}|^2dx \qquad [\text{for all } \alpha\in(0,1)] \\ &\ge \frac{\varepsilon\alpha}{1+\varepsilon}\sum_{j=1}^c\int_\Omega |V_j-V_j^\infty|^2dx - \varepsilon\alpha\sum_{j=1}^c\int_\Omega |V_j^\infty-\overline{V_j}|^2dx,\end{aligned} \tag{27}$$

for all $\varepsilon>0$. Now we claim that there exists a constant $L>0$ such that

$$\left|\begin{pmatrix}V_1^\infty\\ \vdots\\ V_c^\infty\end{pmatrix} - \begin{pmatrix}\overline{V}_1\\ \vdots\\ \overline{V}_c\end{pmatrix}\right| \le L\left|\begin{pmatrix}\overline{V}_{c+1}\\ \vdots\\ \overline{V}_N\end{pmatrix}\right|. \tag{28}$$

In order to prove (28), we observe that (20) can be written (for all times $t \geq 0$ because of Lemma 2.1) as

$$\frac{1}{|\Omega|}\begin{pmatrix} l_1 \\ \vdots \\ l_c \end{pmatrix} = (E^j)^T S E^T \overline{V} = G\begin{pmatrix} \overline{V}_1 \\ \vdots \\ \overline{V}_c \end{pmatrix} + H\begin{pmatrix} \overline{V}_{c+1} \\ \vdots \\ \overline{V}_N \end{pmatrix},$$

where G is like in (17) and the $c \times (N-c)$ matrix H is defined by

$$H := \begin{bmatrix} E^1 \\ \vdots \\ E^c \end{bmatrix} S \begin{bmatrix} (E^{c+1})^T & \dots & (E^N)^T \end{bmatrix}.$$

These formulas, together with (17), imply

$$F\begin{pmatrix} V_1^\infty \\ \vdots \\ V_c^\infty \end{pmatrix} = G\begin{pmatrix} \overline{V}_1 \\ \vdots \\ \overline{V}_c \end{pmatrix} + H\begin{pmatrix} \overline{V}_{c+1} \\ \vdots \\ \overline{V}_N \end{pmatrix}$$

which proves (28) for a suitable constant L depending on E and S. We now plug estimate (28) into (27) to obtain

$$\sum_{j=1}^{c} \|V_j - \overline{V_j}\|^2_{L^2(\Omega)} \geq \frac{\varepsilon\alpha}{1+\varepsilon} \sum_{j=1}^{c} \int_\Omega |V_j - V_j^\infty|^2 dx - \varepsilon\alpha L \sum_{j=c+1}^{N} \int_\Omega |\overline{V_j}|^2 dx$$

and, by the Cauchy–Schwartz inequality,

$$\sum_{j=1}^{c} \|V_j - \overline{V_j}\|^2_{L^2(\Omega)} \geq \frac{\varepsilon\alpha}{1+\varepsilon} \sum_{j=1}^{c} \int_\Omega |V_j - V_j^\infty|^2 dx - \varepsilon\alpha L \sum_{j=c+1}^{N} \int_\Omega |V_j|^2 dx. \tag{29}$$

By combining (26) and (29) we obtain

$$\frac{d}{dt}\mathcal{E}[U(t) - U^\infty] \leq -K\frac{\varepsilon\alpha}{1+\varepsilon} \sum_{j=1}^{d} \int_\Omega |V_j - V_j^\infty|^2 dx - (\underline{r} - \varepsilon\alpha L) \sum_{j=d+1}^{N} \int_\Omega |V_j|^2.$$

A straightforward optimisation procedure or the constants above implies

$$\frac{d}{dt}\mathcal{E}[U(t) - U^\infty] \leq -\left(\frac{K\underline{r}}{K+L} - \delta\right) \sum_{j=1}^{N} \int_\Omega |V_j - V_j^\infty|^2 dx$$

for an arbitrary $0 < \delta \ll 1$, and the desired estimate (21) easily follows by going back to the original variables U, by using the trivial estimate

$$\mathcal{E}[U(t) - U^\infty] \leq \overline{s}\|V(t) - V^\infty\|^2_{L^2(\Omega)}$$

and by using Gronwall inequality. □

3. Confined linear systems on $\mathbb{R}^n$ with integrable equilibria

In this section we deal with linear reaction–convection–diffusion systems posed on the whole of $\mathbb{R}^n$. The main difficulty with respect to the previous section consists of the presence of a *confinement matrix* and the spatial inhomogeneity of the integrable stationary states. As we shall see in this case, a theory applicable to a significant class of examples can only be developed only when the symmetrizer for the reaction matrix has variable coefficients.

Instead, the systems under consideration in this section read as

$$\partial_t U = \operatorname{div}(D(x)\nabla(S(x)U)) + R(x)U, \tag{30}$$

with $U = U(x,t) \in \mathbb{R}^N$, $d \geq 1$, $t \geq 0$, $x \in \mathbb{R}^n$. We prescribe the initial datum

$$U(x,0) = U_0(x). \tag{31}$$

As explained in the example 3.1 below, system (30) generalises a model of N reacting–diffusing species convected by N external potentials. In the sequel we shall first develop

a general theory for $N \times N$ systems including the exponential asymptotic stability result in Theorem 3.1. Then, we shall focus on the 2×2 case, where improved results can be obtained under less restrictive assumptions.

3.1. **General theory for N species.** In this subsection we shall study system (30) coupled with the initial datum (31), posed on the whole of $\mathbb{R}^n$. We shall assume the following structural hypotheses:

(B1) *Essentially convex and diagonal confinement:* The confinement matrix is of the form $S(x) = \mathrm{diag}(s_1(x), \dots, s_N(x))$ and such that there exist two functions $\underline{s}, \overline{s} > 0$ and a constant $K > 0$ such that $\underline{s}^{-1} \in L^1_+(\mathbb{R}^N)$ and, for all j's,

$$\underline{s}(x) \le s_j(x) \le \overline{s}(x), \qquad \overline{s}(x) \le K\underline{s}(x). \tag{32}$$

Moreover, we shall assume that the functions $\mathbb{R}^n \ni x \mapsto \log s_j(x)$ are L^∞ perturbations of uniformly convex functions for all $j = 1, \dots, N$.

(B2) *Positive definite diffusion matrix:* The diffusion matrix $D(x)$ is symmetric and positive definite. Moreover, the following inequality is satisfied

$$\xi^T D(x)\xi \ge d_0 \xi^T S^{-1}(x)\xi, \quad \text{for all } \xi \in \mathbb{R}^N, \tag{33}$$

for a certain positive constant d_0.

(B3) *Confinement compatible reaction matrix:* The confinement matrix $S(x)$ is a symmetrizer for the reaction matrix $R(x)$, i. e.

$$\widetilde{R}(x) := S(x)R(x) \qquad \text{is symmetric.}$$

Moreover, the symmetrised reaction matrix $\widetilde{R}(x)$ has eigenvalues

$$\lambda_1(x), \dots, \lambda_N(x)$$

satisfying $\lambda_1(x) \equiv \dots \equiv \lambda_d(x) \equiv 0$, $\lambda_j(x) \le 0$ for all $j \in \{d+1, \dots, N\}$ and there exists two positive constant C_1 and C_2 such that

$$C_1 s_j(x) \le -\lambda_j(x) \le C_2 s_j(x), \tag{34}$$

for all $x \in \mathbb{R}^N$ and for all $j \in \{d+1, \dots, N\}$.

(B4) *Bounded eigenvector matrix E of the reaction matrix $\widetilde{R}$:* The orthogonal eigenvector matrix $E(x)$ of $\widetilde{R}(x)$ satisfies

$$E(x)\widetilde{R}(x)E(x)^T = \Lambda(x), \quad \Lambda(x) := \mathrm{diag}(\underbrace{0, \dots, 0}_{d}, \lambda_{d+1}(x), \dots, \lambda_N(x)).$$

Moreover, $E(x)$ has uniformly bounded coefficients $e_{ij}(x)$, $i, j = 1, \dots, N$, such that $|e_{ij}| \ge C_E > 0$ for all i, j.

(B5) *Conservation laws:* The reaction matrix $R(x)$ has d constant left zero–eigenvectors, i. e. there exist $F_1, \dots, F_d \in \mathbb{R}^N$ such that

$$F_j^T R(x) = 0, \qquad j \in \{1, \dots, d\}.$$

Remark 3.1 (Degeneracy of the diffusion matrix). Unlike in the previous section, in the present case we are not including possible degeneracy of the diffusion matrix $D(x)$ in the assumptions. This choice is motivated mostly by the sake of avoiding further restrictions in the set of assumptions and further complications in the notation. However, we are confident that it would be possible to allow degeneracies of $D(x)$ and still achieve a unique, asymptotically stable equilibrium by some sort of orthogonality condition in the spirit of (7). This appears clearly in the estimate (48) below, where the last term on the right hand side (which has nonnegative sign and it is not needed in the estimate) would not appear in case of a suitably chosen degeneracy of $D(x)$.

Remark 3.2 (Poincaré inequalities). The convexity assumption in (B1) assures that each one of the L^1 densities $s_j(x)^{-1}$, $j = 1, \dots, N$, satisfies the *weighted Poincaré inequality*

$$\int_{\mathbb{R}^n} f^2(x) s_j(x)^{-1} dx \le C_P \int_{\mathbb{R}^n} |\nabla f(x)|^2 s_j(x)^{-1} dx$$

for all $f \in L^2(s_j^{-1}\, dx)$ such that $\int_{\mathbb{R}^n} f(x) s_j(x)^{-1} dx = 0$. Weighted Poincaré inequalities on the whole space are subject of a very wide literature, we mention for instance the

papers [AMTU01, AD05, ABD07]. More refined conditions could be prescribed involving the diffusion matrix $D(x)$ (see the above references).

