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FAST-REACTION LIMIT FOR THE INHOMOGENEOUS AIZENMAN-BAK MODEL

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Abstract. Solutions of the spatially inhomogeneous diffusive Aizenmann-Bak model for clustering within a bounded domain with homogeneous Neumann boundary conditions are shown to stabilize, in the fast reaction limit, towards local equilibria determined by their monomer density. Moreover, the sequence of monomer densities converges to the solution of a nonlinear diffusion equation whose nonlinearity depends on the size-dependent diffusion coefficient. Initial data are assumed to be integrable, bounded and with a certain number of moments in size. The number density of clusters for the solutions is assumed to verify uniform bounds away from zero and infinity independently of the scale parameter.

1. **Introduction.** In this work, we will analyze the fast reaction asymptotics of the spatially inhomogeneous Aizenman-Bak model for clustering with spatial diffusion given by

$$\partial_t f - a(y) \triangle_x f = Q(f, f) \,. \tag{1}$$

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Here, $f = f(t,x,y)$ is the concentration of clusters with size $y \geq 0$ at time $t \geq 0$ which are spatially diffusing in $x \in \Omega \subset \mathbb{R}^d$, $d \geq 1$ with normalized volume, i.e., $|\Omega| = 1$. Homogeneous Neumann boundary condition:

$$\nabla_x f(t,x,y) \cdot \nu(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where ν denotes the outward unit normal to Ω, are imposed in order to preserve the total number of aggregates. As in [3], we assume that the diffusion coefficient $a(y)$ is non-degenerate

$$0 < a_* \leq a(y) \leq a^* \tag{3}$$

with $a_*, a^* \in \mathbb{R}^+$. The collision operator $Q(f,f)$ takes into account cluster coagulation and fragmentation/break-up, and it reads as

$$Q(f,f) = Q_c(f,f) + Q_b(f,f) \tag{4}$$

with

$$Q_c(f,f) := \int_0^y f(t,x,y-y')f(t,x,y')\,dy' - 2f(t,x,y)\int_0^\infty f(t,x,y')\,dy',$$

and

$$Q_b(f,f) := Q_b^+(f,f) - Q_b^-(f,f) := 2\int_y^\infty f(t,x,y')\,dy' - y\,f(t,x,y).$$

These models appear in applications such as polymerization [1], cluster aggregation in aerosols [1], [2], [5], cell physiology [12], population dynamics [11], astrophysics [14] and blood thrombi formation [6].

A basic formal property of solutions is the conservation of mass, i.e. the total number of monomers. Since the reaction term (4) satisfies $\int_0^\infty y\,Q(f,f)\,dy = 0$, we have (formally) for all $t \geq 0$,

$$\int_\Omega N(t,x)\,dx = \int_\Omega N_{in}(x)\,dx := N_\infty, \qquad \text{where } N(t,x) := \int_0^\infty y\,f(t,x,y)\,dy. \tag{5}$$

Another macroscopic quantity of interest is the number density of polymers,

$$M(t,x) := \int_0^\infty f(t,x,y)\,dy, \tag{6}$$

that together with the monomer density $N(t,x)$ satisfies the reaction-diffusion type system

$$\partial_t N - \triangle_x \left(\int_0^\infty y a(y)\, f(t,x,y)\, dy \right) = 0\,, \tag{7}$$

$$\partial_t M - \triangle_x \left(\int_0^\infty a(y)\, f(t,x,y)\, dy \right) = N - M^2\,. \tag{8}$$

The other important property is the dissipation of the corresponding entropy functional. We will consider the weak definition of the action of the collision operator (4) given by

$$\begin{aligned}\int_0^\infty Q(f,f)\,\varphi\, dy = -2\int_0^\infty \varphi(y) f(y)\, dy \int_0^\infty f(y')\, dy' \\ + \int_0^\infty \int_0^\infty f(y) f(y') \varphi(y'')\, dy\, dy' \\ + 2\int_0^\infty f(y)\,\Phi(y)\, dy - \int_0^\infty y\, f(y)\,\varphi(y)\, dy\end{aligned} \tag{9}$$

for any smooth function $\varphi(y)$, where $y'' = y + y'$ and with the function Φ being the primitive of φ (i.e. $\partial_y \Phi = \varphi$) such that $\Phi(0) = 0$. Let us consider the entropy functional associated to any positive density f as

$$H(f)(t,x) = \int_0^\infty \left(f \ln f - f \right)\, dy\,,$$

with the relative entropy $H(f|g) = H(f) - H(g)$ between two states f and g not necessarily with the same L^1_y-norm. Then, the entropy formally dissipates as

$$\begin{aligned}\frac{d}{dt}\int_\Omega H(f)\, dx = -\int_\Omega \int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\, dy\, dx \\ - \int_\Omega \int_0^\infty \int_0^\infty (f'' - f f') \ln\left(\frac{f''}{f f'}\right) dy\, dy' dx := -D_H(f).\end{aligned} \tag{10}$$

