Discrete and Algebraic Structures

Master Study Mathematics

MAT.402

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(Due to organisational reasons register for the course at TU and KFU)

Part I (edited by Mihyun Kang)

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Part II (edited by Karin Baur)

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revised by Karin Baur, Mihyun Kang and Philipp Sprüssel

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Standard Methods of Enumerative Combinatorics

Lecture 1 on 8.10.2018

1.1 Basics

1.1.1 Binomial Coefficient and Binomial Theorem

Definition 1 (Set of integers)**.**

$$[n] := \{1,2,\dots,n\}$$

Definition 2 (Factorial)**.** For any $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} n! &:= n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=0}^{n-1} (n-i) \\ 0! &:= 1 \end{aligned}$$

Definition 3 (Falling factorial)**.** For any $n,k \in \mathbb{N} \cup \{0\}$ satisfying $k \le n$,

$$\begin{aligned} (n)_k &:= n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \prod_{i=0}^{k-1} (n-i) && \text{for } 0 < k \le n \\ (n)_0 &:= 1 \end{aligned}$$

Definition 4 (Binomial coefficient)**.** For any $n,k \in \mathbb{N} \cup \{0\}$,

$$\binom{n}{k} := \begin{cases} \frac{n!}{(n-k)!k!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} & \text{if } k \le n \\ 0 & \text{if } k > n \end{cases}$$

Definition 5 (Multinomial coefficient)**.** For any $n, k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\}$ satisfying $n = k_1 + k_2 + \ldots + k_m$,

$$\binom{n}{k_1, k_2, \ldots, k_m} := \frac{n!}{k_1!\,k_2!\,\ldots\,k_m!}$$

In the notation of Definition 5, the binomial coefficient is written as $\binom{n}{k, n-k}$.

Properties 1. *For any $n, k \in \mathbb{N} \cup \{0\}$ satisfying $k \leq n$,*

$$\binom{n}{k} = \binom{n}{n-k}$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

For any $k, m, n \in \mathbb{N}$, we have

$$\binom{n+m}{k} = \sum_{\ell=0}^{k} \binom{n}{\ell} \cdot \binom{m}{k-\ell}.$$

Theorem 1 (Binomial theorem)**.** *For any $n \in \mathbb{N}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where x is a formal variable or $x \in \mathbb{C}$.

Theorem 2 (Multinomial theorem)**.** *For any $n \in \mathbb{N}$,*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_2}$$

where the sum is over all $k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\}$ satisfying $n = k_1 + k_2 + \ldots + k_m$.

Applying the binomial theorem (Theorem 1) with $x = 1$ and $x = -1$ we have

$$2^n = \sum_{i=0}^{n} \binom{n}{k}$$
$$0 = \sum_{i=0}^{n} (-1)^k \binom{n}{k}$$

Example 1 (Subsets)**.** *The number of all subsets of $[n]$ is equal to*

$$\sum_{i=0}^{n} \binom{n}{k} = 2^n$$

Example 2 (Subsets of odd sizes)**.**

To calculate the number of subsets of $[n]$ of odd sizes, we observe that

$$\begin{aligned} 2^n \;=\; 2^n + 0 \;&=\; \sum_{i=0}^{n} \binom{n}{k} + \sum_{i=0}^{n} (-1)^k \binom{n}{k} \\ &=\; 2\left[\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\cdot\lfloor n/2 \rfloor}\right] \end{aligned}$$

This implies that the number of subsets of $[n]$ of even sizes is 2^{n-1}, and therefore the number of subsets of $[n]$ of odd sizes is equal to

$$2^n - 2^{n-1} = 2^{n-1}$$

Definition 6 (Generalized binomial coefficient)**.** For any $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$,

$$\binom{\alpha}{k} \;:=\; \frac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$$

(1) Note that the denominator has k terms.
(2) If α is in $\mathbb{N}$ and $k > \alpha$, $\binom{\alpha}{k} = 0$ as in Definition 4.