The above assumptions generalise the main class of examples we have in mind, arising from the applications, i. e. the following linear system of N convection–reaction–diffusion equations with inhomogeneous equilibria.

Example 3.1 ($N \times N$ reaction drift–diffusion system on $\mathbb{R}^n$)**.** Let us consider the system

$$\partial_t u_j = \mathrm{div}(\nabla u_j + u_j \nabla V_j) + R^j(x) \cdot U, \quad j = 1, \dots N, \tag{35}$$

where $U(x,t) = (u_1(x,t), \dots, u_N(x,t))$, $x \in \mathbb{R}^n$, $t \geq 0$, R^j denotes the j–th row of a reaction matrix R and V_j is a *confining potential* acting on the j–th species u_j. A simple computation shows that the system (35) can be written in the above form (30) with

$$S(x) = \mathrm{diag}(e^{V_1(x)}, \dots, e^{V_N(x)}), \qquad D(x) := S(x)^{-1}$$

and $R(x)$ any reaction matrix such that $S(x)R(x)$ is symmetric. The assumption (B1) can be fulfilled by assuming that all potentials $V_j(x)$ are all continuous and convex with equal asymptotic behaviour of the tails as $|x| \to +\infty$. However, this assumption on the tails is often too restrictive. Subsection 3.2 will show how to relax it in 2× example system.

For future use, we write the system (30) in symmetrised form

$$\partial_t S(x)U = S(x)\mathrm{div}\,(D(x)\nabla(S(x)U)) + \widetilde{R}(x)U, \tag{36}$$

The proof of the following Lemma is straightforward (see the proof of Lemma 2.1).

Lemma 3.1 (Conserved quantities)**.** *Let $U(x,t)$ be a classical solution to system (30) such that U decays rapidly at $|x| \to +\infty$. Then, for every $j \in \{1, \dots, d\}$, the quantity*

$$l_j := \int_{\mathbb{R}^N} (F^j)^T U(x,t) dx$$

is conserved for all times $t \geq 0$.

Proposition 3.1 (Stationary solutions)**.** *For every fixed set of real numbers $l_1, \dots, l_d$, there exists a unique stationary solution $U^\infty(x)$ to (30) such that*

$$l_j = \int_{\mathbb{R}^N} F^j U^\infty(x) dx \tag{37}$$

for all $j \in \{1, \dots, d\}$. Moreover, $U^\infty(x)$ has the form

$$U^\infty(x) = S^{-1}(x) C^\infty \tag{38}$$

for a certain constant vector $C^\infty = (C_1^\infty, \dots, C_N^\infty)^T$.

Proof. First we prove that stationary solutions of (30) can only be of the form (38). Let us multiply the stationary version of the equation (30) by $(U(x)^\infty)^T S(x)$ from the left and integrate on $\mathbb{R}^N$. After integration by parts, we get

$$0 = -\int_{\mathbb{R}^N} \nabla(S(x)U^\infty(x)) : D(x)\nabla(S(x)U^\infty(x)) dx + \int_{\mathbb{R}^N} U^\infty(x)^T \widetilde{R}(x) U^\infty(x) dx.$$

Analogously to the proof of Proposition 2.1, the assumptions (B2) and (B3) imply that $\nabla S(x) U^\infty(x) = 0$ and therefore U^∞ satisfies (38). In particular, we get

$$\widetilde{R}(x) S^{-1}(x) C^\infty = 0 \tag{39}$$

and, plugging (38) into (37) implies

$$\int_{\mathbb{R}^N} F^j S(x)^{-1} C^\infty dx = l_j, \qquad j \in \{1, \dots, d\}. \tag{40}$$

The two identities (39) and (40) constitute a linear system of N equations in the N-dimensional unknown C^∞. Such system has a unique solution due to the assumptions in (B1). □

We introduce the following variables

$$W(x,t) := E(x)U(x,t), \qquad Z(x,t) := S(x)U(x,t) = S(x)E^T(x)W(x,t).$$

Similarly to Section 2, we shall consider the *energy functional*

$$\mathcal{E}[U] := \frac{1}{2}\int_{\mathbb{R}^N} U^T S(x) U dx = \frac{1}{2}\int_{\mathbb{R}^N} Z^T S^{-1}(x) Z dx.$$

Theorem 3.1 (Exponential convergence towards inhomogeneous equilibria)**.** *Let $U(x,t)$ be a classical solution to system (30) with initial datum U_0, such that U decays rapidly at $|x| \to +\infty$. Suppose*

$$l_j = \int_{\mathbb{R}^N} (F^j)^T U_0(x) dx, \quad j \in \{1,\dots,d\}, \tag{41}$$

for fixed quantities $l_1,\dots,l_d \in \mathbb{R}$. Let the stationary state U^∞ be uniquely determined by the l_j's as in Proposition 3.1. Then, there exists a fixed constant $\varepsilon > 0$ (depending on the structural assumptions (B1)–(B5)) such that

$$\mathcal{E}[U(t) - U^\infty] \le \mathcal{E}[U_0 - U^\infty] e^{-\varepsilon t} \tag{42}$$

for all $t \ge 0$.

Proof. Due to the linearity of the problem, we can assume without loss of generality $U^\infty(x) \equiv 0$. The general case can be treated by considering the new variable $V = U - U^\infty$. Therefore, the conserved quantities l_j defined above can be taken as follows

$$l_1 = \ldots = l_d = 0.$$

Hence, assumption (41) plus Lemma 3.1 imply

$$0 = (F^j)^T \int_{\mathbb{R}^N} S(x)^{-1} Z(x,t) dx, \quad j \in \{1,\dots,d\}, \tag{43}$$

We compute the evolution of the aforementioned energy (centred at the zero equilibrium state). After integration by parts and due to the assumption (33) we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)] &= -\int_{\mathbb{R}^N} \nabla(S(x)U) : D(x)\nabla(S(x)U) dx + \int_{\mathbb{R}^N} U^T \widetilde{R}(x) U dx \\ &\le -d_0 \int_{\mathbb{R}^N} \nabla(S(x)U) : S^{-1}(x)\nabla(S(x)U) dx + \int_{\mathbb{R}^N} U^T \widetilde{R}(x) U dx.\end{aligned}$$

Now we use the vector valued version of the weighted Poincaré inequality (cf. [AMTU01]) to get

$$\frac{d}{dt}\mathcal{E}[U(t)] \le -\frac{d_0}{C_P}\int_{\mathbb{R}^N} \left(Z - \bar{Z}\right)^T S^{-1}(x) \left(Z - \bar{Z}\right) + \int_{\mathbb{R}^N} U^T \widetilde{R}(x) U dx \tag{44}$$

where

$$\bar{Z} := \left(\frac{\int_{\mathbb{R}^N} Z_1 s_1(x)^{-1} dx}{\int_{\mathbb{R}^N} s_1(x)^{-1} dx}, \dots, \frac{\int_{\mathbb{R}^N} Z_N s_N(x)^{-1} dx}{\int_{\mathbb{R}^N} s_N(x)^{-1} dx} \right)^T.$$

We rephrase the above definition in the following vector form

$$\bar{Z} = I^{-1} \int_{\mathbb{R}^N} S^{-1}(x) Z dx, \quad I := \operatorname{diag}\left(\int_{\mathbb{R}^N} s_1(x)^{-1} dx, \dots, \int_{\mathbb{R}^N} s_N(x)^{-1} dx \right) \tag{45}$$

In terms of the variable W, the estimate (44) reads

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)] \le &-\frac{d_0}{C_P}\int_{\mathbb{R}^N} (W - \widetilde{W})^T E(x) S(x) E(x)^T (W - \widetilde{W}) dx \\ &+ \int_{\mathbb{R}^N} W^T \Lambda(x) W dx \le -\frac{d_0}{K C_P}\int_{\mathbb{R}^N} (W - \widetilde{W})^T S(x) (W - \widetilde{W}) dx \\ &+ \int_{\mathbb{R}^N} W^T \Lambda(x) W dx =: -\frac{d_0}{K C_P} D_D - D_R,\end{aligned} \tag{46}$$

where

$$\widetilde{W} = E(x) S(x)^{-1} \bar{Z}$$

and we have used assumption (B1) and the orthogonality of $E(x)$. Similarly to the simpler case described in section 2, the two dissipation terms D_D and D_R both contribute to a decay of the energy, but they lead the solution $U(t)$ towards two different limiting manifold which intersect at the unique equilibrium state. We shall therefore try to use the same strategy as in the proof of Theorem 2.1. The main difficulty here lies in the term $\widetilde{W}$, which does not coincides simply with a weighted average of W. First we fix some notations:

$$\begin{aligned} &S^I(x) := \mathrm{diag}(s_1(x), \dots, s_d(x)), && S^{II}(x) := \mathrm{diag}(s_{d+1}(x), \dots, s_N(x)) \\ &(F^I)^T := [F_1, \dots, F_d] && (F^{II})^T := [F_{d+1}, \dots, F_N] \\ &(E^I)^T := [E_1, \dots, E_d] && (E^{II})^T := [E_{d+1}, \dots, E_N]. \end{aligned}$$