Global-in-time weak solutions to (1)-(2) satisfying the entropy dissipation inequality

$$\int_\Omega H(f(t))\, dx + \int_0^t D_H(f(s))\, ds \le \int_\Omega H(f_0)\, dx$$

for all $t \ge 0$, were obtained in [8]. The equilibrium states for which the entropy dissipation vanishes are given by:

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}}\,,$$

where N_∞ is uniquely identified by the conservation of mass (5). It is also proved in [8] that f_∞ attracts all global weak solutions in $L^1(\Omega \times (0,\infty))$ of (1)-(2) but no time decay rate is obtained. Exponential rate of decay for this problem was recently studied in [3] in the one dimensional spatial case. We refer to [8], [3] for extensive literature related to these problems.

Our present aim is to study the fast-reaction asymptotics, i.e. the limit $\varepsilon \to 0$ of the rescaled problem:

$$\begin{cases} \partial_t f^\varepsilon - a(y)\triangle_x f^\varepsilon = \dfrac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)\,, \text{for } x \in \Omega,\ y \in (0,\infty), t>0\,, \\ \nabla_x f^\varepsilon(t,x,y)\cdot \nu(x) = 0\,, \qquad \text{for } x \in \partial\Omega,\ y \in (0,\infty), t>0\,, \\ f^\varepsilon(t=0,x,y) = f_{in}(x,y) \geq 0\,, \ \text{for } x \in \Omega,\ y \in (0,\infty)\,, \end{cases} \tag{11}$$

where we shall assume that $(1 + y + \ln f_{in}) f_{in} \in L^1(\Omega \times (0,\infty))$. This asymptotic regime is called the fast reaction limit since the reaction term is dominant as ε gets smaller. In fact, letting formally $\varepsilon \to 0$, we expect $f^\varepsilon \to f^0$ satisfying $Q(f^0, f^0) = 0$, i.e.

$$f^\varepsilon \to e^{-\frac{y}{\sqrt{N^0}}}\,,$$

where the limiting monomer density $N^0(t,x)$ diffuses according to the limit of the moment equation (7):

$$\partial_t N^0 - \triangle_x n(N^0) = 0\,, \tag{12}$$

where $n(N)$ denotes the function

$$n(N) := \int_0^\infty a(y) y e^{-\frac{y}{\sqrt{N}}}\, dy\,. \tag{13}$$

Under assumption (3), equation (12) is a nonlinear, non-degenerate diffusion equation satisfying

$$0 < a_* N \leq n(N) \leq a^* N\,, \qquad 0 < a_* \leq n'(N) \leq a^*\,.$$

Our main goal is a complete rigorous justification of this formal limit. As a first step however, we will show in this work an "if-theorem". We will assume in the following that the number density M^ε given by (6) is bounded away from zero and infinity uniformly in $\varepsilon > 0$. More precisely, our assumptions are the existence of constants $0 < \mathcal{M}_* \leq \mathcal{M}^* < \infty$ such that

Hypothesis **(HMBB)**, M^ε is bounded from below: $\quad M^\varepsilon(t,x) \geq \mathcal{M}_*$

and

Hypothesis **(HMBA)**, M^ε is bounded from above: $\quad M^\varepsilon(t,x) \leq \mathcal{M}^*$

for all $t \geq 0$, $x \in \Omega$ and $\varepsilon > 0$. The main result of this work is the following:

Theorem 1.1. *Let Ω be a bounded smooth subset of $\mathbb{R}^d$ with normalized volume $|\Omega| = 1$ and let the diffusion coefficient $a(y)$ satisfies (3). Assume non-negative initial data $f^\varepsilon(t=0,x,y) = f_{in}(x,y) \geq 0$ such that $(1 + y^6 + \ln f_{in})f_{in} \in L^1(\Omega \times (0,\infty))$ and $f_{in} \in L^\infty(\Omega \times (0,\infty))$. Let us assume that the solutions of the rescaled problem (11) verify the hypotheses* **(HMBB)** *and* **(HMBA)**. *Then, the monomer density N^ε converges in $L^2((0,T)\times\Omega)$ to the unique solution N of the Neumann problem for the nonlinear diffusion equation*

$$\begin{cases} \partial_t N - \triangle_x n(N) = 0\,, \\ \nabla_x N \cdot \nu(x)|_{\partial\Omega} = 0, \end{cases} \tag{14}$$

with initial data $N_{in} = \int_0^\infty y f_{in}\, dy$, for any $T > 0$, and where the nonlinearity $n(N)$ is given by (13).

