DUALITY- AND ENTROPY METHODS FOR REVERSIBLE REACTION-DIFFUSION EQUATIONS WITH DEGENERATE DIFFUSION

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Abstract. A combination of entropy- and duality-methods has been successfully used in the past to study existence and smoothness of reaction-diffusion equations arising in reversible chemistry in many situations, but in general under the requirement that the diffusion is non-degenerate (or at least that all diffusion coefficients are strictly positive a.e.).

Here, we wish to describe how these methods also enable to treat reaction-diffusion systems which are strongly degenerate in the sense that one or more of the species do not diffuse at all.

1. Introduction

Reaction-Diffusion Systems for Reversible Chemistry. The evolution of a mixture of diffusive species $\mathcal{A}_i, i = 1, 2, \dots, q$, undergoing a reversible reaction of the type

$$\alpha_1 \mathcal{A}_1 + \cdots + \alpha_q \mathcal{A}_q \rightleftharpoons \beta_1 \mathcal{A}_1 + \cdots + \beta_q \mathcal{A}_q, \qquad \alpha_i, \beta_i \in \mathbb{N}$$

is modelled using mass-action kinetics (see [1, 7, 2, 3] for a derivation from basic principles) in the following way:

$$\partial_t a_i - d_i\, \Delta_x a_i = (\beta_i - \alpha_i)\left(l \prod_{j=1}^{q} a_j^{\alpha_j} - k \prod_{j=1}^{q} a_j^{\beta_j} \right), \tag{1}$$

where $a_i := a_i(t,x) \geq 0$ denotes the concentration at time t and point x of the species A_i and $d_i \geq 0$ are non-negative diffusion coefficients, which we shall assume constant.

We suppose that $x \in \Omega$, where Ω is a bounded domain of $I\!R^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$ (either $\partial\Omega \in C^{2+\alpha}$ for any $\alpha > 0$ or $\partial\Omega \in C^\infty$ in order to get C^∞ solutions), and complement system (1) by homogeneous Neumann boundary conditions:

$$n(x)\cdot \nabla_x a_i(t,x) = 0\,, \qquad \forall t \geq 0, \quad x \in \partial\Omega, \tag{2}$$

where $n(x)$ is the outer normal unit vector at point x of $\partial\Omega$.

Moreover, after a suitable rescaling, it is possible to set $l = 1$ (or equally $k = 1$) and to consider a normalised domain Ω, i.e. $|\Omega| = 1$.

The particular case $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3 + \mathcal{A}_4$ (that is, when $q = 4$ with $\alpha_1 = \alpha_2 = 1$, $\beta_3 = \beta_4 = 1$, $\alpha_3 = \alpha_4 = 0$, and $\beta_1 = \beta_2 = 0$) has lately received a lot of attention as a prototypical model system featuring quadratic nonlinearities, see e.g. [11, 13, 5, 14, 15, 6, 16, 2, 3]. In order to further simplify the notation we

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choose $k = 1$ ($k \neq 1$ works analogous with minor modifications). We thus consider the particular case of system (1), which writes as

$$\left\{\begin{array}{l} \partial_t a_1 - d_1\,\Delta_x a_1 = a_3\,a_4 - a_1\,a_2, \\ \partial_t a_2 - d_2\,\Delta_x a_2 = a_3\,a_4 - a_1\,a_2, \\ \partial_t a_3 - d_3\,\Delta_x a_3 = a_1\,a_2 - a_3\,a_4, \\ \partial_t a_4 - d_4\,\Delta_x a_4 = a_1\,a_2 - a_3\,a_4, \end{array}\right. \tag{3}$$

together with the homogeneous Neumann boundary conditions (2).

It is proven that whenever $d_1, d_2, d_3, d_4 > 0$, there exists a global smooth solution for dimensions $N = 1, 2$, see [9, 13, 5]. Those solutions decay exponentially fast (with explicit rates of decay) when $t \to +\infty$ towards the unique, positive, constant equilibrium given in (8) below.

For the cases $N > 2$, one can prove [11] the existence of global weak ($L^2\,(\log L)^2$) solutions, which converge exponentially to the equilibrium in L^2 [10]. In general, the existence of smooth solutions in dimensions $N > 2$ constitutes an open problem. In the case of the diffusion coefficients being sufficiently close to each other (depending on the space dimension), global smooth solutions follow from an improved duality argument in [5].

The aim of the present paper is to study the system (3)–(2) when at least one of the constant diffusion coefficients d_i is zero.

The paper is organized as follows.

Plan of the paper. We start in Section 2 by presenting the *a priori* bounds for our system and by overviewing the analytical tools which are available.

In Section 3, we present results for the simpler system $\mathcal{A}_1 + \mathcal{A}_2 \rightleftharpoons \mathcal{A}_3$ discussing the cases that a subset of diffusion coefficients $\{d_1, d_2, d_3\}$ is zero.

Then, starting with Section 4, each of the following sections of this work is devoted to cases in which a certain subset of $\{d_1, d_2, d_3, d_4\}$ is constituted of zeroes. All possible cases are studied (up to symmetries), except the cases when all diffusion coefficients d_i are strictly positive (already treated in [11, 13, 5]) and when all the d_i are zero (the system is then only constituted of ODEs). Section 4 is concerned with the case $d_1 = 0$, while the cases $d_1 = d_2 = 0$ and $d_1 = d_3 = 0$ are treated in Section 5. Then, Section 6 is devoted to the situation when $d_1 = d_2 = d_3 = 0$.

2. A priori estimates and analytical tools

2.1. **A priori estimates and equilibrium.** Two basic *a priori* estimates, coming naturally out of the thermodynamics, can be written for this system.

(i) The conservation of the number of atoms implies (formally, that is for all smooth solutions $(a_i)_{i=1,..,4}$ of (3) with Neumann condition (2)) that for all $t \geq 0$,

$$\left\{\begin{array}{l} M_{13} := \int_\Omega (a_1(t,x) + a_3(t,x))\,dx = \int_\Omega (a_1(0,x) + a_3(0,x))\,dx, \\ M_{14} := \int_\Omega (a_1(t,x) + a_4(t,x))\,dx = \int_\Omega (a_1(0,x) + a_4(0,x))\,dx, \\ M_{23} := \int_\Omega (a_2(t,x) + a_3(t,x))\,dx = \int_\Omega (a_2(0,x) + a_3(0,x))\,dx, \\ M_{24} := \int_\Omega (a_2(t,x) + a_4(t,x))\,dx = \int_\Omega (a_2(0,x) + a_4(0,x))\,dx. \end{array}\right. \tag{4}$$

Note that only three of the above four conservation laws are linearly independent.