Moreover, for any vector $V \in \mathbb{R}^N$ we shall use the notation $V = (V^I, V^{II})$ where $V^I := (V_1, \dots, V_d)$ and $V^{II} := (V_{d+1}, \dots, V_N)$. We observe that (43) and (45) imply

$$0 = F^I I \bar{Z} = F^I I S(x) E^T(x) \widetilde{W}.$$

The previous identity represents a linear system of d scalar equations in the N dimensional unknown $\widetilde{W}^I$, as it can be also written as

$$0 = F^I I S(x)[(E^I(x))^T \widetilde{W}^I + (E^{II}(x))^T \widetilde{W}^{II}].$$

Due to the assumptions (B1) and (B4) and to the non-degeneracy of the constant matrices F and I, it is possible to find a positive constant B such that

$$|\widetilde{W}^I| \le B |\widetilde{W}^{II}| \tag{47}$$

The above inequality is crucial in the present approach, and it can be used together with (46) in a similar fashion as (28) in the proof of Theorem 2.1. In order to detect dissipation of energy functional $\mathcal{E}[U(t)]$ in the right hand side of (46), we estimate the term D_D in as follows

$$\begin{aligned} D_D &= \int_{\mathbb{R}^N} (W^I - \widetilde{W}^I)^T S^I(x)(W^I - \widetilde{W}^I) dx \\ &\quad + \int_{\mathbb{R}^N} (W^{II} - \widetilde{W}^{II})^T S^{II}(x)(W^{II} - \widetilde{W}^{II}) dx \\ &= \sum_{j=1}^{d} \int_{\mathbb{R}^N} s_j(x)(W_j - \widetilde{W}_j)^2 dx + \sum_{j=d+1}^{N} \int_{\mathbb{R}^N} s_j(x)(W_j - \widetilde{W}_j)^2 dx \\ &\ge (1-\beta) \sum_{j=1}^{d} \int_{\mathbb{R}^N} s_j(x) W_j^2 dx - \left(\frac{1}{\beta} - 1\right) \sum_{j=1}^{d} \int_{\mathbb{R}^N} s_j(x) \widetilde{W}_j^2 dx \\ &\quad + \sum_{j=d+1}^{N} \int_{\mathbb{R}^N} s_j(x)(W_j - \widetilde{W}_j)^2 dx, \qquad \text{for all } \beta \in (0,1). \end{aligned} \tag{48}$$

This time (unlike (28)), the reaction dissipation term D_R does not control directly the term with the minus sign in (48). However, we can perform the estimate

$$\begin{aligned} D_R &= -\sum_{j=d+1}^{N} \int_{\mathbb{R}^N} \lambda_j(x) W_j^2 dx = -\sum_{j=d+1}^{N} \int_{\mathbb{R}^N} \lambda_j(x)(W_j - \widetilde{W}_j + \widetilde{W}_j)^2 dx \\ &\ge -(1-\gamma) \sum_{j=d+1}^{N} \int_{\mathbb{R}^N} \lambda_j(x) \widetilde{W}_j^2 dx + \left(\frac{1}{\gamma} - 1\right) \sum_{j=d+1}^{N} \int_{\mathbb{R}^N} \lambda_j(x)(W_j - \widetilde{W}_j)^2 dx \end{aligned} \tag{49}$$

for all $\gamma \in (0,1)$. Since we want to use part or the term $-D_R$ in the estimate (46) (because -D_R induces dissipation of the energy $\mathcal{E}[U(t)]$), for an arbitrary $\alpha \in (0,1)$ we write

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)] \leq & -\frac{d_0(1-\beta)}{KC_P}\int_{\mathbb{R}^N}(W^I)^T S^I(x)W^I dx + \alpha\sum_{j=d+1}^{N}\int_{\mathbb{R}^N}\lambda_j(x)W_j^2 dx\\ &+\frac{d_0}{KC_P}\left(\frac{1}{\beta}-1\right)\sum_{j=1}^{d}\int_{\mathbb{R}^N}s_j(x)\widetilde{W}_j^2 dx\\ &+(1-\alpha)(1-\gamma)\sum_{j=d+1}^{N}\int_{\mathbb{R}^N}\lambda_j(x)\widetilde{W}_j^2 dx\\ &-(1-\alpha)\left(\frac{1}{\gamma}-1\right)\sum_{j=d+1}^{N}\int_{\mathbb{R}^N}\lambda_j(x)(W_j-\widetilde{W}_j)^2 dx\\ &-\frac{d_0}{KC_P}\int_{\mathbb{R}^N}(W^{II}-\widetilde{W}^{II})^T S^{II}(x)(W^{II}-\widetilde{W}^{II})dx.\end{aligned}$$

Then, due to the assumptions (32), (34) and thanks to the estimate (47), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[U(t)] \leq & -\min\left\{\frac{d_0(1-\beta)}{KC_P}, \alpha C_1\right\}\mathcal{E}[U(t)]\\ &+\left[\frac{B^2\, d_0}{K^2C_P}\left(\frac{1}{\beta}-1\right)-C_1(1-\alpha)(1-\gamma)\right]\int_{\mathbb{R}^N}\underline{s}(x)|(\widetilde{W}^{II})|^2 dx\\ &+\left[C_2(1-\alpha)\left(\frac{1}{\gamma}-1\right)-\frac{d_0}{KC_P}\right]\sum_{j=d+1}^{N}\int_{\mathbb{R}^N}s_j(x)(W_j-\widetilde{W}_j)^2 dx.\end{aligned}\tag{50}$$

A suitable choice of the constants $\alpha, \beta, \gamma \in (0,1)$ implies the desired exponential decay (42). The proof is complete. □

Remark 3.3. The last estimate (50) shows that the exponential rate of convergence of the relative energy $\mathcal{E}[U(t)]$ is computable. The final result depends on the constants d_0, K, C_1, C_2, and C_E defined in the structural assumptions (B1)-(B5) plus constants involving uniform bounds for the matrices F and $E(x)$ and the Poincaré constant C_P of the domain Ω. However, the calculation procedure is pretty involved and it will not be performed in full generality, although in the next subsection we shall be able to produce an explicit rate of convergence in view of a more simple set of assumptions.

3.2. **A typical 2×2 example case.** In this subsection we shall detail a 2×2 whole space drift–diffusion–recombination system, which is in fact the linearisation of the nonlinear semiconductor model (63) to be studied in section 4.

$$\begin{cases}\partial_t u = \nabla\cdot(\nabla u + u\nabla V_1) - F(n_\infty, p_\infty, x)(p_\infty u + n_\infty v)\\ \partial_t v = \nabla\cdot(\nabla v + v\nabla V_2) - F(n_\infty, p_\infty, x)(p_\infty u + n_\infty v).\end{cases}\tag{51}$$

where

$$\begin{aligned}&n_\infty(x) := e^{-V_1(x)}, \quad p_\infty(x) := e^{-V_2(x)}, \qquad V_1,\ V_2 \in C^2(\mathbb{R}^n),\\ &V_1 \text{ and } V_2 \text{ are } L^\infty \text{ perturbation of uniformly convex functions}\end{aligned}\tag{52}$$

which implies in particular that $n_\infty, p_\infty \in L^1_+(\mathbb{R}^n)$. As stated in Remark 3.2 in the previous subsection, the set of assumptions (52) ensures the validity of a suitable weighted Poincaré inequality. The reaction rate $F(\cdot,\cdot,\cdot)$ shall denote a continuous function (further assumptions on F will be specified below) and is typically thought to be of *Shockley–Read–Hall* type

$$F(n,p,x) = (r_1 + r_2 n + r_3 p)^{-1},\tag{53}$$

for some positive, bounded below functions $r_1(x), r_2(x), r_3(x) \geq r > 0$.

With the vector variable $U = (u, v)^T$ the above system (51) is of the form (30) with

$$S(x) = \begin{bmatrix} n_\infty(x)^{-1} & 0 \\ 0 & p_\infty(x)^{-1} \end{bmatrix}, \quad D(x) \equiv S(x)^{-1}, \tag{54}$$

and the matrix $R(x)$

$$R(x) := F(n_\infty, p_\infty, x) \begin{bmatrix} p_\infty & n_\infty \\ p_\infty & n_\infty \end{bmatrix}. \tag{55}$$

For the present 2×2 system we are able to follow a different line of the estimates compared to the previous subsection and obtain improved result, under less restrictive assumptions than the set (B1)–(B5).

Note that the considered example is a good representant for a general two–species theory. Indeed, one can easily verify that the set of assumptions on the reaction matrix in (B1)–(B5) forces the above choice (55) for R in the 2×2 case, except that one could have a left eigenvector corresponding to the zero eigenvalue different than $(1, -1)$. A more general left eigenvector can be obtained by multiplying one of the two equations in (51) by an arbitrary constant, this does not imply any significant change in the computations below. Moreover, it is easily seen that, in order to have the matrix SR symmetric, R must be of the form (55) up to a scalar multiplier. Our result in the 2×2 case will require less restrictive assumptions on the scalar multiplier F than those inferred by the set of assumptions (B1)–(B5). Finally, we could easily generalise the following computations to the case of a more general D satisfying assumption (B1). However, in order to simplify the presentation, we shall consider $D = S^{-1}$ as in (54).