Let us remark that the hypotheses **(HMBB)** and **(HMBA)** cannot be obtained by the estimates in [3] since they lead to ε dependent bounds. Bounds from below depending on ε of the density function could be obtained by adapting the arguments in [10]. It is an open problem to show these ε uniform bounds in this generality, although a perturbative setting around global equilibrium is under current investigation.

Next Sections below are the main steps in the proof of the previous Theorem. Section 2 is devoted to show that the entropy dissipation tends to 0 as $\varepsilon \to 0$, which in return shows local stabilization of the distribution function in L^1 in phase-space. Section 3 collects several estimates on moments, L^∞-bounds of f^ε and L^p-bounds of N^ε, which allow, in Section 4, to prove local stabilization in L^2 in space at the cost of a lower exponent of ε controlling uniformly the rest of the ε-expansion of f^ε. Finally, by an L^2 duality arguments, Section 4 finishes the proof of Theorem 1.1 by passing to the limit in the nonlinear nonlocal diffusion equations.

Notation. We will use various short-cuts like $L^p_x = L^p(\Omega)$, $L^p_y = L^p((0,\infty))$, and $L^2_t(L^1_{x,y}) = L^2((0,\infty), L^1(\Omega \times (0,\infty)))$.

2. Entropy Dissipation: L^1-Trend to Local Equilibria.

In this section, we prove an ε independent L^1_x-bound of M^ε, which allows to show that the limiting solution f^ε equilibrates asymptotically at a local equilibrium of the form:

$$f_{N^\varepsilon} := e^{-\frac{y}{\sqrt{N^\varepsilon}}}\,.$$

For notational convenience, we will work (in this and the next section) on the equivalent time-scaled problem with $t = \varepsilon\tau$:

$$\begin{cases} \partial_\tau f^\varepsilon - \varepsilon a(y) \triangle_x f^\varepsilon = Q(f^\varepsilon, f^\varepsilon)\,, & \text{for } x \in \Omega,\ y \in (0,\infty), \tau > 0\,, \\ \nabla_x f^\varepsilon(\tau, x, y) \cdot \nu(x) = 0\,, & \text{for } x \in \partial\Omega,\ y \in (0,\infty), \tau > 0\,, \\ f^\varepsilon(\tau = 0, x, y) = f_{in}(x,y)\,, & \text{for } x \in \Omega,\ y \in (0,\infty). \end{cases} \tag{15}$$

Moreover in this section, it is sufficient to assume initial data $f^\varepsilon(\tau = 0, x, y) = f_{in}(x,y) \geq 0$ such that $(1 + y + \ln f_{in}) f_{in} \in L^1(\Omega \times (0,\infty))$.

We start by deriving the L^1_x-bound of M^ε by integrating equality (8), obtaining

$$\begin{aligned} \frac{d}{d\tau} \int_\Omega M^\varepsilon(\tau,x)\,dx &= \int_\Omega N^\varepsilon(\tau,x)\,dx - \int_\Omega M^\varepsilon(\tau,x)^2\,dx \\ &\leq \int_\Omega N_{in}(x)\,dx - \left(\int_\Omega M^\varepsilon(\tau,x)\,dx\right)^2 \end{aligned}$$

by the conservation of mass (5) and by Hölder's inequality. Therefore, for all $\tau \geq 0$ and $\varepsilon > 0$, we have

$$\int_\Omega M^\varepsilon(\tau,x)\,dx \leq \max\left\{\int_\Omega M_{in}(x)\,dx, \left(\int_\Omega N_{in}(x)\,dx\right)^{1/2}\right\} := \mathcal{M}_0^*. \tag{16}$$

We remark that a bound like (16) also follows clearly for the hypothesis **(HMBA)**, which we nevertheless like to avoid whenever we know how to.