(ii) Secondly, introducing the nonnegative entropy (free energy) functional $E(a_i)$ and the entropy dissipation $D((a_i)_{i=1,..,4}) = -\frac{d}{dt}E((a_i)_{i=1,..,4})$ associated to (3) :

$$E(a_i(t,x)_{i=1,..,4}) = \sum_{i=1}^{4} \int_\Omega \Big(a_i(t,x)\log(a_i(t,x)) - a_i(t,x) + 1 \Big)\, dx\,, \tag{5}$$

$$\begin{aligned} D(a_i(t,x)_{i=1,..,4}) = &\sum_{i=1}^{4} \int_\Omega 4\, d_i\, |\nabla_x \sqrt{a_i(t,x)}|^2\, dx \\ &+ \int_\Omega (a_1\, a_2 - a_3\, a_4) \log\left(\frac{a_1\, a_2}{a_3\, a_4}\right)(t,x)\, dx\,, \end{aligned} \tag{6}$$

it is easy to verify that the following *a priori* estimate holds (still for smooth solutions $(a_i)_{i=1,..,4}$ of (3) with (2)) for all $t \ge 0$

$$E(a_i(t,x)_{i=1,..,4}) + \int_0^t D(a_i(s,x)_{i=1,..,4})\, ds = E(a_i(0,x)_{i=1,..,4})\,. \tag{7}$$

Finally, observe that when at most one diffusion coefficient d_i is equal to 0, there exists a unique constant equilibrium state $(a_{i,\infty})_{i=1,..,4}$ (for which the entropy dissipation vanishes). It is defined by the unique positive constants solving $a_{1,\infty}\, a_{2,\infty} = a_{3,\infty}\, a_{4,\infty}$ provided $a_{j,\infty} + a_{k,\infty} = M_{jk}$ for $(j,k) \in (\{1,2\},\{3,4\})$, that is:

$$\begin{cases} a_{1,\infty} = \frac{M_{13}M_{14}}{M}\,, \qquad a_{3,\infty} = M_{13} - \frac{M_{13}M_{14}}{M} = \frac{M_{13}M_{23}}{M}\,, \\ a_{2,\infty} = \frac{M_{23}M_{24}}{M}\,, \qquad a_{4,\infty} = M_{14} - \frac{M_{13}M_{14}}{M} = \frac{M_{14}M_{24}}{M}\,, \end{cases} \tag{8}$$

where M denotes the total initial mass $M = M_{13} + M_{24} = M_{14} + M_{23}$.

2.2. **Analytical tools.** In the sequel, the following tools will systematically be used:

– First, the entropy decay (7) estimate will be exploited as much as possible. Being nonincreasing, the entropy functional will ensure that

$$a_i \in L^\infty([0,+\infty[; L\log L(\Omega)) \qquad \forall i = 1,..,4. \tag{9}$$

Considering that in (7), the time integral of the entropy dissipation is bounded, its first component will provide the estimate

$$\sqrt{a_i} \in L^2([0,+\infty[; H^1(\Omega)), \qquad \forall i = 1,..,4 \quad \text{provided that } d_i > 0. \tag{10}$$

Finally, the second component of the time integral of the entropy dissipation will ensure that, provided $a_3\, a_4 \in L^1_{loc}([0,+\infty[\times\bar\Omega)$, then $a_1\, a_2 \in L^1_{loc}([0,+\infty[\times\bar\Omega)$ too. This comes out of the following classical inequality (cf. [12]), which holds for any $\kappa > 1$,

$$a_1\, a_2 \le \kappa\, a_3\, a_4 + \frac{1}{\log\kappa}\,(a_1\, a_2 - a_3\, a_4)\, \log\left(\frac{a_1\, a_2}{a_3\, a_4}\right). \tag{11}$$

Note that by letting κ be as large as necessary, this inequality also allows to prove that an approximating sequence $a_1^n\, a_2^n$ is (locally in time) weakly compact in L^1 if the sequence $a_3^n\, a_4^n$ is also weakly compact in L^1 (and when estimate (7) holds uniformly with respect to n).

– Secondly, we shall use the duality method as presented, for instance, by Pierre and Schmitt [17], see also [18] and the references therein. This method ensures that whenever some quantity $u := u(t,x) \geq 0$ satisfies (in the sense of distributions)

$$\begin{cases} \partial_t u - \Delta_x[M\,u] \leq 0, & \forall t \in [0,T], \quad x \in \Omega, \\ n \cdot \nabla_x u = 0, & \forall t \in [0,T], \quad x \in \partial\Omega, \\ u(0,\cdot) \in L^2(\Omega), \end{cases} \tag{12}$$

where $0 < a \leq M(t,x) \leq b < +\infty$, then u lies in $L^2([0,T]\times\Omega)$.

We shall also use a variant of this estimate adapted to degenerate diffusions: Whenever $\alpha_1 := \alpha_1(t,x),\ldots,\ \alpha_k := \alpha_k(t,x) \geq 0$ satisfy for some $d_i \geq 0$

$$\begin{cases} \partial_t \left(\sum_{i=1}^k \alpha_i\right) - \Delta_x \left(\sum_{i=1}^k d_i\,\alpha_i\right) \leq 0, & \forall t \in [0,T], \quad x \in \Omega, \\ n \cdot \nabla_x \alpha_i = 0, & \forall t \in [0,T], \quad x \in \partial\Omega, \\ \alpha_i(0,\cdot) \in L^2(\Omega), \end{cases} \tag{13}$$

then $(\sum_{i=1}^k \alpha_i)\,(\sum_{i=1}^k d_i\,\alpha_i) \in L^1([0,T]\times\Omega)$, see [11].

Still another variant of this estimate is useful: If $\alpha_1 := \alpha_1(t,x)$, $\alpha_2 := \alpha_2(t,x) \geq 0$ satisfy for some $d_1, d_2 > 0$

$$\begin{cases} \partial_t \left(\sum_{i=1}^2 \alpha_i\right) - \Delta_x \left(\sum_{i=1}^2 d_i\,\alpha_i\right) \leq 0, & \forall t \in [0,T], \quad x \in \Omega, \\ n \cdot \nabla_x \alpha_i = 0, & \forall t \in [0,T], \quad x \in \partial\Omega, \\ \alpha_i(0,\cdot) \in L^p(\Omega), \qquad p < +\infty, \end{cases} \tag{14}$$

and if $\alpha_1 \in L^p([0,T]\times\Omega)$, then $\alpha_2 \in L^p([0,T]\times\Omega)$ and vice versa, see [18].

– Thirdly, the above duality estimate (12) can be improved by the following proposition from providing L^2-bounds to estimates in higher L^p-norms at the cost of an additional restraint on the bounds of the coefficient function $M(t,x)$, see [5]:

Proposition 2.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth (e.g. $C^{2+\alpha}$, $\alpha > 0$) boundary $\partial\Omega$, $T > 0$, and $p \in]2,+\infty[$. We consider a coefficient function $M := M(t,x)$ satisfying*

$$0 < a \leq M(t,x) \leq b < +\infty \quad \text{for } (t,x) \in \Omega_T, \tag{15}$$

for some $0 < a < b < +\infty$, and an initial datum $u_0 \in L^p(\Omega)$.

