ON THE FINITE TIME BLOW-UP FOR FILTRATION PROBLEMS WITH NONLINEAR REACTION

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ABSTRACT. We present results for finite time blow-up for filtration problems with nonlinear reaction under appropriate assumptions on the nonlinearities and the initial data. In particular, we prove first finite time blow up of solutions subject to sufficiently large initial data provided that the reaction term "overpowers" the nonlinear diffusion in a certain sense. Secondly, under related assumptions on the nonlinearities, we show that initial data above positive stationary state solutions will always lead to finite time blow up.

1. INTRODUCTION

In this paper, we shall study the blow-up of solutions $u = u(x, t)$ to the following filtration problem

$$u_t = \Delta K(u) + \lambda f(u), \qquad\qquad x \in \Omega, \quad t > 0, \qquad (1a)$$

$$\mathcal{B}(K(u)) \equiv \hat{n} \cdot \nabla K(u) + \beta(x)K(u) = 0, \qquad x \in \partial\Omega, \quad t > 0, \qquad (1b)$$

$$u(x, 0) = u_0(x) \geq 0, \qquad\qquad x \in \Omega, \qquad (1c)$$

where Ω is a bounded domain of $\mathbb{R}^N$ with sufficient smooth boundary $\partial\Omega$ and $\hat{n}$ denotes the outer unit normal vector.

In problem (1), the non-linear functions f and K, are supposed to be in $C^3(\mathbb{R})$ and to satisfy the following positivity, growth, monotonicity and convexity assumptions:

$$K(0) \geq 0, \ K(s) > 0, \quad \text{for } s \in \mathbb{R}^+, \qquad \text{and} \qquad K'(s), \ K''(s) > 0, \quad \text{for } s \in \mathbb{R}_0^+, \tag{2}$$

$$f(s) > 0, \ f'(s) > 0, \ f''(s) > 0, \quad \text{for } s \in \mathbb{R}_0^+, \tag{3}$$

$$\int_0^\infty \frac{K'(s)}{f(s)} ds < \infty, \quad \text{which implies} \quad \int_0^\infty \frac{ds}{f(s)} < \infty. \tag{4}$$

Moreover, we assume the constant $\lambda > 0$ to be positive and the coefficient $0 \leq \beta(x) \leq \infty$ to be in $C^{1+\alpha}(\partial\Omega)$ for $\alpha > 0$ whenever it is bounded. Note that $\beta \equiv 0$, $\beta \equiv \infty$ and $0 < \beta < \infty$ specify homogeneous Neumann, Dirichlet and Robin boundary condition, respectively.

We remark that by imposing non-negative initial data $u_0(x) \geq 0$, the assumed positivity $f(u) > 0$ implies the positivity of solutions to (1), i.e. $u > 0$ in Ω for $t > 0$. For the existence of a unique classical local solution $u \in C^{2,1}(\Omega_T)$ to problem (1)–(3) we refer to [8, 9, 11, 12, 13, 14] and the references therein.

Under the Osgood type condition (4), which is necessary for the blow-up of solutions, the filtration problem (1) will exhibit a blow-up behaviour if the forcing term is sufficiently "strong" and the initial data are sufficiently "large". This can be illustrated by the following example, which applies Kaplan's method, i.e. it investigates the evolution of the Fourier coefficient

$$B(t) := \int_\Omega u(x,t)\, \phi(x)\, dx, \qquad \dot{B}(t) = \int_\Omega \Delta K(u)\, \phi\, dx + \lambda \int_\Omega f\phi\, dx, \qquad (5)$$

where the assumptions on β ensure that $\phi(x)$ can be chosen as the positive (in Ω) and L^1-normalised (i.e. $\|\phi\|_1 = 1$) eigenfunction corresponding to the first eigenvalue μ of the following auxiliary elliptic problem with Robin boundary conditions:

$$\begin{cases} \Delta\phi + \mu\phi = 0, & x \in \Omega, \\ \hat{n} \cdot \nabla\phi + \beta(x)\phi = 0, & x \text{ on } \partial\Omega. \end{cases} \qquad (6)$$