The trend to local equilibrium follows now from the dissipation of the entropy, which is better understood by using the remarkable inequality proven in [1, Propositions 4.2 and 4.3], implying that [3]

$$\int_0^\infty \int_0^\infty (f'' - f f') \ln\left(\frac{f''}{f f'}\right) dy\,dy' \geq M\, H(f|f_N) + 2(M - \sqrt{N})^2\,. \tag{17}$$

Thus, the decay of the entropy functional $H(f^\varepsilon) = \int_0^\infty (f^\varepsilon \ln f^\varepsilon - f^\varepsilon)\,dy$ is estimated using inequality (17) as

$$\begin{aligned} -\frac{d}{d\tau} \int_\Omega H(f^\varepsilon)\,dx \geq\ & \varepsilon \int_\Omega \int_0^\infty a(y) \frac{|\nabla_x f^\varepsilon|^2}{f^\varepsilon}\,dy\,dx \\ &+ \int_\Omega \left[M^\varepsilon H(f^\varepsilon | f_{N^\varepsilon}) + 2(M^\varepsilon - \sqrt{N^\varepsilon})^2\right] dx\,. \end{aligned} \tag{18}$$

Taking into account the Csiszar-Kullback inequality as in [3, Lemma 3], we conclude

$$\begin{aligned}\|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\|^2_{L^1_{x,y}} &\le 2\left\{\int_\Omega \left[M^\varepsilon(\tau,x)+\sqrt{N^\varepsilon}(\tau,x)\right]\,dx\right\}\int_\Omega H(f^\varepsilon|f_{N^\varepsilon})\,dx\\ &\le 2\left\{\mathcal{M}_0^* + \sqrt{N_\infty}\right\}\int_\Omega H(f^\varepsilon|f_{N^\varepsilon})\,dx \qquad (19)\end{aligned}$$

by Hölder's inequality, conservation of mass (5) and the above bound (16). Hence, the dissipation of entropy in (18) and (19) implies the following equilibration of the density function f^ε:

Lemma 2.1. *There exists C independent of ε such that*

$$\int_0^\infty \int_\Omega M^\varepsilon H(f^\varepsilon|f_{N^\varepsilon})\,dx\,d\tau \le C\,, \qquad (20)$$

and thus, using the assumption **(HBMB)***, that*

$$\|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\|^2_{L^2_\tau(L^1_{x,y})} \le C(\mathcal{M}_*)\,,$$

or equivalently,

$$\|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\|^2_{L^2_t(L^1_{x,y})} \le \varepsilon C(\mathcal{M}_*)\,, \qquad (21)$$

for a constant C depending on $\mathcal{M}_*$ but not on ε.

The notation $L^2_\tau(L^1_{x,y})$ refers to the space of functions in the scaled space (τ,x,y) belonging to $L^2((0,\infty),L^1(\Omega\times(0,\infty)))$.

3. **A priori Estimates.** In this section, we show further uniform in ε apriori estimates to be interpolated with (21) in proving Theorem 1.1 in the following section.

We start by showing the uniform control in time and $\varepsilon < 1$ of all moments with respect to y of the solutions provided they are initially finite. Let us define the moment of order $p>1$ by

$$M_p^\varepsilon(f^\varepsilon)(\tau) := \int_\Omega \int_0^\infty y^p\, f^\varepsilon(\tau,x,y)\,dy\,dx$$

for all $\tau \ge 0$. Then, the following Lemma holds:

Lemma 3.1. *Let $f_{in} \ge 0$ be a nonnegative initial datum such that $(1+y^p)f_{in} \in L^1(\Omega\times(0,\infty))$ with $p>1$. Assume that the hypothesis* **(HMBA)** *holds. Then, the solution f^ε of (15) has moments $M_p^\varepsilon(f^\varepsilon)(\tau)$ uniformly bounded in time $\tau \ge 0$ and all $\varepsilon < 1$, i.e., there exist explicit constants $\mathcal{M}_p^*(f_{in},\mathcal{M}^*,p)$ such that*

$$M_p^\varepsilon(f^\varepsilon)(\tau) \le \mathcal{M}_p^*\,, \qquad \text{for a.e. } \tau \ge 0. \qquad (22)$$

Proof. Using the weak formulation (9), it is easy to check that

$$\int_0^\infty Q(f^\varepsilon, f^\varepsilon)\, y^p\, dy = -2 \left(\int_0^\infty y^p f^\varepsilon(y)\, dy \right) M^\varepsilon + \int_0^\infty \int_0^\infty f^\varepsilon(y) f^\varepsilon(z) (y+z)^p \, dy\, dz - \frac{p-1}{p+1} \int_0^\infty f^\varepsilon(y)\, y^{p+1}\, dy.$$