Then, any weak solution u of the parabolic system:

$$\begin{cases} \partial_t u - \Delta_x(Mu) = 0 & \text{on} \quad \Omega_T, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \Omega, \\ n(x) \cdot \nabla_x u = 0 & \text{on} \quad [0,T]\times\partial\Omega, \end{cases} \tag{16}$$

satisfies the estimate (where $p' < 2$ denotes the Hölder conjugate exponent of p)

$$\|u\|_{L^p([0,T]\times\Omega)} \leq (1 + b\,D_{a,b,p'})\,T^{1/p}\,\|u_0\|_{L^p(\Omega)}, \tag{17}$$

and where for any $a, b > 0$, $q \in]1,2[$

$$D_{a,b,q} := \frac{C_{\frac{a+b}{2},q}}{1 - C_{\frac{a+b}{2},q}\frac{b-a}{2}}, \tag{18}$$

provided that the following condition holds

$$C_{\frac{a+b}{2},p'}\,\frac{b-a}{2} < 1. \tag{19}$$

Here, the constant $C_{m,q} > 0$ is defined for $m > 0$, $q \in]1,2[$ as the best (that is, smallest) constant in the parabolic regularity estimate

$$\|\Delta_x v\|_{L^q([0,T]\times\Omega)} \le C_{m,q}\,\|f\|_{L^q([0,T]\times\Omega)}, \tag{20}$$

where $v : [0,T]\times\Omega \to I\!R$ is the solution of the backward heat equation with homogeneous Neumann boundary conditions:

$$\begin{cases} \partial_t v + m\,\Delta_x v = f & \text{on} \quad \Omega_T, \\ v(T,x) = 0 & \text{for} \quad x \in \Omega, \\ n(x)\cdot\nabla_x v = 0 & \text{on} \quad [0,T]\times\partial\Omega. \end{cases} \tag{21}$$

We recall that one has $C_{m,q} < \infty$ for $m > 0$, $q \in]1,2[$ and in particular $C_{m,2} \le \frac{1}{m}$. Note that the constant $C_{m,q}$ may depend (besides on m and q) also on the domain Ω and the space dimension N, but does not depend on the time T.

– Finally, we shall use the properties of the heat kernel in $[0,T]\times\Omega$. We recall (e.g. from [5]) that if $f \in L^p([0,T]\times\Omega)$, then the solution $u := u(t,x)$ of

$$\begin{cases} \partial_t u - \Delta_x u = f, & \forall t \in [0,T], \quad x \in \Omega, \\ n\cdot\nabla_x u = 0, & \forall t \in [0,T], \quad x \in \partial\Omega, \\ u(0,\cdot) \in L^\infty(\Omega), \end{cases} \tag{22}$$

lies in $L^q([0,T]\times\Omega)$ for all $q \in [1,+\infty]$ being determined by convolution with the heat kernel in $L^{\frac{2+N}{N}-\varepsilon}$ for all $\varepsilon > 0$, i.e.

$$\frac{1}{p} + \frac{N}{N+2} - 1 < \frac{1}{q} \quad \Longrightarrow \quad \frac{1}{q} > \frac{1}{p} - \frac{2}{N+2}. \tag{23}$$

We shall also use the monotonicity of the heat kernel: $f \ge 0 \Rightarrow u \ge 0$.

3. Existence and smoothness: the case of a three-species reaction

In this section we present brief proofs of theorems of existence and smoothness in the case of the simpler chemical reaction $\mathcal{A}+\mathcal{B} \rightleftharpoons \mathcal{C}$. The corresponding system writes

$$\begin{cases} \partial_t a - d_1\,\Delta_x a = c - a\,b, \\ \partial_t b - d_2\,\Delta_x b = c - a\,b, \\ \partial_t c - d_3\,\Delta_x c = a\,b - c, \end{cases} \tag{24}$$

together with the boundary condition

$$n\cdot\nabla_x a = 0, \quad n\cdot\nabla_x b = 0, \quad n\cdot\nabla_x c = 0\,, \qquad \forall t \ge 0, \quad x \in \partial\Omega. \tag{25}$$

Remark 3.1. *We remark that in order to keep the presentation and the proofs of the results in this section short, we only present the* a priori *estimates. The approximation arguments use to rigorously build up solutions are similar to those used in the Sections 4–6.*

3.1. **One diffusion missing, first case.** We consider the system (24)–(25) in the case when $d_1, d_3 > 0$ and $d_2 = 0$. We prove the

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain of $\mathbb{R}^N$ and d_1, $d_3 > 0$.*

Then, there exists a smooth ($(C^2([0,+\infty[\times\bar\Omega))^3$ or $(C^\infty([0,+\infty[\times\bar\Omega))^3$) solution to system (24) – (25) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 3.1. We consider a smooth solution to system (24), (25). We observe that

$$\partial_t\Big((a\,\log a - a + 1) + (b\,\log b - b + 1) + (c\,\log c - c + 1)\Big)$$
$$- \Delta_x\Big(d_1\,(a\,\log a - a + 1) + d_2\,(b\,\log b - b + 1) + d_3\,(c\,\log c - c + 1)\Big) \le 0.$$

Then, the method of duality (13) yields

$$\Big[d_1\,(a\,\log a - a + 1) + d_2\,(b\,\log b - b + 1) + d_3\,(c\,\log c - c + 1)\Big]$$
$$\times\Big[(a\,\log a - a + 1) + (b\,\log b - b + 1) + (c\,\log c - c + 1)\Big] \in L^1_{\mathrm{loc}}([0,+\infty[\times\bar\Omega), \tag{26}$$

which implies that $a, c \in L^2(\log L)^2_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$.

Next, $\partial_t b \le c$ implies that also $b \in L^2(\log L)^2_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$. This is enough to define weak solutions to the system. We now present three methods in order to get strong solutions, which work under less and less stringent conditions on the dimension N.

We first suppose that $N \le 5$. Thanks to the properties of the heat kernel (23), we know that $a \in L^{q_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ with $\frac{1}{q_0} > \frac{N-2}{2(N+2)}$ (the number q_0 can be set to ∞ for $N = 1$ and replaced by any $q_0 < +\infty$ when $N = 2$). As a consequence, $a\,b \in L^{r_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ for $\frac{1}{r_0} > \frac{N}{N+2}$ and again thanks to the properties of the heat kernel, $c \in L^{s_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ for $\frac{1}{s_0} > \frac{N-2}{N+2}$.

We now present the bootstrap argument only in the case $N = 5$ (the cases $N = 1, 2, 3, 4$ being easier). We know that $a \in L^{q_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $q_0^{-1} > 3/14$, $c \in L^{s_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $s_0^{-1} > 3/7$ and $\partial_t b \le c$, so that $b \in L^{s_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $s_0^{-1} > 3/7$.

Then, by using the heat kernel first for a, we get $a \in L^{q_1}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $q_1^{-1} > 1/7$, so that $a\,b \in L^{r_1}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $r_1^{-1} > 4/7$ and by using it for c, we obtain that $c \in L^{s_1}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $s_1^{-1} > 2/7$. As a consequence, $b \in L^{s_1}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$.

Again, thanks to the heat kernel properties, $a \in L^{q_2}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ for all $q_2 \in [1,+\infty[$, so that $a\,b \in L^{r_2}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $r_2^{-1} > 2/7$ and $c \in L^{s_2}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ for all $s_2 \in [1,+\infty[$. Thus, so does b and it remains to use a last time the heat kernel and $\partial_t b \le c$ to get the boundedness of a, b, c. Smoothness is then easy to prove.