After integration by parts in (5), the following prototypical comparison condition between f and K

$$\forall t > 0, \quad \int_\Omega \left(f(u(x,t)) - K(u(x,t))\right) \phi(x)\, dx \geq 0 \qquad (7)$$

yields for $\lambda > \mu$ and by applying Jensen's inequality,

$$\dot{B}(t) \geq (\lambda - \mu) \int_\Omega f(u(x,t))\, \phi(x)\, dx \geq (\lambda - \mu) f(B).$$

Thus, the Osgood type condition (4) implies the finite-time blow-up of $B(t)$(cf. [8, Section 4.1]). Moreover, as a by-product of Kaplan's method, the first eigenvalue μ provides a lower bound $\mu < \lambda$, above which blow-up occurs.

These phenomena are connected with the existence of solutions to the steady-state problem corresponding to (1),

$$\begin{cases} \Delta(K(w)) + \lambda f(w) = 0, & x \in \Omega, \\ \mathcal{B}(K(w)) = 0, & x \in \partial\Omega, \end{cases} \qquad (8)$$

It has been shown in [8] that problem (8) exhibits a critical (i.e maximal) value of the parameter λ, say λ^*, such that (any kind of) solution to (8) does not exist for $\lambda > \lambda^*$, while there exist bounded solutions to (8) for all $0 < \lambda \leq \lambda^*$. In fact, there exist at least two solutions of (8) for $0 < \lambda < \lambda^*$ (due to the symmetry properties of (8)). The case of steady-state solutions to the critical parameter λ^* is more intricate, see [8, 9]. Since it always holds that $\mu \geq \lambda^*$ (see [3, 5] or [8] for a proof under the additional assumption (7)), Kaplan's method is in general not sharp enough to treat the full supercritical range $\lambda > \lambda^*$. Nevertheless, in [8, Section 4.2] it has been shown under a mild extra condition on f and K that for all $\lambda > \lambda^*$ blow-up of solutions of (1) subject to any initial data $u_0 \geq 0$ occurs (see also [1, 2, 6]).

The aim of this work is to complement the analysis of blow-up of solutions for the above filtration problem (1). In a first result (see Theorem 1 below), we shall assume generalised comparison conditions and characterise sufficiently large initial data such that the solution to (1) blows up in finite time even in the subcritical region $\lambda < \lambda^*$. Moreover, in Theorem 3, we shall prove that solutions of (1) subject to initial data above the positive solution of the steady state problem (which exists since $\lambda < \lambda^*$) blow up in finite time.

2. Blow-up analysis

In the subcritical case $\lambda < \lambda^*$, the large-time behaviour of solutions to the semi-linear filtration problem (1)–(3) depends strongly on the initial data u_0. In particular, as one would expect, blow-up occurs for large enough initial data. A first result in this direction can again be proven by Kaplan's method. More precisely, we can state the following:

Theorem 1 (Subcritical Blow-Up for Large Initial Data).
Let the assumptions (2), (3) *and* (4) *hold. Let μ be the first eigenvalue of the problem* (6) *and ϕ the corresponding positive normalised eigenfunction. Let $u(x,t)$ denote the solution of* (1). *Assume either that there exists a* strictly concave *function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\liminf_{s \to 0} \frac{\psi(s)}{s} > \frac{\lambda}{\mu}$ such that*

$$\int_\Omega \left[\psi(f(u(x,t))) - K(u(x,t)) \right] \phi(x) \, dx \geq 0, \qquad \forall t > 0, \tag{9}$$

or that there exists a strictly convex *function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\varphi(0) = 0$ and $\limsup_{r \to 0} \frac{\varphi(r)}{r} < \frac{\mu}{\lambda}$ such that*

$$\int_\Omega \left[f(u(x,t)) - \varphi(K(u(x,t))) \right] \phi(x) \, dx \geq 0, \quad \forall t > 0, \quad and \quad \int_1^\infty \frac{ds}{K(s)} < \infty. \tag{10}$$