Taking into account Hypothesis **(HMBA)** and $(y+z)^p \le C_p' \,(y^p + z^p)$, we deduce

$$\int_0^\infty Q(f^\varepsilon, f^\varepsilon)\, y^p\, dy \le\, 2C_p' \mathcal{M}^* \int_0^\infty y^p f^\varepsilon(y)\, dy - \frac{p-1}{p+1} \int_0^\infty f^\varepsilon(y)\, y^{p+1}\, dy$$

for all $p > 1$. Integrating in space, we find that the evolution of the moment of order $p > 1$ is given by

$$\frac{d}{d\tau} M_p^\varepsilon(f^\varepsilon)(\tau) \le 2C_p' \mathcal{M}^*\, M_p^\varepsilon(f^\varepsilon)(\tau) - \frac{p-1}{p+1} M_{p+1}^\varepsilon(f^\varepsilon)(\tau).$$

Trivial interpolation of the $p+1$-order moment with the moment of order one implies

$$M_p^\varepsilon(f^\varepsilon)(\tau) \le \frac{1}{\delta^{p-1}} \int_\Omega N_{in}(x)\, dx + \delta\, M_{p+1}^\varepsilon(f^\varepsilon)(\tau)$$

for all $\delta > 0$, and thus

$$\frac{d}{d\tau} M_p^\varepsilon(f^\varepsilon)(\tau) \le 2C_p' \mathcal{M}^*\, M_p^\varepsilon(f^\varepsilon)(\tau) - \frac{p-1}{p+1} \frac{1}{\delta} M_p^\varepsilon(f^\varepsilon)(\tau) + D_\delta$$

for a certain constant D_δ (of order δ^{-p}). Choosing $\delta > 0$ such that

$$2C_p' \mathcal{M}^* - \frac{p-1}{p+1} \frac{1}{\delta} \le -\frac{1}{10\delta}$$

we obtain

$$\frac{d}{d\tau} M_p^\varepsilon(f^\varepsilon)(\tau) \le -\frac{1}{10\delta} M_p^\varepsilon(f^\varepsilon)(\tau) + D_\delta$$

for all $t > 0$, from which

$$M_p^\varepsilon(f^\varepsilon)(\tau) \le \min(M_p^\varepsilon(f^\varepsilon)(0), 10\delta D_\delta),$$

ending the proof. □

We can also control uniformly the distribution function f^ε.

Lemma 3.2. *Let $f_{in} \geq 0$ be a nonnegative initial datum such that $f_{in} \in L^\infty(\Omega \times (0,\infty))$. Then, the solution f^ε of (15) is uniformly bounded in time $\tau \geq 0$ and all $\varepsilon < 1$, i.e., there exists an explicit constant $\mathcal{K}(f_{in})$ such that*

$$\|f^\varepsilon(\tau)\|_{L^\infty_{x,y}} \leq \mathcal{K}\,, \qquad \text{for a.e. } \tau \geq 0. \tag{23}$$

Proof. We use [8, Lemma 3.5] with $\varphi(r) = (r-K)_+$ with $K \geq \|f_{in}\|_{L^\infty_{x,y}}$ to obtain

$$\int_\Omega \int_0^\infty Q_c(f^\varepsilon, f^\varepsilon)\, \varphi'(f)\, dydx \leq -\int_\Omega \int_0^\infty \int_0^\infty \varphi'(f^\varepsilon(x,y)) f^\varepsilon(x,y) f^\varepsilon(x,y')\, dy'\, dy\, dx$$

for all $\tau \geq 0$. Let us remind the main ideas of the proof of [8, Lemma 3.5] for the sake of the reader, see also [9]. Assume first φ is differentiable and convex such that $0 \leq \varphi(r) \leq r\varphi'(r)$ for all $r > 0$. The action of the coagulation operator can be written as

$$\begin{aligned} I &:= \int_\Omega \int_0^\infty Q_c(f^\varepsilon, f^\varepsilon)\, \varphi'(f)\, dydx \\ &= -2 \int_\Omega \int_0^\infty \int_0^\infty \varphi'(f(y)) f(y) f(y')\, dy'\, dy\, dx \\ &\quad + \int_\Omega \int_0^\infty \int_0^y f(y-y') f(y') \varphi'(f(y))\, dy'\, dy\, dx. \end{aligned}$$