Example 3.

$$\begin{aligned} \binom{\frac{1}{2}}{k} \;&:=\; \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\ldots\cdot\left(\frac{1}{2}-k+1\right)}{k!} \\ &=\; \frac{1\cdot(1-2)\cdot\ldots\cdot(1-2(k-1))}{2^k\, k!} \\ &=\; \frac{(-1)^{k-1}}{4^k\,(2k-1)}\binom{2k}{k} \end{aligned}$$

Theorem 3 (Generalized binomial theorem)**.** *For any $\alpha \in \mathbb{C}$ and $x \in \mathbb{C}$ with $|x| < 1$ (or x a formal variable),*

$$(1+x)^{\alpha} \;=\; \sum_{k=0}^{\infty} \binom{\alpha}{k}\, x^k$$

Example 4.

$$\begin{aligned} (1-x)^{\frac{1}{2}} \;&=\; \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}\,(-x)^k \\ &=\; \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{4^k\,(2k-1)}\binom{2k}{k}\,(-x)^k \\ &=\; -\sum_{k=0}^{\infty} \frac{1}{4^k\,(2k-1)}\binom{2k}{k}\,x^k \end{aligned}$$

1.1.2 Landau- and Asymptotic Notations

(1) $O(g(n)) := \{f(n): \quad \exists c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \leq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty\right\}$

$f(n) = O(g(n))$
"f(n) is big oh of g(n)"
"f(n) grows at most as fast as g(n)"'

(2) $\Omega(g(n)) := \{f(n): \exists c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \geq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \liminf_{n\to\infty} \frac{|f(n)|}{|g(n)|} > 0\right\}$

$f(n) = \Omega(g(n))$
"f(n) is big omega of g(n)"
"f(n) grows at least as fast as g(n)"'

(3) $\Theta(g(n)) := O(g(n)) \cap \Omega(g(n))$

$f(n) = \Theta(g(n))$
"f(n) is big theta of g(n)"
"f(n) grows as fast as g(n)"'

(4) $o(g(n)) := \{f(n): \forall c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \leq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0\right\}$

$f(n) = o(g(n))$ *or* $f(n) \ll g(n)$
"f(n) is small oh of g(n)"
"f(n) grows slower than g(n)"'

(5) $\omega(g(n)) := \{f(n): \forall c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \geq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = \infty\right\}$

$f(n) = \omega(g(n))$ *or* $f(n) \gg g(n)$
"f(n) is small omega of g(n)"
"f(n) grows faster than g(n)"'

(6)

$$f(n) \sim g(n) \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

"$f(n)$ is of order of $g(n)$"

"$f(n)$ is asymptotically equal to $g(n)$"'

Example 5.

$$1 + \tfrac{1}{n} = 1 + o(1) \quad \text{vs} \quad 1 + \tfrac{1}{n^2} = 1 + o(1) \quad \text{vs} \quad 1 + \tfrac{1}{n^2+1000} = 1 + o(1)$$

1.1.3 Useful Inequalities and Asymptotic Estimates

Theorem 4 (GM-AM-RMS inequality)**.** *For any $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$,*

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

In words, "the geometric mean is smaller than the arithmetic mean, which is smaller than the root mean square".

Theorem 5 (Stirlings formula)**.** *For any $n \in \mathbb{N}$,*

$$\begin{aligned} n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right) \\ &= (1 + o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \end{aligned}$$

Lemma 6 (Binomial coefficient approximation)**.** *For any $n, k \in \mathbb{N}$ satisfying $k \leq n$,*

$$\begin{aligned} \left(\frac{n}{k}\right)^k &\leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k \\ \binom{n}{k} &= 2^{H\left(\frac{n}{k}\right)n + O(\log k)} \end{aligned}$$

where H is the Entropy function

$$H(x) := -x\log_2 x - (1-x)\log_2(1-x)$$

Lecture 2 on 9.10.2018

1.2 Recursions

1.2.1 Recursions

Example 6. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 & = 0 \\ a_{n+1} & = 2a_n + 1, \quad n \geq 0 \end{cases}$$

The sequence begins with $0, 1, 3, 7, 15, 31, \ldots$

We add 1 on both sides of the latter equation to obtain

$$a_{n+1} + 1 = 2(a_n + 1), \quad n \geq 0.$$

Then we have $a_n + 1 = 2^n(a_0 + 1) = 2^n$ *and therefore the sequence is explicitly given by*

$$a_n = 2^n - 1, \quad n \geq 0.$$

Example 7. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 & = 0 \\ a_1 & = 1 \\ a_{n+2} & = a_{n+1} + a_n, \quad n \geq 0. \end{cases}$$

This sequence is called Fibonacci sequence and begins with $0, 1, 1, 2, 3, 5, 8, 13, \ldots$*.*

How can we derive the explicit form of the sequence?