We shall use the entropy (or free energy) functional

$$E(u, v) := \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{u^2}{n_\infty} + \frac{v^2}{p_\infty} \right) dx$$

which coincides with $\int_{\mathbb{R}^N} U^T S(x) U dx$ in the vector notation, consistently with the strategy developed in the previous sections. After introducing the variables $z_1 = \frac{u}{n_\infty}$ and $z_2 = \frac{v}{p_\infty}$ and multiplying equation (51) with the vector (z_1, z_2) we obtain

$$\frac{d}{dt} E(z_1, z_2) = \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^N} z_1^2 \, dn_\infty + \int_{\mathbb{R}^N} z_2^2 \, dp_\infty \right) = -D(z_1, z_2),$$

with

$$D(z_1, z_2) = \int_{\mathbb{R}^N} |\nabla z_1|^2 dn_\infty + \int_{\mathbb{R}^N} |\nabla z_2|^2 dp_\infty + \int_{\mathbb{R}^N} F \, n_\infty p_\infty (z_1 + z_2)^2 \, dx,$$

and the measures $dn_\infty = n_\infty dx$, $dp_\infty = p_\infty dx$. Denoting further the normalised measures

$$d\xi_1 = \frac{dn_\infty}{N_\infty}, \quad N_\infty = \int_{\mathbb{R}^N} dn_\infty, \qquad d\xi_2 = \frac{dp_\infty}{\Pi}, \quad \Pi = \int_{\mathbb{R}^N} dp_\infty,$$

we estimate the Fisher-information terms using a weighted Poincaré inequality (cf. [AMTU01])

$$\int_{\mathbb{R}^N} |\nabla z_i|^2 d\xi_i \geq P_i \int_{\mathbb{R}^N} (z_i - \overline{z_i})^2 \, d\xi_i, \quad \text{with } \overline{z_i} := \int z_i \, d\xi_i.$$

The constants P_1 and P_2 are the whole space Poincaré constants with respect to ξ_1 and ξ_2 respectively, therefore they depend on V_1 and V_2. Then, for a suitable constant $C > 0$, we are looking for the following entropy–entropy dissipation estimate

$$\begin{aligned} D &\geq P_1 \int_{\mathbb{R}^N} (z_1 - \overline{z_1})^2 dn_\infty + P_2 \int_{\mathbb{R}^N} (z_2 - \overline{z_2})^2 dp_\infty + \int_{\mathbb{R}^N} F \, n_\infty p_\infty \, (z_1 + z_1)^2 \, dx \\ &\geq C \, E = \frac{C}{2} \left(\int_{\mathbb{R}^N} z_1^2 dn_\infty + \int_{\mathbb{R}^N} z_2^2 dp_\infty \right), \end{aligned}$$

under the constraint for the conservation of mass :

$$\int_{\mathbb{R}^N} z_1 \, dn_\infty = \int_{\mathbb{R}^N} z_2 \, dp_\infty. \tag{56}$$

Lemma 3.2.
Consider measurable functions z_i, $i = 1, 2$ such that (56) holds. Let $F(n_\infty, p_\infty)$ be integrable with respect to the measure $n_\infty p_\infty dx$ and satisfy

$$Fn_\infty \leq \frac{1}{\tau_n}\,, \qquad Fp_\infty \leq \frac{1}{\tau_p}\,. \tag{57}$$

Then

$$D \geq \frac{K(1-\varepsilon)}{K_1} E$$

holds provided that

$$K \leq \min\left\{ \frac{P_1}{\frac{2}{\tau_n}\frac{1-\varepsilon}{\varepsilon} + \frac{1}{2k_1}}, \frac{P_2}{\frac{2}{\tau_p}\frac{1-\varepsilon}{\varepsilon} + \frac{1}{2k_1}} \right\}, \tag{58}$$

with K_1 defined in (61) and where $0 < \varepsilon < 1$ can be chosen in order to maximise the constant K.

Remark 3.4. In contrast to Theorem 3.1, the strategy of the proof presented here uses the entropy dissipation coming from the reaction to construct a lower bound of the entropy dissipation in terms of the relative entropy. This is done crucially for the special case of constant z_i using the conservation law and the integrability of F w.r.t. the measure $n_\infty p_\infty\, dx$. To obtain the entropy-entropy dissipation estimate for general z_i we employ an ansatz around constant states and take a sufficiently small fraction of the reaction-dissipation such that the remainders are controlled by the dissipation coming from the diffusion. To control the remainders, the bounds (57) are required naturally. Nevertheless, for general systems like stated in Theorem 3.1 this approach seems at least clumsy for formalise.

Proof. Step 1 - We consider the particular situation when z_1 and z_2 are constant. We then calculate the relative entropy as

$$E = \frac{N_\infty + \Pi}{2}\left(\overline{z_1}^2 + \overline{z_2}^2\right), \tag{59}$$

while for the entropy dissipation we have

$$D = \mathcal{F}_0\left(\overline{z_1} + \overline{z_2}\right)^2, \qquad \text{with } \mathcal{F}_0 := \int_{\mathbb{R}^N} F(n_\infty, p_\infty)\, n_\infty p_\infty\, dx \tag{60}$$

Thus, using the conservation law $N_\infty \overline{z_1} = \Pi \overline{z_2}$ to express one $\overline{z_i}$ in terms of the other, we find by comparing (59) with (60) (symmetrised in terms of $\overline{z_1}$ and $\overline{z_2}$) that

$$E(\overline{z_1}, \overline{z_2}) = K_1 D(\overline{z_1}, \overline{z_2})\,, \quad \text{with } K_1 = \frac{1}{2\mathcal{F}_0} \frac{N_\infty + \Pi + \frac{N_\infty^2}{\Pi} + \frac{\Pi^2}{N_\infty}}{\left(1 + \frac{N_\infty^2}{\Pi^2}\right)^2 + \left(1 + \frac{\Pi^2}{N_\infty^2}\right)^2}\,. \tag{61}$$

Step 2 - We employ the ansatz $z_i = \overline{z_i} + \delta_i$, $i = 1, 2$ with $\overline{\delta_i} = 0$. Expanding the relative entropy yields

$$E = \frac{1}{2}\overline{\delta_1^2} + \frac{N_\infty}{2}\overline{z_1}^2 + \frac{1}{2}\overline{\delta_2^2} + \frac{\Pi}{2}\overline{z_2}^2\,. \tag{62}$$

On the other hand expanding the entropy dissipation, we estimate the contribution coming from the reaction using Young's inequality for a $\varepsilon < 1$

$$\begin{aligned}\int_{\mathbb{R}^N} F(n_\infty, p_\infty)\, n_\infty p_\infty \left[\overline{z_1} + \overline{z_2} + (\delta_1 + \delta_2)\right]^2 dx \geq (1-\varepsilon)\mathcal{F}_0(\overline{z_1} + \overline{z_2})^2 \\ - \frac{1-\varepsilon}{\varepsilon}\int_{\mathbb{R}^N} F(n_\infty, p_\infty)\, n_\infty p_\infty\, (\delta_1 + \delta_2)^2\, dx\,.\end{aligned}$$

Then, using (57) we continue to estimate again with Young's inequality

$$\begin{aligned}\int_{\mathbb{R}^N} F(n_\infty, p_\infty)\, n_\infty p_\infty \left[\overline{z_1} + \overline{z_2} + (\delta_1 + \delta_2)\right]^2 dx \geq (1-\varepsilon)\mathcal{F}_0(\overline{z_1} + \overline{z_2})^2 \\ - 2\frac{1-\varepsilon}{\varepsilon}\left[\frac{1}{\tau_p}\int_{\mathbb{R}^N} \delta_1^2\, dn_\infty + \frac{1}{\tau_n}\int_{\mathbb{R}^N} \delta_2^2\, dp_\infty\right].\end{aligned}$$

Step 3 - By the previous steps we may now estimate for a constant $K < 1$

$$\begin{aligned} D &\geq P_1(\Omega)\int_{\mathbb{R}^N}\delta_1^2\,dn_\infty + P_2(\Omega)\int_{\mathbb{R}^N}\delta_2^2\,dp_\infty + K\int_{\mathbb{R}^N} Fn_\infty p_\infty(z_1+z_1)\,dx \\ &\geq (1-\varepsilon)\mathcal{F}_0(\overline{z_1}+\overline{z_2})^2 + \left[P_1(\Omega) - \frac{2}{\tau_p}\frac{1-\varepsilon}{\varepsilon}\right]\int_{\mathbb{R}^N}\delta_1^2\,dn_\infty \\ &\quad + \left[P_2(\Omega) - \frac{2}{\tau_n}\frac{1-\varepsilon}{\varepsilon}\right]\int_{\mathbb{R}^N}\delta_2^2\,dp_\infty \\ &\geq \frac{K(1-\varepsilon)}{K_1}E + \left[P_1(\Omega) - K\left(\frac{2}{\tau_p}\frac{1-\varepsilon}{\varepsilon}+\frac{1}{2k_1}\right)\right]\int_{\mathbb{R}^N}\delta_1^2\,dn_\infty \\ &\quad + \left[P_2(\Omega) - K\left(\frac{2}{\tau_n}\frac{1-\varepsilon}{\varepsilon}+\frac{1}{2k_1}\right)\right]\int_{\mathbb{R}^N}\delta_2^2\,dp_\infty\,, \end{aligned}$$

and statement of the lemma follows. □

We have therefore proven the following

Theorem 3.2. *Let n_∞, p_∞, V_1 and V_2 satisfy (52) and let assumption (57) on F hold. Let $(u(t), v(t))$ be a classical solution to the system* (51) *with initial datum (u_0, v_0) having finite energy $\displaystyle\int_{\mathbb{R}^N}\left[\frac{u_0^2}{n_\infty}+\frac{v_0^2}{p_\infty}\right]dx$. Then,*

$$\int_{\mathbb{R}^N}\left[\frac{u(x,t)^2}{n_\infty}+\frac{v(x,t)^2}{p_\infty}\right]dx \leq \exp\left(-\frac{K_1}{K(1-\varepsilon)}t\right)\int_{\mathbb{R}^N}\left[\frac{u_0^2}{n_\infty}+\frac{v_0^2}{p_\infty}\right]dx$$

with the constants K, K_1 and ε as in Lemma 3.2.