Using the convexity of φ in the last term,
$\varphi(f(y')) \geq \varphi(f(y)) + \varphi'(f(y))(f(y') - f(y))$, we get

$$\begin{aligned} I \leq &-2 \int_\Omega \int_0^\infty \int_0^\infty \varphi'(f(y)) f(y) f(y')\, dy'\, dy\, dx \\ &+ \int_\Omega \int_0^\infty \int_0^y f(y-y') \varphi(f(y'))\, dy'\, dy\, dx \\ &+ \int_\Omega \int_0^\infty \int_0^y f(y-y')[f(y)\varphi'(f(y)) - \varphi(f(y))]\, dy'\, dy\, dx. \end{aligned}$$

Changing variables in the second and third term of the right-hand side as $(y, y') \mapsto (y, z = y - y')$ and $(y, y') \mapsto (y', z = y - y')$ respectively, we obtain

$$
\begin{aligned}
I \le -2 &\int_\Omega \int_0^\infty \int_0^\infty \varphi'(f(y)) f(y) f(y')\, dy'\, dy\, dx \\
&+ \int_\Omega \int_0^\infty \int_0^\infty f(z)\varphi(f(y'))\, dz\, dy'\, dx \\
&+ \int_\Omega \int_0^\infty \int_0^\infty \chi_{[0,y]}(z) f(z)[f(y)\varphi'(f(y)) - \varphi(f(y))]\, dz\, dy\, dx \\
= - &\int_\Omega \int_0^\infty \int_0^\infty \varphi'(f(y)) f(y) f(y')\, dy'\, dy\, dx \\
&+ \int_\Omega \int_0^\infty \int_0^\infty [\chi_{[0,y]}(z) f(z) - f(z)]\, [f(y)\varphi'(f(y)) - \varphi(f(y))]\, dz\, dy\, dx,
\end{aligned}
$$

where $\chi_{[0,y]}(z)$ is the characteristic function of the interval $[0, y]$. It is easy to observe that the last term is non-positive from which the stated inequality on the contribution of the coagulation operator results. The proof for $\varphi(r) = (r - K)_+$ follows by approximation by smooth differentiable convex functions verifying $0 \le \varphi(r) \le r\varphi'(r)$ for all $r > 0$.

Next, we estimate the gain part of the fragmentation kernel to deduce

$$
\begin{aligned}
&\int_\Omega \int_0^\infty Q_b^+(f^\varepsilon, f^\varepsilon)\varphi'(f)\, dydx \\
&\qquad \le \frac{2}{K} \int_\Omega \int_0^\infty \int_0^\infty \varphi'(f^\varepsilon(x,y)) f^\varepsilon(x,y) f^\varepsilon(x,y')\, dy'\, dy\, dx
\end{aligned}
$$

where we use that $K\varphi'(r) \le r\varphi'(r)$ for $r \ge K$ and otherwise $\varphi'(r) = 0$. Putting these terms together and disregarding the non-positive contribution of Q_b^-, we get

$$
\begin{aligned}
&\frac{d}{d\tau} \int_\Omega \int_0^\infty \varphi(f)\, dy\, dx \\
&\qquad \le \left(\frac{2}{K} - 1\right) \int_\Omega \int_0^\infty \int_0^\infty \varphi'(f^\varepsilon(x,y)) f^\varepsilon(x,y) f^\varepsilon(x,y')\, dy'\, dy\, dx.
\end{aligned}
$$

Then, the result follows by taking $K = \mathcal{K}(f_{in}) = \max\{2, \|f_{in}\|_{L^\infty_{x,y}}\}$. □

Finally in this section, we show an interpolation inequality.

Lemma 3.3. *Let $f \ge 0$, $f \in L^\infty(\Omega \times (0,\infty))$ such that $(1 + y^r)f \in L^1(\Omega \times (0,\infty))$. Let $p > 1$ and any $k > 0$ such that $r > pk$ and $pk + 1 > 2p$. Then, for a constant C*

$$
\left\| \int_0^\infty y f(x,y)\, dy \right\|_{L^p_x} \le C \, \|f\|_{L^\infty_{x,y}}^{1/p'} \, \|(1 + y^r) f\|_{L^1_{x,y}}^{1/p}.
$$

As a consequence, the monomer density N^ε of the solution f^ε of (15) with suitable initial data satisfies an explicit bound $\mathcal{N}(f_{in}, \mathcal{M}^*, \mathcal{M}_r^*)$ such that

$$\|N^\varepsilon(\tau)\|_{L^p_x} \le \mathcal{N}\,, \qquad \text{for a.e. } \tau \ge 0. \tag{24}$$