1.2.2 Linear recurrences

Definition 7. (1) We say that a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfies a k-th order recurrence (or recursion) if a_n can be written as

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}), \quad n \geq k$$

for a function $f : \mathbb{C}^k \to \mathbb{C}$.
(2) The sequence $(a_n)_{n\geq 0}$ is said to be an *order k linear recurrence with constant coefficients* $c_1, \ldots, c_k$, if it can be written as

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where the c_i are constants.
(3) The *characteristic polynomial* of an order k linear recurrence as in (2) is the polynomial

$$q(z) = z^k - c_1 \cdot z^{k-1} - c_2 \cdot z^{k-2} - \ldots - c_k.$$

First-order linear recurrences

Theorem 7. *A first-order linear recurrence with constant coefficients of the form*

$$\begin{cases} a_0 = 0 \\ a_n = c_n \cdot a_{n-1} + d_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

with $c_i \neq 0$ *has an explicit solution,*

$$a_n = d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n.$$

Proof. Divide both sides of the recurrence in (1.1) by $c_n c_{n-1} \cdots c_1$:

$$\frac{a_n}{c_n c_{n-1} \cdots c_1} = \frac{a_{n-1}}{c_{n-1} c_{n-2} \cdots c_1} + \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Change of variables: let

$$b_n = \frac{a_n}{c_n c_{n-1} \cdots c_1}.$$

We get the difference relation

$$b_n = b_{n-1} + \frac{d_n}{c_n c_{n-1} \cdots c_1},$$

that is,

$$b_n - b_{n-1} = \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Summing up the differences $b_m - b_{m-1}$ for $m = 1, \ldots, n$ (noting that $b_0 = 0$), we get

$$b_n = \sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1}.$$

Thus, for a_n ($n \geq 1$) we have

$$\begin{aligned} a_n &= c_n c_{n-1} \cdots c_1 \cdot \left(\sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1} \right) \\ &= d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n. \end{aligned}$$

□

Example 8. *Returning to Example 6 we have*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0, \end{cases}$$

so taking $c_i = 2, d_i = 1$ *for all i in Theorem 7 we have*

$$a_n = 1 + \sum_{i=1}^{n-1} 2^{n-i} = 1 + \frac{2(1-2^{n-1})}{1-2} = 1 + 2^n - 2,$$

because $\sum_{i=1}^{m} r^i = \frac{r(1-r^m)}{1-r}$ *for any* $r \neq 1$.

Higher order linear recurrences with constant coefficients

Theorem 8. *All solutions to the k-th order linear recurrence with constant coefficients* $c_1, c_2, \ldots, c_k$

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}, \quad n \geq k,$$

can be expressed as linear combinations of terms of the form

$$\alpha^n, n\alpha^n, n^2\alpha^n, \ldots, n^{m-1}\alpha^n$$

where α *is a root of order m of the characteristic polynomial*

$$q(z) = z^k - c_1 \cdot z^{k-1} - c_2 \cdot z^{k-2} - \ldots - c_k.$$

Remark 1. If $q(z) = (z-\alpha)^m \cdot p(z)$ for some polynomial $p(z)$ where $(z-\alpha) \nmid p(z)$, i.e. α is a root of multiplicity m, then $q(\alpha) = q'(\alpha) = \ldots = q^{(m-1)}(\alpha) = 0$.

Proof of Theorem 8. Let α be a simple root of $q(z)$ (i.e. the multiplicity is 1) and $a_n = \alpha^n$. We want to check that a_n is a solution to the recurrence, i.e.

$$\begin{aligned} 0 &\overset{!}{=} \alpha^n - c_1\alpha^{n-1} - c_2\alpha^{n-2} - \ldots - c_k\alpha^{n-k} \\ &= \alpha^{n-k} \cdot \left(\alpha^k - c_1\alpha^{k-1} - \ldots - c_k\right) \\ &= \alpha^{n-k} \cdot q(\alpha) = 0. \end{aligned}$$

Let α now be a root of multiplicity of m of $q(z)$. Then, for $0 \leq i \leq m-1$, $a_n = n^i\alpha^n$ is a solution to the recurrence because

$$\begin{aligned} 0 &\overset{!}{=} a_n - c_1 \cdot a_{n-1} - c_2 \cdot a_{n-2} - \ldots - c_k a_{n-k} \\ &= n^i\alpha^n - c_1(n-1)^i\alpha^{n-1} - c_2(n-2)^i\alpha^{n-2} - \ldots c_k(n-k)^i\alpha^{n-k} \\ &= \alpha^{n-k} \cdot \left(n^i\alpha^k - c_1(n-1)^i\alpha^{k-1} - \ldots - c_k(n-k)^i\right) \\ &= \alpha^{n-k} \cdot \big((n-k)^i q(\alpha) + \alpha\left((n-k)^i - (n-k-1)^i\right) q'(\alpha) + \\ &\qquad \alpha^2\left(b_0(n-k)^i + b_1(n-k-1)^i + b_2(n-k-2)^i\right) q''(\alpha) + \ldots\big) \\ &= \alpha^{n-k} \cdot \left(\sum_{0 \leq j \leq i} \alpha^j \cdot \left(\sum_{0 \leq \ell \leq j} b_{j,l}(n-k-\ell)^i\right) q^{(j)}(\alpha)\right) = 0, \end{aligned}$$

because of the previous remark; the $b_{j,\ell}$ are constants.