We introduce a different type of arguments in the case when $5 < N \le 12$. Let us assume that $c \in L^{s_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$. Then, the same holds for b, and thanks to the properties of the heat kernel (23), $a \in L^{q_0}_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$, $q_0^{-1} > \frac{N+2-2s_0}{s_0(N+2)}$. As a consequence, $c\,b^{s_0-1} \in L^1_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ and $c\,a^\alpha \in L^1_{\mathrm{loc}}([0,+\infty[\times\bar\Omega)$ for $\alpha <$

$\frac{s_0-1}{1-(2s_0)/(N+2)}$. Then, we notice that for all $p \in [1, +\infty[$,

$$\partial_t(b^p/p) + a\, b^p = b^{p-1}\, c,$$
$$\partial_t(a^p/p) + b\, a^p + d_1\, (p-1)a^{p-2}|\nabla_x a|^2 - d_1\, \nabla_x \cdot (a^{p-1}\nabla_x a) = a^{p-1}\, c.$$

Since a and b are nonnegative, this ensures for all $T > 0$,

$$\int_0^T\!\!\int_\Omega a\, b^p\, dxdt \le \int_0^T\!\!\int_\Omega c\, b^{p-1}\, dxdt + \int_\Omega \frac{b^p(0,\cdot)}{p}\, dx,$$
$$\int_0^T\!\!\int_\Omega a^p\, b\, dxdt \le \int_0^T\!\!\int_\Omega c\, a^{p-1}\, dxdt + \int_\Omega \frac{a^p(0,\cdot)}{p}\, dx,$$

so that first (using $p = s_0$) $a\, b^{s_0} \in L^1_{\rm loc}([0, +\infty[\times\bar\Omega)$ and secondly (with $p = \alpha + 1$) $a^{1+\alpha}\, b \in L^1_{\rm loc}([0, +\infty[\times\bar\Omega)$. By interpolation, we end up with $a\, b \in L^{r_0}$, $r_0^{-1} > \frac{2\,(1-s_0/(N+2))}{1+s_0 N/(N+2)}$. Finally, thanks to the properties of the heat kernel, $c \in L^{s_0}_{\rm loc}([0, +\infty[\times\bar\Omega)$, with

$$\frac{1}{s_1} > \frac{2\,(1 - s_0/(N+2))}{1 + s_0\, N/(N+2)} - \frac{2}{N+2}.$$

This leads to the following induction formula: $c \in L^{s_n}_{\rm loc}([0, +\infty[\times\bar\Omega)$, where

$$s_0 = 2, \qquad \frac{1}{s_{n+1}} > \frac{2\,(1 - s_n/(N+2))}{1 + s_n N/(N+2)} - \frac{2}{N+2}.$$

It can be verified than when $N \le 12$, the sequence s_n is increasing (after a finite number of steps, the estimates on the heat kernel imply that $c \in L^\infty$). Smoothness is then easy to prove. Note that the method above applied to the non-degenerate case ($d_1, d_2, d_3 > 0$) leads to the existence of smooth solutions when $N \le 16$.

We finally introduce a proof which holds for any dimension N. According to the variant (14) of the duality estimate in Section 2.2 applied to

$$\partial_t(a + c) - \Delta_x(d_1\, a + d_3\, c) = 0,$$

we know that any estimate in $L^{q_0}_{loc}$ for a can be transfered to c. Thanks to the properties of the heat kernel used for a, we see that $a \in L^{q_0}_{loc} \Rightarrow c \in L^{q_0}_{loc} \Rightarrow a \in L^{q_1}_{loc}$, with $1/q_1 > 1/q_0 - 2/(N + 2)$. A simple induction with $q_0 = 2$ leads after finitely many steps to an L^∞_{loc} estimate for a and c. The same can be obtained for b since $\partial_t b \le c$. Again, smoothness is now easy to prove. This ends the proof of Theorem 3.1. □

3.2. **One diffusion missing, second case.**

Theorem 3.2. *Let $\Omega \subset I\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain, and d_1, $d_2 > 0$, $d_3 = 0$.*

Then, there exists a weak ($(L^2_{\rm loc}([0, +\infty[\times\bar\Omega))^3$) solution to system (24) – (25) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Moreover, if $N \le 3$, then this solution is also smooth ($(C^2([0, +\infty[\times\bar\Omega))^3$ or $(C^\infty([0, +\infty[\times\bar\Omega))^3$).

Proof of Theorem 3.2. We consider a solution to system (24), (25). We observe that (26) still holds, so that $a, b \in L^2(\log L)^2_{\rm loc}([0, +\infty[\times\bar\Omega)$.

Then, $\partial_t c \le a\,b$ also implies that $c \in L\log L_{\rm loc}([0,+\infty[\times\bar\Omega)$. This is enough to define weak solutions of the system.

Moreover, if $N = 1$, then thanks to the properties of the heat kernel (23), $a, b \in L^{q_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, $q_0^{-1} > 1/3$, so that $c \in L^{q_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$. Then, still thanks to the properties of the heat kernel (23), $a, b \in L^{q_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for all $q_1 \in [1,+\infty[$, so that $c \in L^{q_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for all $q_1 \in [1,+\infty[$. A last iteration shows that $a, b, c \in L^{\infty}_{\rm loc}([0,+\infty[\times\bar\Omega)$. Higher smoothness is then easily obtained.

We now propose an alternative method when $N = 2$ or $N = 3$. We present it only in the case $N = 3$ since the case $N = 2$ is easier. We observe that

$$(\partial_t - d_1\,\Delta_x)(a^2/2) = a\,(c - a\,b) - d_1\,|\nabla_x a|^2 \le a\,c.$$

Using the variant (13) of the method of duality, we see that $a\,c \in L^1_{\rm loc}([0,+\infty[\times\bar\Omega)$. The same holds for $b\,c$. Using the properties of the heat kernel, we end up with $a^2, b^2 \in L^{r_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, $r_0^{-1} > 3/5$. But $\partial_t c \le a\,b$, so that $c \in L^{r_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$. We then proceed by induction using the properties of the heat kernel. We obtain first $a, b \in L^{q_0}$, $q_0^{-1} > 1/5$ so that $c \in L^{q_0/2}_{\rm loc}([0,+\infty[\times\bar\Omega)$, and then $a, b \in L^{q_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for all $q_1 < +\infty$. A last iteration enables to conclude the proof of Theorem 3.2. □

3.3. **Two diffusions missing, first case.**

Theorem 3.3. *Let $\Omega \subset I\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain and $d_1 > 0$, $d_2 = d_3 = 0$.*

Then, there exists a smooth $((C^2([0,+\infty[\times\bar\Omega))^3$ or $(C^\infty([0,+\infty[\times\bar\Omega))^3)$ solution to system (24) – (25) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 3.3. It is enough to notice that $\partial_t(b + c) = 0$ so that

$$b(t,x) + c(t,x) = b(0,x) + c(0,x) =: \phi(x).$$

Then, the system can be rewritten

$$\begin{aligned} \partial_t a - d_1\,\Delta_x a &= \phi - b - a\,b, \\ \partial_t b &= \phi - b - a\,b. \end{aligned}$$

Since the right-hand-side of the system is bounded, it is easy to prove existence of smooth solutions. This ends the proof of Theorem 3.3. □

3.4. **Two diffusions missing, second case.**

Theorem 3.4. *Let $\Omega \subset I\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain and $d_3 > 0$, $d_1 = d_2 = 0$.*