Remark 3.5. Comparing the result of Theorem 3.2 with the result in Theorem 3.1 we find the following improvements:

- The exponential rate of convergence is much easier to compute.
- We do not need the two potentials V_1 and V_2 to have equivalent behaviour of the tales at $|x| \to +\infty$, as it was implicitly required in (B1) (cf. Example 3.1).
- The assumptions (57) on F are less restrictive than those implicitly inferred by (B3). In particular, (57) match with the Shockley–Read–Hall reaction rate (53).

4. A reaction–diffusion model with nonlinear reaction

In this section we study the nonlinear model system arising in semiconductor and plasma physics

$$\begin{cases} n_t = \mathrm{div} J_n - R(n,p)\,, & J_n := \nabla n + n\nabla V_n \\ p_t = \mathrm{div} J_p - R(n,p)\,, & J_p := \nabla p + p\nabla V_p \end{cases} \tag{63}$$

where n and p model two species of charged particles subject to confinement and to a recombination–generation mechanism $R(n,p)$. We suppose non-negative initial data

$$n(x,0) = n_I(x) \geq 0\,, \qquad\qquad p(x,0) = p_I(x) \geq 0\,,$$

and the following assumptions:

(NL1) The confining potentials V_n and V_p satisfy

$$D^2 V_i(x) \geq \sigma_i \mathbb{I}_N, \qquad \text{for certain constants } \ \sigma_n, \sigma_p > 0, \quad i \in \{n,p\}.$$

Moreover, $\|\Delta V_i\|_{L^\infty(\mathbb{R}^n)}$ is finite for $i \in \{n,p\}$, and we define $\mu_i := e^{-V_i(x)}$ and introduce the related measures $d\mu_i := \mu_i(x)dx$.

(NL2) The recombination–generation term is of the form $R(n,p) = F(n,p,x)(np - \delta^2\mu_n\mu_p)$ for a constant $\delta > 0$, which – without loss of generality – shall be rescaled as $\delta = 1$. The scalar function $F(n,p,x) \geq 0$ is assumed to be such that

$$R(n,p) \leq A_1 + A_2 n + A_3 p, \tag{64}$$

for constants $A_1 > 0$, $A_2, A_3 \geq 0$, which includes the typical *Shockley–Read–Hall* form (53).

The nonlinear system (63) models recombination–generation mechanisms in semiconductor theory when neglecting the influence of the self–consistent potential. The physical much more relevant and mathematically more interesting case of system (63) coupled to a Poisson equation for a self-consistent field is nevertheless beyond the scope of this paper and will be subject of an upcoming article [DFFM]. For the existence of global solutions of the system (63) (with self–consistent potential), we refer to [MWZ]. We recall that the initial mass $M \in \mathbb{R}$ is conserved for all $t > 0$

$$\int_{\mathbb{R}^N} n(x,t)\,dx - \int_{\mathbb{R}^N} p(x,t)\,dx = M := \int_{\mathbb{R}^N} n_I(x)\,dx - \int_{\mathbb{R}^N} p_I(x)\,dx\,. \tag{65}$$

Given M as fixed, the equilibrium $n = n_\infty$, $p = p_\infty$ is uniquely determined by

$$\begin{cases} n_\infty(x) = C_n\, e^{-V_n(x)}, \quad p_\infty(x) = C_p\, e^{-V_p(x)}, \\ C_n, C_p > 0: \quad C_n C_p = 1, \quad C_n \int e^{-V_n} dx + C_p \int e^{-V_p} dx = M. \end{cases} \tag{66}$$

The *relative entropy* $E = E(n,p)$ of the system (63) with respect to the equilibrium (66)

$$E = \int_{\mathbb{R}^N} \left[n \ln \frac{n}{n_\infty} - (n - n_\infty) + p \ln \frac{p}{p_\infty} - (p - p_\infty) \right] dx \tag{67}$$

dissipates (i.e. $\frac{d}{dt} E(n,p) = -D(n,p)$) with the entropy dissipation

$$D = \int_{\mathbb{R}^N} \frac{|J_n|^2}{n} dx + \int_{\mathbb{R}^N} \frac{|J_p|^2}{p} dx + \int_{\mathbb{R}^N} F\,\mu^2 \left(\frac{n}{n_\infty}\frac{p}{p_\infty} - 1 \right) \ln \left(\frac{n}{n_\infty}\frac{p}{p_\infty} \right) dx\,. \tag{68}$$

Note, that the first two integrals in the entropy dissipation quantify how drift and diffusion tend to make the ratios $\frac{n}{n_\infty}$ and $\frac{p}{p_\infty}$ constant, while the third reaction-type integral drives their product to be equal to one. Hence, the conservation of mass (65)) determines that the entropy dissipation vanishes only for the equilibrium (66).

4.1. **Uniform L^∞ bound.** We shall establish a uniform L^∞ bound for the solution $(n(t), p(t))$ to (63), which will be a used in assumption (NL4) below when studying the large time behaviour of (63). The proof applies a Nash-Moser-type iteration method based on L^r bounds. The cases $r < \infty$ are a refined version of [MWZ, Lemma 4.1], where a global bound for all L^r norms is proven for $r < \infty$ in the (more difficult) case with self–consistent potential. We remark that the L^∞ estimate stated below can be extended to the self–consistent potential (see [DFFM]).

Lemma 4.1 (Uniform $L^1 \cap L^\infty$ bounds)**.** *Assume the initial data n_I, p_I are in $L^1 \cap L^\infty(\mathbb{R}^n)$ with finite entropy $E(n_I, p_I) < +\infty$. Then the solution (n,p) of (63) satisfies*

$$\sup_{t \geq 0} \left[\|n(t)\|_{L^r} + \|p(t)\|_{L^r} \right] < \infty$$

for all $r \in [1, +\infty]$.

Proof. First, we remark that uniform L^1 bounds for $n(t)$ and $p(t)$ follow directly from the dissipation of the relative entropy $\frac{d}{dt} E(n,p) = -D(n,p) \leq 0$ and the Csizsár–Kullback inequality (cf. [UAMT00], for instance for n

$$\|n - n_\infty\|_{L^1}^2 \leq C \int \left(n \log \frac{n}{n_\infty} - (n - n_\infty) \right) dx\,. \tag{69}$$

For $r > 0$, we compute the evolution of the L^{r+1} norm of n and p via integration by parts:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}[n^{r+1}+p^{r+1}]dx = -\frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla n^{\frac{r+1}{2}}\right|^2 dx - \frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla p^{\frac{r+1}{2}}\right|^2 dx\\
&+ r\int_{\mathbb{R}^N} n^{r+1}\Delta V_n dx + r\int_{\mathbb{R}^N} p^{r+1}\Delta V_p dx - (r+1)\int_{\mathbb{R}^N}(n^r+p^r)R(n,p,x)dx\\
&\le -\frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla n^{\frac{r+1}{2}}\right|^2 dx - \frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla p^{\frac{r+1}{2}}\right|^2 dx\\
&+ r\|\Delta V_n\|_{L^\infty}\int_{\mathbb{R}^N} n^{r+1}dx + r\|\Delta V_p\|_{L^\infty}\int_{\mathbb{R}^N} p^{r+1}dx\\
&+ A_1(r+1)\int_{\mathbb{R}^N}(n^r+p^r)dx + 2A_2(r+1)\int_{\mathbb{R}^N} n^{r+1}dx + 2A_3(r+1)\int_{\mathbb{R}^N} p^{r+1}dx.
\end{aligned}$$

Therefore, for a fixed constant C independent on r, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N}[n^{r+1}+p^{r+1}]dx &\le -\frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla n^{\frac{r+1}{2}}\right|^2 dx\\
&- \frac{4r}{r+1}\int_{\mathbb{R}^N}\left|\nabla p^{\frac{r+1}{2}}\right|^2 dx + Cr\int_{\mathbb{R}^N}(n^{r+1}+p^{r+1})dx + \frac{C}{r}.
\end{aligned} \tag{70}$$