Then, the solution $u(x,t)$ of (1)–(3) *blows up in finite time, provided sufficiently large initial data $u_0 \geq 0$.*

Remark 2. *The proof of Theorem 1 entails a sufficient condition on initial data u_0 to yield finite time blow-up of solutions to* (1) *under the above assumptions. More precisely, blow-up occurs for initial data satisfying*

$$B(0) = \int_\Omega u_0 \, \phi \, dx > f^{-1}(s_0), \qquad where \quad s_0 = \max_{s \in \mathbb{R}_+} \{ \lambda s \leq \mu \psi(s) \} > 0,$$

or

$$B(0) = \int_\Omega u_0 \, \phi \, dx > K^{-1}(r_0), \qquad where \quad r_0 = \max_{r \in \mathbb{R}_+} \{ \lambda \varphi(r) \leq \mu r \} > 0.$$

Proof of Theorem 1. At first, by multiplication of (1a) with the eigenfunction ϕ and by integration over Ω, we obtain the following evolution law for the Fourier coefficient $B(t)$,

$$\dot{B}(t) = \frac{d}{dt} \int_\Omega u \, \phi \, dx = \int_\Omega \Delta K(u) \, \phi \, dx + \lambda \int_\Omega f(u) \, \phi \, dx.$$

Assuming by contradiction that u is a global-in-time solution, we apply Green's identity and remark that the boundary terms cancel since the eigenproblem (6) is chosen to satisfy the same boundary condition as the filtration problem (1b). Thus, we obtain

$$\dot{B}(t) = -\mu \int_\Omega K(u) \, \phi \, dx + \lambda \int_\Omega f(u) \, \phi \, dx. \tag{11}$$

We shall start by considering assumption (9). By inserting assumption (9) into (11) and applying Jensens's inequality with respect to the normalised positive measure $\phi(x)\,dx$, we estimate

$$\dot{B}(t) \geq \lambda \int_\Omega f(u)\,\phi\,dx - \mu \int_\Omega \psi(f(u))\,\phi\,dx \geq \lambda \int_\Omega f(u)\,\phi\,dx - \mu\,\psi\left(\int_\Omega f(u)\,\phi\,dx\right).$$
(12)

Next, we define $s(t) := \int_\Omega f(u)\,\phi\,dx$. Due to the convexity of f, we observe that, again by Jensen's inequality,

$$s(t) = \int_\Omega f(u)\,\phi\,dx \geq f\left(\int_\Omega u\,\phi\,dx\right) = f(B(t)).$$
(13)

Then, we denote for any $0 < \varepsilon < \lambda$ by s_ε the largest root of the equality $\lambda s - \mu\psi(s) = \varepsilon s$, i.e.

$$s_\varepsilon := \max_{s\in\mathbb{R}_+}\{(\lambda - \varepsilon)s \leq \mu\,\psi(s)\}.$$
(14)

Note that the concavity of ψ with $\mu \liminf_{s\to 0}\frac{\psi(s)}{s} > \lambda$ implies that $0 < s_\varepsilon < +\infty$ and we have $\lambda s - \mu\,\psi(s) > \varepsilon s$ for all $s > s_\varepsilon$. Therefore, we can continue to estimate (12) by using (13) and (14)

$$\dot{B}(t) \geq \varepsilon s(t) \geq \varepsilon f(B(t)) > 0, \qquad \text{for all} \quad s(t) \geq s_\varepsilon.$$
(15)

The differential inequality (15) implies the finite time blow-up of the Fourier coefficient $B(t)$ provided that the initial value $B(0)$ is chosen large enough such that $s(0) \geq f(B(0)) \geq s_\varepsilon$. Moreover, the blow-up occurs at latest at time $T_\varepsilon^* \leq \frac{1}{\varepsilon}\int_{B(0)}^\infty \frac{ds}{f(s)} < +\infty$, provided that $B(0) \geq f^{-1}(s_\varepsilon)$. Thus, minimisation in ε yields blow-up the latest at time

$$T^* \leq \inf_{0<\varepsilon<\lambda}\left\{\frac{1}{\varepsilon}\int_{f^{-1}(s_\varepsilon)}^\infty \frac{ds}{f(s)}\right\} < +\infty.$$
(16)