Then, there exists a smooth $((C^2([0,+\infty[\times\bar\Omega))^3$ or $(C^\infty([0,+\infty[\times\bar\Omega))^3)$ solution to system (24) – (25) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 3.4. We notice first that $\partial_t(a - b) = 0$ and, thus (for all times $t \ge 0$)

$$a(t,x) - b(t,x) = a(0,x) - b(0,x) =: -\phi(x).$$

Then, the system can be rewritten as

$$\partial_t a = c - a\,(a+\phi),$$
$$\partial_t c - d_3\,\Delta_x c = a\,(a+\phi) - c.$$

Thanks to the duality method, we already know that $c \in L^2_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and hence (thanks to the first equation) that $a \in L^2_{\text{loc}}([0,+\infty[\times\bar\Omega)$. Then, the first equation also implies that

$$\partial_t(a^3/3) \le c\,a^2 - a^4 - a^3\,\phi \le c^2/2 - a^4/2 - a^3\phi\,.$$

As a consequence (integrating w.r.t. time), we see that $a \in L^4_{\text{loc}}([0,+\infty[\times\bar\Omega)$. Then, using the properties of the heat kernel, $c \in L^{s_0}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ with $\frac{1}{s_0} > \frac{N-2}{2(N+2)}$ (the exponent is replaced by ∞ when $N=1$ and by any $s_0 \in [1,+\infty[$ if $N=2$).

In fact, if we know that $c \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$, then

$$\begin{aligned}\partial_t(a^{2p-1}/(2p-1)) + a^{2p} &\le c\,a^{2p-2} - \phi\,a^{2p-1}\\ &\le c^p/p + a^{2p}\,(1-1/p) - \phi\,a^{2p-1},\end{aligned}$$

using Young's inequality. As a consequence, $a \in L^{2p}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ and using the properties of the heat kernel, $c \in L^{s_1}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ with $1/s_1 > 1/s_0 - 2/(N+2)$. An immediate induction shows that $a, c \in L^\infty_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and this is enough to guarantee the smoothness of the solution. This ends the proof of Theorem 3.4. □

4. Existence and smoothness, one diffusion coefficient missing

This section is devoted to the study of system (3), (2) in the case when $d_1 = 0$, $d_2,\, d_3,\, d_4 > 0$.

Theorem 4.1. *Let $\Omega \subset I\!\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain and $d_1 = 0,\, d_2,\, d_3,\, d_4 > 0$.*

Then, there exists a weak $(L^1_{loc}([0,+\infty]\times\bar\Omega)\times(L^2(\log L)^2_{loc}([0,+\infty]\times\bar\Omega)^3)$ solution to system (3), (2) for all smooth (and compatible with the Neumann boundary condition) nonnegative initial data.

Remark 4.1. *The smoothness of the solutions constructed in Theorem 4.1 is considered in Theorem 4.2 below.*

Proof of Theorem 4.1. We consider a (strong) solution of system (3), (2) and observe that

$$\partial_t\left(\sum_{i=1}^4 (a_i\,\log a_i - a_i + 1)\right) - \Delta_x\left(\sum_{i=1}^4 d_i\,(a_i\,\log a_i - a_i + 1)\right) \le 0. \tag{27}$$

Using the method of duality, we get the estimate

$$\left[\sum_{i=2}^4 d_i\,(a_i\,\log a_i - a_i + 1)\right]\left[\sum_{i=1}^4 (a_i\,\log a_i - a_i + 1)\right] \in L^1_{loc}([0,+\infty[\times\bar\Omega).$$

Thus, as a consequence and since $d_2,\, d_3,\, d_4 > 0$, we deduce that $a_2,\, a_3,\, a_4 \in L^2(\log L)^2_{\text{loc}}([0,+\infty[\times\bar\Omega)$. In particular, we have $a_3\,a_4 \in L\log L_{\text{loc}}([0,+\infty[\times\bar\Omega)$. Then, one uses inequality (11) in order to see that $a_1\,a_2 \in L^1_{\text{loc}}([0,+\infty[\times\bar\Omega)$.

Since these bounds enable to define all the terms in system (3), (2), existence can then be proven using the following approximating system:

$$\left\{\begin{array}{l} \partial_t a_1^n - d_1\,\Delta_x a_1^n = \frac{a_3^n\,a_4^n - a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}, \\ \partial_t a_2^n - d_2\,\Delta_x a_2^n = \frac{a_3^n\,a_4^n - a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}, \\ \partial_t a_3^n - d_3\,\Delta_x a_3^n = \frac{a_1^n\,a_2^n - a_3^n\,a_4^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}, \\ \partial_t a_4^n - d_4\,\Delta_x a_4^n = \frac{a_1^n\,a_2^n - a_3^n\,a_4^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}, \end{array}\right. \qquad \forall t \in I\!R_+, \quad x \in \Omega, \tag{28}$$

$$n\cdot\nabla_x a_i^n = 0, \qquad \forall t \in I\!R_+, \quad x \in \partial\Omega, \tag{29}$$

$$a_i^n(0,x) = a_i(0,x), \qquad \forall x \in \Omega. \tag{30}$$

Existence of a smooth solution for system (28), (29), (30) (for all given $n \in I\!N^*$ and for $d_i \ge 0$) is a consequence of the existence of smooth solutions $U : I\!R_+ \times \bar\Omega \to I\!R^d$ of systems like

$$\partial_t U - D\,\Delta_x U = f(U),$$

where D is a diagonal constant matrix whose entries are all nonnegative, and f is globally Lipschitz-continuous. Those solutions are limits of the sequence U_k, where

$$\partial_t U_{k+1} - D\,\Delta_x U_{k+1} = f(U_k).$$

For details we refer for example to [8].

Then, we show that it is possible to pass to the limit in (28), (29), (30) in order to recover a (weak) solution to (3), (2). This is done by verifying that the *a priori* estimates proven above for the solutions of (3), (2) still hold (uniformly with respect to n) for the approximated system (28), (29), (30). Note first that the method of duality ensures that

$$\left[\sum_{i=2}^4 d_i\,(a_i^n\,\log a_i^n - a_i^n + 1)\right]\left[\sum_{i=1}^4 (a_i^n\,\log a_i^n - a_i^n + 1)\right]$$

is bounded in $L^1_{loc}([0,+\infty[\times\bar\Omega)$ and, as a consequence, that $a_3^n\,a_4^n$ is bounded in $L\log L_{\rm loc}([0,+\infty[\times\bar\Omega)$. Then, the entropy dissipation estimate writes (for the approximated system):

$$\sup_{t\in[0,T]}\int_\Omega \sum_{i=1}^4 (a_i^n\log a_i^n - a_i^n + 1)\,dx + \int_0^T\!\!\int_\Omega \left[\sum_{i=1}^4 d_i \frac{|\nabla_x a_i^n(s,x)|^2}{a_i^n(s,x)}\right] dxds$$
$$+ \int_0^T\!\!\int_\Omega \frac{a_1^n(s,x)\,a_2^n(s,x) - a_3^n(s,x)\,a_4^n(s,x)}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n(s,x)}\,\log\left(\frac{a_1^n(s,x)\,a_2^n(s,x)}{a_3^n(s,x)\,a_4^n(s,x)}\right) dxds \le C,$$

for a constant C independent of n. The first term in this estimate ensures that the $(a_i^n)_{i=1,..,4}$ are weakly compact in $L^1_{\rm loc}([0,+\infty[\times\bar\Omega)$. Up to extraction of a subsequence, the (a_i^n) converge therefore to some limit a_i for $i = 1,..,4$.