We recall the classical Nash inequality (cf. [Nas58, CL93])

$$\|f\|_{L^2}^{1+\frac{2}{N}} \le C_N\|f\|_{L^1}^{2/N}\|\nabla f\|_{L^2}, \tag{71}$$

which is valid for all $f \in L^1(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$. Applying (71) to $f = n^{\frac{r+1}{2}}$ and $f = p^{\frac{r+1}{2}}$, we obtain, for $\epsilon > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^N}(n^{r+1}+p^{r+1})dx &\le \epsilon\int_{\mathbb{R}^N}\left[\left|\nabla n^{\frac{r+1}{2}}\right|^2 + \left|\nabla p^{\frac{r+1}{2}}\right|^2\right]dx\\
&+ \frac{\widetilde{C}}{\epsilon}\left[\int_{\mathbb{R}^N}(n^{\frac{r+1}{2}}+p^{\frac{r+1}{2}})dx\right]^2
\end{aligned} \tag{72}$$

where $\widetilde{C} > 0$ does not depend on r. Now, for nonnegative integer k let $\lambda_k = 2^k - 1$, $r = \lambda_k$ and $\epsilon_k := \frac{A}{\lambda_k}$ for a certain positive constant A. Then, it is easy to prove that there exists $A > 0$ such that $\epsilon_k = A/\lambda_k$ satisfies

$$\epsilon_k(C\lambda_k + \epsilon_k) \le \frac{4\lambda_k}{\lambda_k+1}$$

where C is as in (70). Therefore, multiplying (72) (with $\epsilon = \epsilon_k$) by $C(\lambda_k+\epsilon_k)$ and using the resulting inequality in (70) with $r = \lambda_k$ yields

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N}[n^{\lambda_k+1}+p^{\lambda_k+1}]dx &\le -\epsilon_k\int_{\mathbb{R}^N}[n^{\lambda_k+1}+p^{\lambda_k+1}]dx\\
&+ B\lambda_k(\lambda_k+\epsilon_k)\sup_{0\le\tau\le t}\left[\int_{\mathbb{R}^N}(n^{\frac{\lambda_k+1}{2}}+p^{\frac{\lambda_k+1}{2}})dx\right]^2 + \frac{C}{\lambda_k}
\end{aligned} \tag{73}$$

where $B = C\widetilde{C}/A$. We state now the following Lemma, which is an adapted version of [Kow05, Lemma 5.1] and [Ali79, Lemma 3.2]

Lemma 4.2. *Let $W_k = W_k(t)$ be a sequence of nonnegative functions (of the time variable $t \ge 0$) satisfying*

$$\frac{d}{dt}W_{\lambda_k+1}(t) \le -\epsilon_k W_{\lambda_k+1}(t) + B\lambda_k(\lambda_k+\epsilon_k)\left[\sup_{0\le\tau\le t} W_{\frac{\lambda_k+1}{2}}(t)\right]^2 + \frac{C}{\lambda_k}$$

with $\lambda_k = 2^k - 1$, $\epsilon_k := \frac{A}{\lambda_k}$, $\epsilon_k(C\lambda_k+\epsilon_k)$, A, B, C fixed positive, k positive integer. Suppose that

$$W_{\lambda_k+1}(0) \le K^{\lambda_k+1}, \qquad \sup_{t\ge0}(W_{\lambda_0+1}(t)) \le K \tag{74}$$

for all k. Then, there exists a fixed constant $a > 0$ such that

$$\sup_{t\geq 0}\left[\limsup_{k\to+\infty}\left(W_k(t)\right)^{1/(\lambda_k+1)}\right] \leq aK. \tag{75}$$

Proof. We denote

$$E := \left[\sup_{0\leq\tau\leq t} W_{\frac{\lambda_k+1}{2}}(t)\right]^2, \qquad \delta_k := \frac{\max\{B,C\}\lambda_k(\lambda_k+\epsilon_k)}{\epsilon_k}.$$

The differential inequality in the assumptions becomes

$$\frac{dW_{\lambda_k+1}(t)}{dt} \leq -\epsilon_k W_{\lambda_k+1}(t) + \epsilon_k\left[\delta_k E + \frac{C}{A}\right] \tag{76}$$

and we also have $\delta_k \geq \frac{C}{A}$ by definition of δ_k and ϵ_k. Solving (76) yields

$$W(t) \leq e^{-\epsilon_k t}\left[W(0) + \left(\delta_k E + \frac{C}{A}\right)\right] \leq 2\max\left\{K^{\lambda_k+1}, \delta_k E + \frac{C}{A}\right\}.$$

Now, it is easy to prove by induction that the previous expression implies (due to all the assumptions above)

$$W_{\lambda_k+1}(t) \leq 4^{2^k}\delta_k\delta_{k-1}^2\ldots\delta_1^{2^{k-1}}K^{\lambda_k+1}. \tag{77}$$

A simple computation shows

$$\delta_k \leq D2^{3k}, \qquad D := \frac{\max\{B,C\}\max\{A,1\}}{A}. \tag{78}$$

Now, due the well known formulas $1 + r + r^2 + \ldots + r^k = \frac{r^{k+1}-1}{r-1}$ and $k \leq (3/2)^k$, the inequality (78) implies

$$\begin{aligned}\delta_k\delta_{k-1}^2\ldots\delta_1^{2^{k-1}} &\leq D^{1+2+2^2+\ldots+2^{k-1}}2^{3\left(k+2(k-1)+2^2(k-2)+\ldots+2^{k-2}2+2^{k-1}\right)}\\ &\leq D^{2^k-1}2^{3\cdot 2^k\left(\left(\frac{3}{4}\right)^k+\left(\frac{3}{4}\right)^{k-1}+\ldots+\left(\frac{3}{4}\right)\right)} \leq D^{2^k-1}2^{12\cdot 2^k}.\end{aligned}$$

Therefore, by taking the 2^k–th root of both sides of (77) and by using the last estimate, after sending $k \to +\infty$ we obtain the desired estimate (75) with $a = D\,2^{14}$. □

To show uniform L^∞ bounds, we apply now the result of the previous lemma 4.1 with $W_{\lambda_k+1}(t) = \int_{\mathbb{R}^N}\left(n^{\lambda_k+1} + p^{\lambda_k+1}\right)dx$ and the following choice of K:

$$K := \max\{1, \sup_{t\geq 0}\|w(\cdot,t)\|_{L^1}, \|w(\cdot,0)\|_{L^\infty}\}. \tag{79}$$

We recall that

$$W_{\lambda_k+1}(0) \leq \|w(\cdot,0)\|_{L^1}\|w(\cdot,0)\|_{L^\infty}^{\lambda_k} \leq K^{\lambda_k+1},$$

where K is defined by (79). Moreover, we recall that, due to the uniform bound in L^1 for n and p, we have $\sup_{t\geq 0} W_{\lambda_0+1}(t) < \infty$. Hence, the result in (75) clearly implies that $\|n(\cdot,t)\|_{L^\infty} + \|p(\cdot,t)\|_{L^\infty}$ is uniformly bounded in t. □

4.2. **Exponential convergence to equilibrium.** In the following we show exponential convergence (with constants that can all be made explicit) towards the unique equilibrium states n_∞, p_∞ as defined in (66). In addition to the assumptions (NL1) and (NL2) we will suppose that :

- (NL3) The confining potentials are equal $V := V_n = V_p$ and $\mu := e^{-V(x)}$ is – without loss of generality – normalised with $\int_{\mathbb{R}^N} d\mu = \int_{\mathbb{R}^N}\mu\,dx = 1$.
- (NL4) There exists a constant lower bound $F(n,p)\mu(x) \geq C_F(\|n\|_\infty^{-1}, \|p\|_\infty^{-1})$ and moreover, due to the bounds of Lemma 4.1, $C_F(\|n\|_\infty^{-1}, \|p\|_\infty^{-1}) \geq C_F > 0$.

Notations: We rewrite the drift-diffusion fluxes J_n and J_p in terms of their stationary state μ, i.e.

$$J_n = \mu\,\nabla\left(\frac{n}{\mu}\right), \quad J_p = \mu\,\nabla\left(\frac{p}{\mu}\right),$$

and J_n and J_p vanish iff n and p are proportional to μ. Moreover, we introduce the shortcuts

$$\overline{n} = \int_{\mathbb{R}^N} n\,dx\,, \quad \overline{p} = \int_{\mathbb{R}^N} p\,dx\,,$$

and observe that $C_n = \int_{\mathbb{R}^N} n_\infty\,dx$ and $C_p = \int_{\mathbb{R}^N} p_\infty\,dx$ due to $\int_{\mathbb{R}^N} \mu\,dx = 1$.

Lemma 4.3 (Entropy entropy-dissipation inequality)**.** *Let n and p be nonnegative functions in $L^1 \cap L^\infty(\mathbb{R}^n)$ satisfying the conservation law $N - P = C_n - C_p$ as given in* (65)*. Suppose the assumptions (NL1), (NL3), and (NL4) hold. Then, the following inequality holds for a constant K depending only on the stated quantities*

$$E(n,p) \le K(\|n\|_1, \|p\|_1, C_F)\, D(n,p)\,. \tag{80}$$

Proof. Step 1: Upper bound for the relative entropy.
We recall the logarithmic Sobolev inequality for the a normalised reference measure $d\mu = \mu\,dx$ corresponding to the strictly convex confining potential $\mu = e^{-V}$ (see [HS87] to also permit L^∞-bounded perturbation)

$$\int_{\mathbb{R}^N} \frac{|J_n|^2}{n}\,dx = 4\int_{\mathbb{R}^N} \left|\nabla\left(\sqrt{\frac{n}{\mu}}\right)\right|^2 d\mu \ge C_L(\sigma_n)\int_{\mathbb{R}^N} n\ln\left(\frac{n}{\overline{n}\,\mu}\right)dx\,,$$

with σ_n as in assumption (NL1) and an analog inequality holds for p with respect to σ_p. Thus split the entropy density as $\ln\frac{n}{n_\infty} = \ln\frac{n}{\overline{n}\,\mu} + \ln\frac{\overline{n}}{C_n}$, we obtain

$$\begin{aligned} E \le C_L &\left[\int_{\mathbb{R}^N} \frac{|J_n|^2}{n}\,dx + \int_{\mathbb{R}^N} \frac{|J_p|^2}{p}\,dx\right] \\ &+ C_n\left[\frac{\overline{n}}{C_n}\ln\frac{\overline{n}}{C_n} - \left(\frac{\overline{n}}{C_n} - 1\right)\right] + C_p\left[\frac{\overline{p}}{C_p}\ln\frac{\overline{p}}{C_p} - \left(\frac{\overline{p}}{C_p} - 1\right)\right]. \end{aligned}$$