We shall now consider assumption (10) and proceed in an analog way as above. Using (10) in (11) and applying Jensen's inequality for the convex function φ yields

$$\dot{B}(t) \geq \lambda \int_\Omega \varphi(K(u))\,\phi\,dx - \mu \int_\Omega K(u)\,\phi\,dx \geq \lambda\,\varphi\left(\int_\Omega K(u)\,\phi\,dx\right) - \mu \int_\Omega K(u)\,\phi\,dx.$$
(17)

Similar to above, we define $r(t) := \int_\Omega K(u)\,\phi\,dx$ and observe as in (13) that the convexity of K and Jensen's inequality with respect to the normalised positive measure ϕ imply $r(t) \geq K(B(t))$.

Then, we denote for any $0 < \varepsilon$ by r_ε the largest root of the equality $\lambda\varphi(r) - \mu r = \varepsilon r$. Note that the convexity of $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\varphi(0) = 0$ and $\lambda \limsup_{r\to 0}\frac{\varphi(r)}{r} < \mu$ implies that $0 < r_\varepsilon < +\infty$ and we have that $\lambda\varphi(r) - \mu r > \varepsilon r$ for all $r > r_\varepsilon$. Therefore, we can continue to estimate (17) as

$$\dot{B}(t) \geq \lambda\,\varphi(r(t)) - \mu\,r(t) > \varepsilon\,r(t) \geq \varepsilon\,K(B(t)) > 0, \qquad \text{for all} \quad r(t) \geq r_\varepsilon.$$
(18)

Thus, analog to above, we obtain finite time blow-up latest at time

$$T^* \leq \inf_{0<\varepsilon}\left\{\frac{1}{\varepsilon}\int_{K^{-1}(r_\varepsilon)}^\infty \frac{ds}{K(s)}\right\} < +\infty.$$
(19)

This finishes the proof.   □

3. Blow-up for initial data above positive steady state

In this section, we will follow ideas of [6] and [13] to prove that solutions of the filtration problem (1) subject to initial data u_0 larger than a positive solution of the associated stationary problem will blow-up in finite time provided suitable conditions on the involved nonlinearities, in particular that the function $g(s) := f(K^{-1}(s))$ is convex. Thus, under this assumption, we are able to sharpen the result of Theorem 1. We remark also that solutions being initially below a positive steady state remain naturally bounded by the steady state solution due to a comparison principle (see [9]) and thus exist globally in time.

In the following, we say that $w = w(x) > 0$ is a classical solution of (8), if $z = z(x) = K(w(x))$ is a classical solution of

$$\begin{cases} \Delta z + \lambda g(z) = 0, & x \in \Omega, \\ \mathcal{B}(z) = 0, & x \in \partial\Omega, \end{cases} \tag{20}$$

where $g(z) = f(K^{-1}(z)) = f(w)$ with $z = K(w)$. The assumptions (2), in particular the monotonicity property of K implies that both the above steady-state problems are equivalent with respect to the existence and the number of solutions. For a positive steady-state solution $w > 0$ of (8) (see [9, Section 3.2]), the eigenvalue problem associated to the linearisation of the filtration problem (1) around w (i.e. we expand (1) according to $u(t,x) = w(x) + \Phi(t,x)$ and decompose $\Phi(t,x) = e^{-\mu t}\phi(x)$), reads as

$$\begin{cases} -\Delta[K'(w)\phi] = \lambda f'(w)\phi + \mu\phi, & \text{in } \Omega, \\ \mathcal{B}(K'(w)\phi) = 0, & \text{on } \partial\Omega. \end{cases} \tag{21}$$

We remark that the linearisation eigenvalue problem (21) has a solution for each $\lambda \in (0, \lambda^*]$ (cfr. [9, Theorem 8]). Moreover, it has been shown that the first eigenvalue $\mu(\lambda)$ of (21) is negative if the function $g(s) = f(K^{-1}(s))$ is convex, for instance.