Using the following variant of inequality (11):

$$\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n} \le \kappa\,\frac{a_3^n\,a_4^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n} + \frac{1}{\log\kappa}\,\frac{(a_1^n\,a_2^n - a_3^n\,a_4^n)\,\log\left(\frac{a_1^n\,a_2^n}{a_3^n\,a_4^n}\right)}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n},$$

we see (thanks to Dunford-Pettis' criterion, cf. [4] for example) that the quantity

$$\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\,\sum_{i=1}^4 a_i^n}$$

lies in a weak compact set of $L^1_{\text{loc}}([0,+\infty[\times\bar\Omega)$. As a consequence, $\partial_t a_i^n - d_i\,\Delta_x a_i^n$ lies in a weak compact set of $L^1_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and the properties of the heat kernel ensure that the sequences $(a_i^n)_{i=2,3,4}$ converge a.e. towards a_i (up to a subsequence). We then rewrite the first equation of (28) as

$$a_1^n(t,x) = a_1^n(0,x)\, e^{-\int_0^t \frac{a_2^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma} + \int_0^t \frac{a_3^n(s,x)\,a_4^n(s,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(s,x)}\, e^{-\int_s^t \frac{a_2^n(\sigma,x)}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma}\,ds.$$

Then, $\frac{1}{n}\sum_{i=1}^4 a_i^n$ converges a.e. to 0, so that

$$\frac{a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n} \to a_2 \quad a.e. \quad \text{and} \quad \frac{a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n} \to a_3\,a_4 \quad a.e.$$

Thanks to the bounds on $(a_i^n)_{i=2,3,4}$ in $L^2(\log L)^2_{\text{loc}}([0,+\infty[\times\bar\Omega)$, we see that a_1^n converges a.e. to a_1. Finally, the weak compactness of $\frac{a_1^n\,a_2^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}$ and $\frac{a_3^n\,a_4^n}{1+\frac{1}{n}\sum_{i=1}^4 a_i^n}$ in $L^1_{\text{loc}}([0,+\infty[\times\bar\Omega)$ enables to pass to the limit in (the weak formulation) of (28), (29), (30).

This ends the proof of Theorem 4.1. □

Theorem 4.2. *Let $\Omega \subset I\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain, and $d_1 = 0$, d_2, d_3, $d_4 > 0$.*

Then, there exists a classical ($(C^2([0,+\infty]\times\bar\Omega)^4$ or $(C^\infty([0,+\infty]\times\bar\Omega)^4$) solution to system (3), (2) for all smooth (and compatible with the Neumann boundary condition) nonnegative initial data in the case $N=1$.

For $N>1$, such a classical solution also exists provided $a_3, a_4 \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$ with $p > \frac{3}{4}(N+2)$. This last condition holds, for instance, when d_2, d_3 and d_4 are close enough (that is, if there exists $\delta > 0$ such that if $|d_i - d_j| < \delta$ for all $i,j \in \{2,3,4\}$, then $a_3, a_4 \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$ for some $p > \frac{3}{4}(N+2)$ by application of the improved duality estimate Proposition 2.1 to the equations of $a_2 + a_3$ and $a_2 + a_4$).

Finally, such a classical solution also exists if $a_2, a_3 \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$ (or, $a_2, a_4 \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$) with $p > N+2$. This last condition holds when d_2, d_3 are close enough (that is, if there exists $\delta > 0$ such that if $|d_2 - d_3| < \delta$, then $a_2, a_3 \in L^p_{\text{loc}}([0,+\infty[\times\bar\Omega)$ for some $p > N+2$, again by application of the improved duality estimate Proposition 2.1).

Proof. In the one-dimensional case $N=1$, it is in fact possible to show that the solutions of Theroem 4.1 are strong. This follows by observing that a_2, a_3, a_4 satisfy a heat equation with a r.h.s. in $L^1_{\text{loc}}([0,+\infty[\times\bar\Omega)$. As a consequence (23), a_2, a_3, a_4 lie in $L^{q_0}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ with $q_0^{-1} > 1/3$. Then, a_2 satisfies a heat equation with a r.h.s. in $L^{q_0/2}_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and it lies therefore in $L^{q_1}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ for all $q_1 \in [1,+\infty[$. Moreover, we observe that

$$\partial_t a_1 \le a_3\,a_4 \in L^{r_0}_{\text{loc}}([0,+\infty[\times\bar\Omega), \qquad \text{with} \quad r_0^{-1} > 2/3, \tag{31}$$

so that $a_1\,a_2 \in L^{r_0}_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and a_3, a_4 satisfies a heat equation with a r.h.s. in $L^{r_0}_{\text{loc}}([0,+\infty[\times\bar\Omega)$, and thus lie in $L^{q_1}_{\text{loc}}([0,+\infty[\times\bar\Omega)$ for all $q_1 \in [1,+\infty[$. Using again (31), this also holds for a_1. A last application of the properties of the heat kernel (23) and of (31) ensures that all a_i are bounded, and the smoothness is an easy consequence of general properties of the heat operator and the regularity with respect to a parameter of solutions of ODEs.

For the cases $N > 1$, we shall first assume that $a_3, a_4 \in L^{p_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $p_0 > \frac{3}{4}\,(N+2)$.

Then $\partial_t a_1 \le a_3\,a_4$, so that $a_1 \in L^{p_0/2}_{\rm loc}([0,+\infty[\times\bar\Omega)$. Also, $\partial_t a_2 - d_2\,\Delta_x a_2 \le a_3\,a_4$, so that thanks to the properties of the heat kernel, $a_2 \in L^{q_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/q_0 > 2/p_0 - 2/(N+2)$. Then $a_1\,a_2 \in L^{r_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/r_0 > 2/p_0 + 1/q_0 = 4/p_0 - 2/(N+2)$. Finally, since $\partial_t a_3 - d_3\,\Delta_x a_3 \le a_1\,a_2$ and $\partial_t a_4 - d_4\,\Delta_x a_4 \le a_1\,a_2$, the properties of the heat kernel imply that $a_3 \in L^{p_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$ and $a_4 \in L^{p_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/p_1 > 1/r_0 - 2/(N+2) = 4/p_0 - 4/(N+2)$.

Iterating this procedure, we see that a_3 and $a_4 \in L^{p_n}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/p_n = 4/p_{n-1} - 4/(N+2)$, for all $n \in I\!N$. Since $p_0 > \frac{3}{4}\,(N+2)$, we see that the sequence p_n is increasing and reaches after finitely many iterations a final index $k \in I\!N$ for which $p_k \ge 0$ while $p_{k+1} < 0$. Then, a last iteration yields $a_3, a_4 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$, and consequently $a_1, a_2 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. It is then easy to show that all a_i are in fact smooth.