Employing an elementary inequality $y\ln y - (y-1) \le C\ln(y)(\sqrt{y}-1)^2$ for $y \in [0,\infty)$ (see [DF08]) yields further that

$$E \le C_L D + C(\ln(\|n\|_1), \ln(\|p\|_1))\left[\left(\sqrt{\frac{\overline{n}}{C_n}} - 1\right)^2 + \left(\sqrt{\frac{\overline{p}}{C_p}} - 1\right)^2\right]. \tag{81}$$

Note that as particular benefit of employing the logarithmic Sobolev inequality individually to the two species we may estimate the $L\log L$ structure of the entropy on the level of the integrals $\overline{n}$ and $\overline{p}$, for which L^1 bounds are sufficient. Alternatively, estimating directly the entropy density would require $\|\ln n\|_\infty$ and $\|\ln p\|_\infty$ bounds and exponential convergence can be obtain in the case of slowly growing L^∞ a-priori estimates (see [DF08]).

Step 2: Mass exchange due to recombination and the conservation law.
We quantify the recombination mass exchange between n and p under the constraint of the conservation law (65), i.e. $\overline{n} - \overline{p} = C_n - C_p$ in terms of a two dimensional functional inequality involving the integrated quantities

$$y_1 = \sqrt{\frac{\overline{n}}{C_n}} = \sqrt{\int_{\mathbb{R}^N} \frac{n}{n_\infty}\,d\mu}\,, \qquad y_2 = \sqrt{\frac{\overline{p}}{C_p}} = \sqrt{\int_{\mathbb{R}^N} \frac{p}{p_\infty}\,d\mu}\,.$$

Precisely, we show that for all $(y_1, y_2) \in [0,\infty)^2$ there exists a constant C such that

$$C_n(y_1-1)^2 + C_p(y_2-1)^2 \le C(y_1y_2 - 1)^2 \tag{82}$$

holds under the conservation law constraint

$$C_n(y_1-1)(y_1+1) = C_p(y_2-1)(y_2+1)\,. \tag{83}$$

We remark, that inequality (82) (together with (85) below) yields the entropy entropy-dissipation estimate (80) in the special situations when $\frac{n}{\mu}$ and $\frac{p}{\mu}$ are constant.

To proof (82) we factorise, for instance with respect to y_2

$$(y_1y_2 - 1) = \left[\frac{y_1-1}{y_2-1}y_2 + 1\right](y_2-1) = \left[\frac{C_p}{C_n}\frac{y_2+1}{y_1+1}y_2 + 1\right](y_2-1)\,,$$

using the constraint (83). Hence, using again (83), the inequality (82) rewrites as

$$C_p(y_2-1)^2\left[\frac{C_p}{C_n}\frac{(y_2+1)^2}{(y_1+1)^2}+1\right]\le C\left[\frac{C_p}{C_n}\frac{y_2+1}{y_1+1}y_2+1\right]^2(y_2-1)^2\,,$$

which holds for a constant C as the fraction

$$\frac{\left[\frac{C_p}{C_n}\frac{(y_2+1)^2}{(y_1+1)^2}+1\right]}{\left[\frac{C_p}{C_n}\frac{y_2+1}{y_1+1}y_2+1\right]^2}=\frac{\left[\frac{C_p}{C_n}+\frac{(y_1+1)^2}{(y_2+1)^2}\right]}{\left[\frac{C_p}{C_n}y_2+\frac{y_1+1}{y_2+1}\right]^2}\le C$$

is bounded on $[0,\infty)^2$ as can be check by distinguishing the cases $1\gtrless\frac{y_2+1}{y_1+1}$. Thus, altogether from (81) and (82), we have that

$$E\le C_L D+C(\|n\|_1,\|p\|_1)\left(\sqrt{\int_{\mathbb{R}^N}\frac{n}{n_\infty}\,d\mu}\sqrt{\int_{\mathbb{R}^N}\frac{p}{p_\infty}\,d\mu}-1\right)^2. \tag{84}$$

Step 3: Lower bound for the entropy dissipation.
Denoting the third, reactive term of the entropy dissipation (68) D_r, we apply the elementary inequality $\ln(y)(y-1)\ge 4(\sqrt{y}-1)^2$, assumption (NL4), and Jensen's inequality to estimate

$$D_r\ge C\int_{\mathbb{R}^N}F\mu\left(\sqrt{\frac{n}{n_\infty}}\sqrt{\frac{p}{p_\infty}}-1\right)^2d\mu\ge C_F\left(\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\sqrt{\frac{p}{p_\infty}}\,d\mu-1\right)^2. \tag{85}$$

Hence, comparing (85) with (84) we are – qualitatively spoken – left to interchange the square root with the integrals. We therefore estimated some fraction (leaving the rest aside for the logarithmic Sobolev inequality) of the drift-diffusion terms of the entropy dissipation (68) by Poincaré's inequality

$$\int_{\mathbb{R}^N}\left|\nabla\left(\sqrt{\frac{n}{n_\infty}}\right)\right|^2d\mu\ge C_P\int_{\mathbb{R}^N}\delta_n^2\,d\mu\,,\quad\text{where }\ \delta_n:=\sqrt{\frac{n}{n_\infty}}-\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu\,,$$

and analog with p and δ_p. Expanding (85) in terms of δ_n and δ_p, we exploit that the first order integrals $\int_{\mathbb{R}^N}\delta_n\,d\mu=\int_{\mathbb{R}^N}\delta_p\,d\mu=0$ vanish and estimate with Young's inequality

$$\begin{aligned}D_r&=C_F\left(\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu\int_{\mathbb{R}^N}\sqrt{\frac{p}{p_\infty}}\,d\mu-\int_{\mathbb{R}^N}\delta_n\delta_p\,d\mu-1\right)^2\\&\ge C_F\left[\left(\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu\int_{\mathbb{R}^N}\sqrt{\frac{p}{p_\infty}}\,d\mu-1\right)^2\right.\\&\qquad\left.-C(\|n\|_1,\|p\|_1)\left(\int_{\mathbb{R}^N}\delta_n^2\,d\mu+\int_{\mathbb{R}^N}\delta_p^2\,d\mu\right)\right],\end{aligned}$$

where $\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu\le C(\|n\|_1)$ by Cauchy-Schwartz and analog for p. We remark that since the entropy dissipation vanishes at the equilibrium in second order in terms of δ_n and δ_p the cancellation of first order integrals (in general $\int_{\mathbb{R}^N}\delta_n\,d\mu_p=\int_{\mathbb{R}^N}\delta_p\,d\mu_n=0$) is crucial for the proof and precisely the point where we are forced to make assumption (NL3).

To quantify what qualitatively was called interchanging square roots with integration, we observe (and again analog for p) that

$$\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu=\sqrt{\int_{\mathbb{R}^N}\frac{n}{n_\infty}\,d\mu}-R_n\int_{\mathbb{R}^N}\delta_n^2\,d\mu \tag{86}$$

and $R_n=(\sqrt{\int_{\mathbb{R}^N}\frac{n}{n_\infty}\,d\mu}+\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu)^{-1}$ is unbounded if and only if $\sqrt{\int_{\mathbb{R}^N}\frac{n}{n_\infty}\,d\mu}\ge\int_{\mathbb{R}^N}\sqrt{\frac{n}{n_\infty}}\,d\mu$ vanishes. We are therefore lead to distinguish two cases: At first, for an $\varepsilon>0$ to be set below, we suppose the

Case $\sqrt{\int_{\mathbb{R}^N} \frac{n}{n_\infty}\,d\mu} \geq \varepsilon$ and $\sqrt{\int_{\mathbb{R}^N} \frac{p}{p_\infty}\,d\mu} \geq \varepsilon$: and expand in terms of (86),

$$\left(\int_{\mathbb{R}^N} \sqrt{\frac{n}{n_\infty}}\,d\mu \int_{\mathbb{R}^N} \sqrt{\frac{p}{p_\infty}}\,d\mu - 1\right)^2 \geq \left(\sqrt{\int_{\mathbb{R}^N} \frac{n}{n_\infty}\,d\mu}\sqrt{\int_{\mathbb{R}^N} \frac{p}{p_\infty}\,d\mu} - 1\right)^2 - C(\|n\|_1, \|p\|_1)\left(\int_{\mathbb{R}^N} \delta_n^2\,d\mu + \int_{\mathbb{R}^N} \delta_p^2\,d\mu\right),$$

and the first term on the right hand side corresponds to (84). Thus, we obtain the entropy entropy-dissipation estimate (80) by choosing a constant $K(\|n\|_1, \|p\|_1, C_F)$ large enough such that the remainder terms of order $\int_{\mathbb{R}^N} \delta_n^2\,d\mu$ and $\int_{\mathbb{R}^N} \delta_p^2\,d\mu$ are compensated by the Poincaré lower bounds of the drift-diffusion dissipation terms (after leaving some fraction aside to employ the logarithmic Sobolev inequality).