Theorem 3 (Blow-up Above Positive Steady States)**.** *Let Ω be a bounded domain of class $C^{2,\alpha}$ and $w > 0$ be a classical positive solution of the stationary-state problem (8). Let $u \in C(\overline{\Omega} \times [0,T))$ be a positive solution of (1a) and $T > 0$ be the maximal time of existence. Assume that the function $g(s) := f(K^{-1}(s))$ is convex. Assume initial data $u_0 \geq w$ in Ω with $u_0 \neq w$. Then, the solution blows-up in finite time.*

Remark 4. *For Neumann and Robin boundary conditions, the proof of Theorem 3 can be generalised to allow for $K'(0) = 0$ (since then we have $m := \inf_{x\in\Omega} w(x) > 0$), which allows to include the porous medium problem.*

Proof of Theorem 3. Let ϕ be the first eigenfunction of problem (21) corresponding to a negative eigenvalue $\mu(\lambda)$. We consider the function $\theta(x,t) := u(x,t) - w(x) \geq 0$, which being initially non-negative remains non-negative thanks to the comparison principle proven in [9]. Thus, the monotonicity assumption on K also ensures $K(u) - K(w) \geq 0$. Moreover, by using the equations (1a) and (8), we have $\theta_t = \Delta(K(u) - K(w)) + \lambda(f(u) - f(w))$.

Next, we test θ_t with $K'(w)\,\phi$ and integrate by parts and use that w is a solution of (21) to calculate

$$
\frac{d}{dt}\int_\Omega \theta K'(w)\,\phi\,dx
$$

$$
=\int_\Omega (K(u)-K(w))\,\Delta[K'(w)\,\phi]\,dx + \lambda \int_\Omega (f(u)-f(w))\,K'(w)\,\phi\,dx
$$

$$
=\int_\Omega (K(u)-K(w))(-\lambda f'(w)\,\phi - \mu\phi)\,dx + \lambda\int_\Omega (f(u)-f(w))\,K'(w)\,\phi\,dx
$$

$$
=\lambda\int_\Omega \big[K'(w)(f(u)-f(w)) - f'(w)(K(u)-K(w))\big]\,\phi\,dx
$$

$$
-\mu\int_\Omega (K(u)-K(w))\,\phi\,dx. \tag{22}
$$

Now, we introduce the function $h(s) := f(s) - f(0) - sf'(0)$. It is easily verified that h satisfies

$$
h(s)>0,\ \ s\in\mathbb{R}^+,\ \ h'(s)>0,\ \ h(0)=h'(0)=0,\ \ h''(s)>0,\ \ s\in\mathbb{R}, \tag{23}
$$

and its minimum is $(0, h(0)) = (0,0)$.

In the following, we apply a dichotomy argument (see also [6, pag. 1355] and [9, proof of Theorem 9] for its generalisation to $K(u)$ satisfying assumptions (2) in the case of supercritical $\lambda > \lambda^*$) in order to prove that there exists a constant $\Lambda > 0$ such that, for any $s \geq 0$ and independently on $x \in \Omega$, the following estimate holds:

$$
F_w(s) := K'(w)(f(w+s)-f(w)) - f'(w)(K(w+s)-K(w)) \geq \Lambda\,h(s). \tag{24}
$$

Clearly it is enough to study the case $s > 0$. To this aim, we start by defining the constants

$$
m := \inf_{x\in\Omega} w(x) \geq 0, \quad M := \sup_{x\in\Omega} w(x) < +\infty, \quad c_1 := \frac{f'(M)}{K'(m)}, \quad c_2 := \frac{f(M)}{K'(M)}.
$$