We conclude this case by pointing out that applying Proposition 2.1 to the equations

$$\begin{cases} \partial_t(a_2+a_3) - \Delta_x\Big(\frac{d_2\,a_2 + d_3\,a_3}{a_2+a_3}\,(a_2+a_3)\Big) = 0, \\ \partial_t(a_2+a_4) - \Delta_x\Big(\frac{d_2\,a_2 + d_4\,a_4}{a_2+a_4}\,(a_2+a_4)\Big) = 0, \end{cases}$$

together with the homogeneous Neumann boundary conditions (2) proves that there exists $\delta > 0$ such that if $|d_i - d_j| < \delta$ for all $i,j \in \{2,3,4\}$, then a_3 and $a_4 \in L^{p_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for some $p_0 > \frac{3}{4}\,(N+2)$.

As an alternative case $N > 1$, we now assume that a_2 and $a_3 \in L^{p}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $p > N+2$. We also consider $a_4 \in L^{s_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$, which holds for $s_0 = 2$.

Then, $\partial_t a_1 \le a_3\,a_4$, so that $a_1 \in L^{q}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/q \ge 1/s_0 + 1/p$, and $a_1\,a_2 \in L^{r}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/r \ge 1/s_0 + 2/p$. Since $\partial_t a_4 - d_4\,\Delta_x a_4 \le a_1\,a_2$, the properties of the heat kernel ensure that $a_4 \in L^{s_1}_{\rm loc}([0,+\infty[\times\bar\Omega)$, with $1/s_1 > 1/s_0 + 2/p - 2/(N+2)$.

Iterating this procedure for $p > N+2$, we obtain that $a_4 \in L^{s_n}_{\rm loc}([0,+\infty[\times\bar\Omega)$ with $1/s_n > 1/s_{n-1} + 2/p - 2/(N+2)$ for all $n \in I\!N$ until after finitely many steps an index $k \in I\!N$ is reached such that $s_k \ge 0$ while $s_{k+1} < 0$. Then, a last iteration yields $a_4 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. As a consequence, a_3 and $a_4 \in L^{p}_{\rm loc}([0,+\infty[\times\bar\Omega)$, and so on one hand $a_1 \in L^{p}_{\rm loc}([0,+\infty[\times\bar\Omega)$, and on the other hand, since $\partial_t a_2 - d_2\,\Delta_x a_2 \le a_3\,a_4$, the properties of the heat kernel ensure that $a_2 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$ (remember that $p > N+2$). Finally, $a_1\,a_2 \in L^{p}_{\rm loc}([0,+\infty[\times\bar\Omega)$, and since $\partial_t a_3 - d_3\,\Delta_x a_3 \le a_1\,a_2$, the properties of the heat kernel ensure that $a_3 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. A last application of the inequality $\partial_t a_1 \le a_3\,a_4$ ensures that $a_1 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. Once again, it is then easy to show that all a_i are in fact smooth.

We conclude by remarking that Proposition 2.1 applied to the equation

$$\partial_t(a_2+a_3) - \Delta_x\Big(\frac{d_2\,a_2 + d_3\,a_3}{a_2+a_3}\,(a_2+a_3)\Big) = 0,$$

with the homogeneous Neumann boundary conditions (2) proves that there exists $\delta > 0$, such that if $|d_2 - d_3| < \delta$, then $a_2, a_3 \in L^{p}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for some $p > N+2$.

This ends the proof of Theorem 4.2. $\square$

5. Existence and smoothness, two diffusion coefficients missing

5.1. **First case.**

Theorem 5.1. *Let $\Omega \subset I\!\!R^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain, and $d_1 = d_3 = 0$, d_2, $d_4 > 0$.*

Then, there exists a smooth ($(C^2([0,+\infty[\times\bar\Omega))^4$ or $(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 5.1. We consider a solution to system (3), (2). Then, $\partial_t(a_1 + a_3) = 0$, so that $a_1, a_3 \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. Moreover, $a_2, a_4 \in L^\infty([0,+\infty[; L^1(\Omega))$ thanks to the conservation laws (4), so that $a_1\, a_2$ and $a_3\, a_4 \in L^1_{\rm loc}([0,+\infty[\times\bar\Omega)$.

By using the properties of the heat kernel (23), we obtain as previously that $a_2, a_4 \in L^{p_0}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for $p_0^{-1} > \frac{N}{N+2}$ and the same is true for $a_1\, a_2$ and $a_3\, a_4$. An immediate induction shows that a_2 and $a_4 \in L^{p_n}_{\rm loc}([0,+\infty[\times\bar\Omega)$ for $(p_n)_{n\in I\!\!N^*}$ such that

$$\frac{1}{p_0} > \frac{N}{N+2}, \qquad \frac{1}{p_{n+1}} > \frac{1}{p_n} - \frac{2}{N+2}.$$

Finally, after finitely many steps, $a_i \in L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$ for all $i = 1,..,4$, and the smoothness can then be easily recovered.

Existence is obtained through the use of the approximated system (28), (29), (30). The proof used above (when applied to a_i^n instead of a_i) shows that the sequences a_i^n are bounded in $L^\infty_{\rm loc}([0,+\infty[\times\bar\Omega)$. Then, a_2^n and a_4^n converge (up to a subsequence) a.e. to some limit a_2, a_4 thanks to the properties of the heat kernel. Denoting $\phi(x) := a_1^n(t,x) + a_3^n(t,x) = a_1(0,x) + a_3(0,x)$, we see that

$$a_1^n(t,x) = \; a_1^n(0,x)\, e^{-\int_0^t \frac{a_2^n(\sigma,x)+a_4^n(\sigma,x)}{1+\frac1n \sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma} + \int_0^t \frac{\phi(x)\, a_4^n(s,x)}{1+\frac1n \sum_{i=1}^4 a_i^n(s,x)}\, e^{-\int_s^t \frac{a_2^n(\sigma,x)+a_4^n(\sigma,x)}{1+\frac1n \sum_{i=1}^4 a_i^n(\sigma,x)}d\sigma}\, ds.$$

Then, $\frac1n \sum_{i=1}^4 a_i^n$ converges a.e. to 0 so that a_1^n converges a.e to some limit a_1 (up to a subsequence). The same holds for a_3^n. As a consequence, it is possible to pass to the limit in (28), (29), (30) and this ends the proof of Theorem 5.1. $\square$

5.2. **Second case.**

Theorem 5.2. *Let $\Omega \subset I\!\!R^N$ be a bounded, regular ($C^{2+\alpha}$ or C^∞) domain and $d_1 = d_2 = 0$, d_3, $d_4 > 0$.*

Then, there exists a weak ($(L^1_{\rm loc}([0,+\infty[\times\bar\Omega))^2 \times (L^2(\log L)^2_{\rm loc}([0,+\infty[\times\bar\Omega))^2$) solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Moreover, in the case that $N = 1$, this solution is also smooth ($C^2([0,+\infty[\times\bar\Omega)^4$ or $C^\infty([0,+\infty[\times\bar\Omega)^4$).