The other cases, e.g. $\int_{\mathbb{R}^N} \sqrt{\frac{n}{n_\infty}}\,d\mu \leq \sqrt{\int_{\mathbb{R}^N} \frac{n}{n_\infty}\,d\mu} < \varepsilon$ in terms of n: In these far–from–equilibrium cases, we chose $\varepsilon = C(\|p\|_1) \leq (2\int_{\mathbb{R}^N} \sqrt{p/p_\infty}\,d\mu)^{-1}$ and find therefore that

$$\left(\int_{\mathbb{R}^N} \sqrt{\frac{n}{n_\infty}}\,d\mu \int_{\mathbb{R}^N} \sqrt{\frac{p}{p_\infty}}\,d\mu - 1\right)^2 \geq \frac{1}{4}\,.$$

Hence since the second term on right hand side of (84) is trivially bounded in terms of a constant $C(\|n\|_1, \|p\|_1)$ we obtain the stated entropy dissipation provided for a constant $K(\|n\|_1, \|p\|_1, C_F)$ large enough. □

Remark 4.1. The above proof of lemma 4.3 obviously does not yield a sharp convergence rate. Nevertheless, it only requires natural L^1 bounds (which very generally follows from the entropy decay), a nondegeneracy assumption on the reaction rate (NL4), which as a consequence of the homogeneity of the nonlinear reaction term $R(n,p)$ includes also the confining measure μ, and suitable confinement assumption (NL1), (NL3). It remains an open problem to proof exponential (or algebraic) convergence without (NL3) or if (NL4) would be relaxed to $F \geq C_F > 0$.

Remark 4.2. An entropy entropy-dissipation estimate for the system (63) coupled to a self-consistent Poisson equation is subject of a forthcoming paper [DFFM].

Theorem 4.1 (Exponential convergence to equilibrium)**.** *Let n and p be solutions to the system* (63) *subject to non-negative initial data n_I and p_I. Suppose the assumptions (NL1)–(NL4) hold. Then, the solution converges exponentially (with explicitly computable constants) to the unique equilibrium state* (66) *and the following estimate holds*

$$\|n(t) - n_\infty\|_1^2 + \|p(t) - p_\infty\|_1^2 \leq C E(n_I, p_I)\, e^{-Kt}\,, \tag{87}$$

with a constant K depending explicitly on the initial masses $\int_{\mathbb{R}^N} n_I$, $\int_{\mathbb{R}^N} p_I$, the initial entropy $E(n_I, p_i)$, the uniform convexity constant of the confinement potential V, and the lower bound in (NL4).

Proof. By the uniform L^1 bound of Lemma 4.1 and the entropy entropy dissipation estimate Lemma 4.3, the results follows directly by integrating

$$\frac{d}{dt}E(n,p) = -D(n,p) \leq -K\,E(n,p)\,,$$

and the Csizsár–Kullback inequality (69). □

References

[ABD07] A. Arnold, J. P. Bartier, and J. Dolbeault, *Interpolation between logarithmic Sobolev and Poincaré inequalities*, Commun. Math. Sci. **5** (2007), no. 4, 971–979.

[AD05] A. Arnold and J. Dolbeault, *Refined convex Sobolev inequalities*, J. Funct. Anal. **225** (2005), no. 2, 337–351.

[Ali79] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations **4** (1979), no. 8, 827–868.

[Ama85] H. Amann, *Global existence for semilinear parabolic systems*, J. Reine Angew. Math. **360** (1985), 47–83.

[AMT00] A. Arnold, P. A. Markowich, and G. Toscani, *On large time asymptotics for drift-diffusion-Poisson systems*, Proceedings of the Fifth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Maui, HI, 1998), vol. 29, 2000, pp. 571–581.

[AMTU01] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations **26** (2001), no. 1-2, 43–100.

[CHS78] E. Conway, D. Hoff, and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction-diffusion equations*, SIAM J. Appl. Math. **35** (1978), no. 1, 1–16.

[CJM+01] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, and A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math. **133** (2001), no. 1, 1–82. MR MR1853037 (2002j:35188)

[CL93] E. A. Carlen and M. Loss, *Sharp constant in Nash's inequality*, Internat. Math. Res. Notices (1993), no. 7, 213–215.

[DF] L. Desvillettes and K. Fellner, *Entropy methods for reaction-diffusion equations with degenerate diffusion arising in reversible chemistry*, preprint.

[DF06] Laurent Desvillettes and Klemens Fellner, *Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations*, J. Math. Anal. Appl. **319** (2006), no. 1, 157–176.

[DF08] L. Desvillettes and K. Fellner, *Entropy methods for reaction-diffusion equations: Slowly growing a-priori bounds*, Rev. Matematica Iberoamericana **24** (2008), no. 2, 407–431.

[DFFM] M. Di Francesco, K. Fellner, and P. A. Markowich, *In preparation.*

[DGJ97] P. Degond, S. Génieys, and A. Jüngel, *Symmetrization and entropy inequality for general diffusion equations*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 9, 963–968.

[DPD02] Manuel Del Pino and Jean Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) **81** (2002), no. 9, 847–875. MR MR1940370 (2003h:35051)

[FHM97] W. B. Fitzgibbon, S. L. Hollis, and J. J. Morgan, *Stability and Lyapunov functions for reaction-diffusion systems*, SIAM J. Math. Anal. **28** (1997), no. 3, 595–610.

[Fis37] R. A. Fisher, *The advance of advantageous genes*, Ann. eugenics **7** (1937), 335–369.

[Fri54] K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–392.

[GGH96] A. Glitzky, K. Gröger, and R. Hünlich, *Free energy and dissipation rate for reaction diffusion processes of electrically charged species*, Appl. Anal. **60** (1996), no. 3-4, 201–217.

[GH97] A. Glitzky and R. Hünlich, *Energetic estimates and asymptotics for electro-reaction-diffusion systems*, Z. Angew. Math. Mech. **77** (1997), no. 11, 823–832.

[Gri91] P. Grindrod, *Patterns and waves*, Oxford Applied Mathematics and Computing Science Series, The Clarendon Press Oxford University Press, New York, 1991, The theory and applications of reaction-diffusion equations.

[HS87] R. Holley and D. Stroock, *Logarithmic Sobolev inequalities and stochastic Ising models*, J. Statist. Phys. **46** (1987), no. 5-6, 1159–1194.

[KL89] H. O. Kreiss and J. Lorenz, *Initial-boundary value problems and the Navier-Stokes equations*, Pure and Applied Mathematics, vol. 136, Academic Press Inc., Boston, MA, 1989.

[Kow05] R. Kowalczyk, *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl. **305** (2005), no. 2, 566–588.

[KPP37] A. N. Kolmogorov, I. G. Petrovsky, and Bulletin Universite dEtat 'a Moscou (Bjul. Moskowskogo Gos. Univ.) Serie internationale A 1 (1937) pp. 1-26. Piskunov, N. S., *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'État à Moscou (Bjul. Moskowskogo Gos. Univ.), Série internationale **A1** (1937), 1–26.

[MRS90] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor equations*, Springer-Verlag, Vienna, 1990.

[Mur03] J. D. Murray, *Mathematical biology. II*, third ed., Interdisciplinary Applied Mathematics, vol. 18, Springer-Verlag, New York, 2003, Spatial models and biomedical applications.

[MWZ] P. A. Markowich, H. Wu, and S. Zheng, *Global existence and asymptotic behavior for a drift–diffusion–poisson model*, To appear on Math. Models Methods Appl. Sci.

[Nas58] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954.

[Ott01] Felix Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26** (2001), no. 1-2, 101–174. MR MR1842429 (2002j:35180)

[Rot84] F. Rothe, *Global solutions of reaction-diffusion systems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1984.

[SK85] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation.*, Hokkaido Math. J. **14** (1985), 249–275.

[Smo83] J. Smoller, *Shock waves and reaction-diffusion equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 258, Springer-Verlag, New York, 1983.

[Tur52] A. M. Turing, *The chemical basis of morphogenesis*, Philo. Trans. Roy. Soc. Lond. Ser. B **237** (1952), 5–72.

[UAMT00] A. Unterreiter, A. Arnold, P. Markowich, and G. Toscani, *On generalized Csiszár-Kullback inequalities*, Monatsh. Math. **131** (2000), no. 3, 235–253.

Marco Di Francesco, Division of Mathematics for Engineering, Department of Pure and Applied Mathematics, University of L'Aquila, P.le E. Pontieri 2, Monteluco di Roio, I-67040 L'Aquila, Italy
E-mail address: mdifrance@gmail.com

Klemens Fellner, Department of Applied Mathematics and Theoretical Physics (DAMTP), Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK
E-mail address: K.Fellner@damtp.cam.ac.uk

ON LEAVE OF ABSENCE FROM THE FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERTSTR. 15, 1090 WIEN, AUSTRIA
E-mail address: Klemens.Fellner@univie.ac.at

PETER A. MARKOWICH, DEPARTMENT OF APPLIED MATHEMATICS AND THEORETICAL PHYSICS (DAMTP), CENTRE FOR MATHEMATICAL SCIENCES, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UK
E-mail address: P.A.Markowich@damtp.cam.ac.uk

PETER A. MARKOWICH, FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERTSTR. 15, 1090 WIEN, AUSTRIA
E-mail address: Peter.Markowich@univie.ac.at