Proof. For $p>1$ and $\|N^\varepsilon(\tau)\|^p_{L^p(\Omega)} = \int_\Omega \left(\int_0^\infty y f^\varepsilon\,dy\right)^p dx$, we use first the L^∞ bound of Lemma 3.2 and further Hölder's inequality, observing that $p'(-k+1) < -1$, to estimate for various constants C

$$\begin{aligned}
\|N^\varepsilon(\tau)\|^p_{L^p_x} &\le C\|f^\varepsilon\|^{p-1}_{L^\infty_{x,y}} \int_\Omega \left(\int_0^\infty \left((1+y)^{pk} f^\varepsilon\right)^{1/p} (1+y)^{-k+1}\,dy\right)^p dx\\
&\le C\|f^\varepsilon\|^{p-1}_{L^\infty_{x,y}} \int_\Omega\int_0^\infty (1+y)^{pk} f^\varepsilon\,dy\,dx\\
&\le C\|f^\varepsilon\|^{p-1}_{L^\infty_{x,y}} \int_\Omega\int_0^\infty (1+y^r) f^\varepsilon\,dy\,dx\,,
\end{aligned}$$

which is bounded by Lemma 3.1, and thus, so is (24). □

4. **Interpolation: Trend to Nonlinear Diffusion in L^2.**

Returning to the original time variable, we gain from the estimate (21) in Lemma 2.1 and the bounds of the Lemmata 3.1, 3.2, and 3.3 the following result:

Lemma 4.1. *Under the assumptions of Theorem 1.1 exists for any $T>0$ a constant C_T independent of ε such that for $\theta = 1/20$*

$$\|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\|_{L^2_{t,x}(L^1_y((1+y)\,dy))} \le \varepsilon^\theta C_T\,, \tag{25}$$

on bounded time intervals $t\in[0,T]$.

Proof. We estimate using the L^∞ bound of Lemma 3.2 and Cauchy-Schwarz that for various constants C

$$\begin{aligned}
&\|f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}}\|^2_{L^2_{t,x}(L^1_y(1+y))}\\
&\quad\le C\left(\|f\|_{L^\infty_{x,y}}+1\right)\\
&\qquad\times \int_0^T\int_\Omega\left(\int_0^\infty (1+y)^2 |f^\varepsilon - e^{-y/\sqrt{N^\varepsilon}}|^{1/2}(1+y)^{-1}\,dy\right)^2 dx\,dt\\
&\quad\le C\int_0^T\int_\Omega\int_0^\infty (1+y)^4 |f^\varepsilon - e^{-y/\sqrt{N^\varepsilon}}|\,dy\,dx\,dt\,.
\end{aligned}$$

Next, for a $A>1$ to be chosen, we split $\int_0^\infty dy = \int_0^A dy + \int_A^\infty dy := I_1 + I_2$. For the first part, we have by Lemma 2.1 in the original time variable that

$$I_1 \le C(1+A)^4 \int_0^T\int_\Omega\int_0^\infty |f^\varepsilon - e^{-y/\sqrt{N^\varepsilon}}|\,dy\,dx\,dt \le CA^4\,T^{1/2}\sqrt{\varepsilon}\,.$$

For the second part, we estimate

$$\begin{aligned} I_2 &\le C\frac{1}{A}\int_0^T\int_\Omega\int_A^\infty (1+y)^5(f^\varepsilon + e^{-y/\sqrt{N^\varepsilon}})\,dy\,dx\,dt \\ &\le \frac{C}{A}\left(T(\mathcal{M}_5+\mathcal{M}_0) + \int_0^T\int_\Omega (\sqrt{N^\varepsilon} + (N^\varepsilon)^3)\,dx\,dt\right) \\ &\le \frac{C}{A}\left(C_T + T\left(\sqrt{N_\infty} + \|N^\varepsilon\|^3_{L^\infty_t(L^3_x)}\right)\right) \le C_T\frac{1}{A}\,, \end{aligned}$$

where we have used Lemma 3.3 for the last term with $p = 3$, $r = 6$ and $5/3 < k < 2$. Thus finally, the statement follows by choosing $A = \varepsilon^{-1/10}$. □

In the following, we will expand f^ε according to (25) as

$$f^\varepsilon = e^{-\frac{y}{\sqrt{N^\varepsilon}}} + \varepsilon^\theta f_1^\varepsilon\,,$$

where f_1^ε is bounded in $L^2_{t,x}(L^1_y((1+y)\,dy))$ and satisfies $\nabla_x f_1^\varepsilon\cdot\nu(x) = 0$ on $\partial\Omega$. This yields the moment equation

$$\partial_t N^\varepsilon - \triangle_x n(N^\varepsilon) = \varepsilon^\theta \triangle_x \int_0^\infty a(y) y f_1^\varepsilon\,dy \;:= \varepsilon^\theta\,\triangle_x g^\varepsilon,$$

where g^ε is uniformly bounded in $L^2_{t,x}$ and satisfies $\nabla_x g^\varepsilon\cdot\nu(x) = 0$.