Using the growth condition (4) we can ensure the existence of $S = S(f, K, \lambda)$, such that

$$
\frac{f(s)}{K'(s)} > 2(c_1\,s + c_2), \quad \forall\,s > S, \tag{25}
$$

it is sufficient indeed to chose $S = \sup\left\{t\ :\ \frac{f(t)}{K'(t)} - 2(c_1\,t + c_2) = 0\right\}$. The dichotomy consist in dealing with the case $s > S$ and $s \leq S$ separately. First, for $s > S$, we can estimate $f(w+s) > 2[c_1(w+s)+c_2]K'(w+s) \geq 2[s\,c_1\,K'(w+s)+f(M)]$. Using the previous inequality, since

$$
f'(w)(K(w+s)-K(w)) = K'(\xi)\,sf'(w) \leq K'(w+s)\,s\,f'(M),
$$

by choosing C such that $0 < \Lambda_1 < K'(m)/2$, we can deduce, again for $s > S$,

$$
\begin{aligned}
F_w(s) \geq &K'(w)(f(w+s)-f(w)) - c_1\,s\,K'(w+s)) \\
\geq &K'(w)(f(w+s)-f(M)) - c_1\,s\,K'(w+s)) \\
\geq &\frac{1}{2}K'(w)\,f(w+s) \geq \frac{1}{2}K'(w)\,f(s) \geq \Lambda_1(f(s)-f(0)-sf'(0)) \\
= &\Lambda_1 h(s) > 0.
\end{aligned} \tag{26}
$$

Now we study the other case, $s \leq S$. We start by using the Taylor expansion up to the second order to rewrite $F_w(s)$ as

$$F_w(s) = \left(f''(w + \eta)K'(w) - K''(w + \eta)f'(w) \right)\frac{s^2}{2},$$

for some $0 < \eta < s$. The hypothesis on $g(s)$ gives

$$\frac{f''(\theta)}{K''(\theta)} > \frac{f'(\theta)}{K'(\theta)} > \frac{f'(w)}{K'(w)}, \quad \theta = w + \eta > 0, \ \text{with} \ 0 \leq \eta \leq S. \qquad (27)$$

Hence, from (27) we have:

$$\begin{aligned}
F_w(s) &= \left[f''(w + \eta)K'(w) - K''(w + \eta)f'(w) \right]\frac{s^2}{2} \\
&\geq \inf_{\eta} \left[f''(w + \eta)K'(w) - K''(w + \eta)f'(w) \right]\frac{s^2}{2} \\
&= \left[f''(w + \tilde{\eta})K'(w) - K''(w + \tilde{\eta})f'(w) \right]\frac{s^2}{2} \\
&\geq \min_{x \in \overline{\Omega}} \left[f''(w + \tilde{\eta})K'(w) - K''(w + \eta_1)f'(w) \right]\frac{s^2}{2} \\
&= \left[f''(w(\tilde{x}) + \tilde{\eta})K'(w(\tilde{x})) - K''(w(\tilde{x}) + \tilde{\eta})f'(w(\tilde{x})) \right]\frac{s^2}{2},
\end{aligned}$$

where we have denoted by $\tilde{\eta} \in [0, S]$ and $\tilde{x} \in \overline{\Omega}$ the points that realize the infimum and the maximum in the previous inequalities. Moreover, thanks to (27) we can easily verify that

$$\Lambda_2 := \frac{1}{2}\left[f''(w(\tilde{x}) + \tilde{\eta})K'(w(\tilde{x})) - K''(w(\tilde{x}) + \tilde{\eta})f'(w(\tilde{x})) \right] > 0.$$

So, since $f''(0) > 0$, there exists a $\Lambda_3 > 0$, small enough such that:

$$F_w(s) \geq \Lambda_2 s^2 \geq \Lambda_3[f(s) - f(0) - sf'(0)] = \Lambda_3 h(s) > 0, \quad \text{if} \ s \leq S. \qquad (28)$$

Chosing $\Lambda = \min\{\Lambda_1, \Lambda_3\}$ and combining (28) and (26) we get (24).