Proof of Theorem 5.2. We consider a solution to system (3), (2). We notice that thanks to the method of duality, inequality (27) leads to the estimate

$$(d_3\, a_3 + d_4\, a_4)\,(a_1 + a_2 + a_3 + a_4) \in L \log L_{\rm loc}([0,+\infty[\times\bar\Omega), \tag{32}$$

so that $a_3, a_4 \in L^2(\log L)^2_{\rm loc}([0,+\infty[\times\bar\Omega)$.

Using now the second term of entropy dissipation integral (7) (that is, inequality (11)), we see that

$$a_3\, a_4 \in L^1_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega}) \qquad \Rightarrow \qquad a_1\, a_2 \in L^1_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})\,,$$

so that all terms in system (3), (2) can be defined. In order to prove existence, we consider the approximated system (28), (29), (30) and verify that the arguments above can be used (uniformly w.r.t. n). Estimate (32) still holds uniformly w.r.t. n, and leads to the boundedness in $L\log L_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$ of the sequence $a_3^n\, a_4^n$. Inequality (11) shows then that $a_1^n\, a_2^n$ is weakly compact in $L^1_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$. Finally, thanks to the properties of the heat kernel, we know that a_3^n and a_4^n converge a.e. to some a_3, a_4.

The equation for a_1^n can then be rewritten as

$$\partial_t a_1^n = \gamma^n - \delta^n\, a_1^n\, (a_1^n - \phi)\,, \tag{33}$$

where $\phi(x) := a_1^n(t,x) - a_2^n(t,x)$ is constant for all n and t. Moreover, $\delta^n(t,x) = (1+\frac{1}{n}\sum_{i=1}^4 a_i^n)^{-1}$ tends (up to a subsequence) to 1 a.e. and lives in $[0,1]$ and $\gamma_n = \delta^n a_3^n\, a_4^n$ converges in $L^1([0,T]\times\Omega)$. We first observe that (up to extraction of a subsequence), the quantities $\int_0^T |\delta^n(t,x)-1|\,dt$ and $\int_0^T |\gamma^n(t,x)-a_3(t,x)\,a_4(t,x)|\,dt$ converge towards 0 for a.e. $x\in\Omega$. Then, for a given $x\in\Omega$, we introduce the sequence of functions of time $\alpha_p^n := \alpha_p^n(t)$ approximating the ODE (33) in the following iterative way:

$$\alpha_{p+1}^n(t) = a_1^n(0)\, e^{-\int_0^t \delta^n(s)\,(\alpha_p^n(s)-\phi)\,ds} + \int_0^t \gamma^n(s)\; e^{-\int_s^t \delta^n(\sigma)\,(\alpha_p^n(\sigma)-\phi)\,d\sigma}\, ds.$$

It is easy to see that (still for a given x), $\sup_{t\in[0,T]}\sup_{p,n\in I\!N}|\alpha_p^n(t)| < \infty$ and $\lim_{p\to\infty}\sup_{n\in I\!N}\sup_{t\in[0,T]}|\alpha_p^n(t)-\alpha^n(t)| = 0$. Then, it is possible to prove by induction on p that (for all p) $\lim_{n\to\infty}|\alpha_p^n(t)-\alpha_p(t)| = 0$, where

$$\alpha_{p+1}(t) = a_1^n(0)\, e^{-\int_0^t (\alpha_p(s)-\phi)\,ds} + \int_0^t a_3(s)\, a_4(s)\, e^{-\int_s^t (\alpha_p(\sigma)-\phi)\,d\sigma}\, ds.$$

It is therefore possible to prove that the sequence $\alpha_p(t)$ is a Cauchy sequence in $L^\infty([0,T])$, and consequently that there exists (still for a given x) a function $a_1 := a_1(s)$ such that $\lim_{p\to\infty}\sup_{t\in[0,T]}|\alpha_p(t)-a_1(t)| = 0$. Thanks to the previous construction, we see that a_1^n converges to a_1 uniformly on $[0,T]$ (for a given x). Therefore, we obtain the convergence a.e. of a_1^n to a_1, and the same holds of course for a_2^n. This is enough to pass to the limit in the system.

Finally, when $N=1$, the properties of the heat kernel imply that $a_3, a_4 \in L^{p_0}_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$ with $p_0^{-1} > 1/3$, which implies that a_3 and $a_4 \in L^{p_0/2}_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$. Then, we observe that (with $\phi(x) := a_1(t,x) - a_2(t,x)$)

$$\begin{aligned}
\partial_t(a_2^{p_0-1}/(p_0-1)) \le\ & a_2^{p_0-2}\, a_3\, a_4 - a_2^{p_0} - a_2^{p_0-1}\,\phi \\
\le\ & a_2^{p_0}\,\frac{p_0-2}{p_0} + (a_3a_4)^{p_0/2}\,\frac{2}{p_0} - a_2^{p_0} - a_2^{p_0-1}\phi \\
\le & -\frac{2}{p_0}a_2^{p_0} - a_2^{p_0-1} + (a_3a_4)^{p_0/2}\,\frac{2}{p_0},
\end{aligned}$$

due to Young's inequality. Thus, after integration in time, $a_2 \in L^{p_0}_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$ and the properties of the heat kernel imply that $a_3\, a_4 \in L^{p_1}_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$ for all $p_1 \in [1,+\infty[$. Finally, a last iteration ensures that all a_i lie in $L^\infty_{\mathrm{loc}}([0,+\infty[\times\bar{\Omega})$, which implies smoothness of the solution without difficulty. This ends the proof of Theorem 5.2 □

6. Existence and smoothness, three diffusion coefficients missing

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded and regular ($C^{2+\alpha}$ or C^∞) domain and $d_1 = d_2 = d_3 = 0$, $d_4 > 0$.*

Then, there exists a smooth ($(C^2([0,+\infty[\times\bar\Omega))^4$ or $(C^\infty([0,+\infty[\times\bar\Omega))^4$) solution to system (3), (2) for all smooth initial nonnegative data (compatible with the Neumann boundary condition).

Proof of Theorem 6.1. We consider a solution to system (3), (2). We first observe that

$$a_1(t,x) + a_3(t,x) = a_1(0,x) + a_3(0,x) := \phi(x) \le C, \qquad \forall t \ge 0, \quad x \in \Omega,$$
$$a_2(t,x) + a_3(t,x) = a_2(0,x) + a_3(0,x) := \psi(x) \le C, \qquad \forall t \ge 0, \quad x \in \Omega,$$

where $C \ge 0$ is defined as

$$C = \sup\Big\{ \|a_1(0,\cdot)\|_{L^\infty} + \|a_3(0,\cdot)\|_{L^\infty}\,,\ \|a_2(0,\cdot)\|_{L^\infty} + \|a_3(0,\cdot)\|_{L^\infty} \Big\}.$$

Then, the system can be rewritten as

$$\partial_t a_3 = (\phi - a_3)\,(\psi - a_4) - a_3\,a_4 \le C^2,$$
$$\partial_t a_4 - d_4\,\Delta_x a_4 = (\phi - a_3)\,(\psi - a_4) - a_3\,a_4 \le C^2.$$

Since the left-hand side is bounded (when a_3, a_4 are nonnegative), it is possible to prove existence of a bounded solution by a simple continuation argument (Cf. [8]). Note that we do not need to use an approximated system here. Smoothness follows then easily. This ends the proof of Theorem 6.1 □

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