Lemma 4.2. *Assume that g^ε is uniformly bounded in $L^2_{t,x}$ and satisfies $\nabla_x g^\varepsilon\cdot\nu(x) = 0$ on $\partial\Omega$. Then, the sequence of solutions N^ε for the nonlinear diffusion equation*

$$\begin{cases} \partial_t N^\varepsilon - \triangle_x n(N^\varepsilon) = \varepsilon^\theta\triangle_x g^\varepsilon\,, \\ \nabla_x N^\varepsilon\cdot\nu(x)|_{\partial\Omega} = 0, \end{cases} \tag{26}$$

with initial data $N_{in}\in L^2_x$ converges as $\varepsilon\to 0$ in $L^2_{t,x}$ to the unique solution N of the nonlinear diffusion equation

$$\begin{cases} \partial_t N - \triangle_x n(N) = 0\,, \\ \nabla_x N\cdot\nu(x)|_{\partial\Omega} = 0, \end{cases} \tag{27}$$

with initial data N_{in}.

Proof. The proof uses a duality argument as in [13]. We first remark that the initial data N_{in} belongs to L^2 by Lemma 3.3. Let us also observe the uniqueness of the Cauchy problem for the limiting nonlinear non-degenerate diffusion equation (27) that follows from standard arguments, see for instance [7], [4]. For any $T > 0$, we consider nonnegative solutions $w \ge 0$ with end data $w(T) = 0$ of the equation

$$-\,\partial_t w - \frac{n(N^\varepsilon) - n(N)}{N^\varepsilon - N}\triangle_x w = H \ge 0\,, \tag{28}$$

with Neumann boundary condition $\nabla_x w \cdot \nu(x)|_{\partial\Omega} = 0$, for nonnegative test functions $H \in C_0^\infty([0,T]\times\Omega)$. These solutions satisfy the estimates

$$\|\triangle_x w\|_{L^2([0,T]\times\Omega)} \le C\|H\|_{L^2([0,T]\times\Omega)} \tag{29}$$

for a constant C. The existence of such solutions follows via smooth approximations of the bounded coefficient $a_* \le \frac{n(N^\varepsilon)-n(N)}{N^\varepsilon - N} \le a^*$, which justify also the following formal calculations: multiplication of (28) with $-\triangle_x w$ and integration by parts yields

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_x w|^2\,dx + a_*\int_\Omega (\triangle_x w)^2\,dx \le -\int_\Omega H(\triangle_x w)\,dx$$
$$\le \frac{a_*}{2}\int_\Omega (\triangle_x w)^2\,dx + \frac{1}{2a_*}\int_\Omega H^2\,dx\,,$$

by Young's inequality. Then, after integration in time over the interval $[0,T]$ and recalling that $w(T)=0$, it follows that

$$\frac{a_*}{2}\int_0^T\int_\Omega (\triangle_x w)^2\,dxdt \le \frac{1}{2a_*}\int_0^T\int_\Omega H^2\,dxdt\,,$$

which gives (29).

To prove the statement of the Lemma, we multiply the difference of equation (26) with (27) by the dual solution w and integrate by parts in time and space:

$$\left|\int_0^T\int_\Omega (N^\varepsilon - N)\,H\,dxdt\right| = \varepsilon^\theta\left|\int_0^T\int_\Omega g^\varepsilon \triangle_x w\,dxdt\right| \le \varepsilon^\theta\|g^\varepsilon\|_{L^2_{t,x}}\,\|H\|_{L^2_{t,x}}.$$

Since H is arbitrary, we deduce that for a constant C

$$\|N^\varepsilon - N\|_{L^2_{t,x}} \le C\,\varepsilon^\theta\,\|g^\varepsilon\|_{L^2_{t,x}} \le C\,\varepsilon^\theta.$$

This ends the proof of the Lemma and Theorem 1.1. □

Remark 4.3. In [13], explicit examples show that equations with discontinuous diffusion (as equation (26) is one) can be ill-posed with a right-hand side in L^q for q close to 1, while well-posed for a right-hand side in L^2. Therefore, the interpolation Lemma 4.1, which allows to obtain a right-hand side in H^{-2}, seems crucial.

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