Finally, in order to prove the statement of the theorem, we denote $A(t) := \int_\Omega \theta K'(w)\phi\,dx$, and continue to estimate (22),

$$\begin{aligned}
A'(t) &\geq \lambda \int_\Omega \left[K'(w)(f(u) - f(w)) - f'(w)(K(u) - K(w)) \right]\phi\,dx \\
&\quad - \mu \int_\Omega (K(u) - K(w))\,\phi\,dx \\
&\geq \frac{\lambda\Lambda}{\|K'(w)\|_\infty} \int_\Omega h(\theta)K'(w)\phi\,dx - \mu \int_\Omega (K(u) - K(w))\,\phi\,dx \\
&\geq \frac{\lambda\Lambda}{\|K'(w)\|_\infty} h\left(\int_\Omega \theta K'(w)\phi\,dx \right) = \lambda\Lambda\, h(A(t)). \qquad (29)
\end{aligned}$$

Here, in the last inequality, we have used Jensen's inequality for the convex function h and the normalized measure $K'(w)\phi\,dx$, i.e. $\int_\Omega K'(w)\phi\,dx = 1$ as well as $\mu < 0$ and $\int_\Omega (K(u) - K(w))\phi\,dx \geq 0$ for all $t \geq 0$.

The inequality (29), together with (24), implies the blow-up of $A(t)$ and since $A(t) \leq \|\theta(\cdot, t)\|$, blow-up of θ at t_θ^* and hence of u at $t^* < \infty$ where $t^* \leq t_\theta^*$, since $u = w + \theta > \theta$ in Ω and w is bounded. $\qquad \square$

References

[1] H. Bellout, *A criterion for blow-up of solutions to semilinear heat equations,* SIAM J. Math. Anal. **18**, (1987), 722–727.

[2] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow-up for $u_t = \Delta u + \lambda g(u)$ revisited,* Adv. Diff. Eq, (1996), 73–90.

[3] M.G. Crandall and P.H. Rabinowitz, *Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems,* Arch. Rational Mech. Anal. **58**, (1975), no. 3, 207–218.

[4] V.A. Galaktionov and J.L. Vazquez, *Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations,* Arch. Rational Mech. Anal. **129**, (1995), 225–244.

[5] H.B. Keller and D.S. Cohen, *Some positone problems suggested by nonlinear heat generation,* J. Math. Mech. **16**, (1967), 1361–1376.

[6] A.A. Lacey, *Mathematical analysis of thermal runaway for spartially inhomogeneous reactions,* SIAM J. Appl. Math. **43**, (1983), 1350–1366.

[7] A.A. Lacey and D.E. Tzanetis, *Global existence and convergence to a singular steady state for a semilinear heat equation,* Proc. Royal Soc. Edinb. **105A**, (1987), 289-305.

[8] E.A. Latos and D.E. Tzanetis, *Existence and blow-up of solutions for a Semilinear Filtration problem,* Electron. J. Diff. Equ., Vol. 2013 (2013), No. **178**, pp. 1-20.

[9] E.A. Latos and D.E. Tzanetis, *Existence and blow-up of solutions for a non-local filtration and Porous Medium problem,* Proc. Edinb. Math. Soc. (2) **53**, (2010), no. 1, 195–209.

[10] H. Levine and P. Sacks, *Some existence and nonexistence theorems for solutiona of degenerate parabolic equations,* J. Diff. Eqns. **52**, (1984), 135–161.

[11] P. Quittner and P. Souplet, Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States, *Birkhäuser Advanced Texts,* (2007).

[12] D.H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems,* Indiana Univ. Math. J. **21**, (1972), 979–1000.

[13] Philippe Souplet, *An optimal Liouville-type theorem for radial entire solutions of the porous medium equation with source,* J. Differential Equations **246** (2009), 3980-4005.

[14] J.L. Vazquez, *The Porous Medium equation: Mathematical theory,* Oxford Mathematical Monographs, Madrid, (2007).

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