Furthermore, a linear combination of $\alpha^n, n \cdot \alpha^n, \ldots, n^{m-1}\alpha^n$ is also a solution to the recurrence.

If $q(z)$ has distinct roots $\alpha_1, \ldots, \alpha_s$ with multiplicities $m_1, \ldots, m_s$ where $m_1 + \ldots + m_s = k$, then a linear combination of all these $n^j \cdot \alpha_i^n$ for $0 \leq j \leq m_i - 1$, $1 \leq i \leq s$, i.e.

$$a_n = \sum_{i=1}^{s} \left(b_{i,0} + b_{i,1}n + \ldots + b_{i,m_i-1}n^{m_i-1}\right) \cdot \alpha_i^n$$

is also a solution to the recurrence.

We claim the opposite is also true. Let R be the set of solutions to the recurrence, the elements of R are sequences in $\{a = (a_n)_n \mid a_n \in \mathbb{R}\}$. Then R is closed under addition and scalar multiplication, and $R \neq \emptyset$. Therefore, R is a vector space.

We claim that R has dimension k. Consider a map f from R to $\mathbb{R}^k$:

$$f : R \to \mathbb{R}^k, \quad a \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}.$$

This is a linear map and also an isomorphism, because any solution to the recurrence is uniquely determined by the k initial values. So, R has dimension k.

Therefore, R is given (generated) by linear combinations of any k linearly independent solutions to the recurrence. Now it suffices to show that the set of k solutions $\{n^j \alpha_i^n \mid 1 \leq j \leq m_i - 1, 1 \leq i \leq s\}$ to the recurrence is linearly independent. But this is true because these solutions have different orders of growth (in particular at ∞). □

Example 9. *Returning to Example 7 we consider the Fibonacci sequence*

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, & n \geq 2 \\ a_0 = 0, a_1 = 1. \end{cases} \tag{1.2}$$

Its characteristic polynomial is

$$q(z) = z^2 - z - 1 = \left(z - \frac{1+\sqrt{5}}{2}\right) \cdot \left(z - \frac{1-\sqrt{5}}{2}\right),$$

and the solution to the recurrence for a_n is

$$\begin{cases} a_n = r_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + r_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \\ a_0 = 0, a_1 = 1. \end{cases}$$

From the initial conditions, we get $r_1 = \frac{1}{\sqrt{5}}$, $r_2 = -\frac{1}{\sqrt{5}}$. The explicit solution to (1.2) *is*

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example 10. *Consider the second order linear recurrence*

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 2$$

with the initial conditions $a_0 = 0, a_1 = 1$.

The characteristic polynomial is

$$q(z) = z^2 - 5z + 6 = (z-2)(z-3).$$

The solution will be of the form

$$a_n = r_1 2^n + r_2 3^n.$$

Due to the initial conditions, $r_1 = -1$ *and* $r_2 = 1$ *and the solution is*

$$a_n = -2^n + 3^n.$$

This can be solved in Maple as follows:

```
rsolve({a(n) = 5 * a(n-1) - 6 * a(n-2), a(0) = 0, a(1) = 1}, a(n));
```

In Mathematica:

```
RSolve[{a[n] == 5 * a[n-1] - 6 * a[n-2], a[0] == 0, a[1] == 1}, a[n], n]
```

Exercise 1. Find initial conditions a_0, a_1, a_2 for which the growth rate of the solution to the recurrence

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n \geq 3$$

is (a) constant, (b) exponential, and (c) fluctuating in sign.

Higher order linear recurrences with non-constant coefficients

Example 11. *Consider the recurrence*

$$\begin{cases} a_n = na_{n-1} + n(n-1)a_{n-2}, & n \geq 2 \\ a_0 = 0, a_1 = 1. \end{cases}$$

Divide by $n!$ *to receive*

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + \frac{a_{n-2}}{(n-2)!}.$$

Change variables: let $b_n = \frac{a_n}{n!}$*. Then we get the recurrence*

$$\begin{cases} b_n = b_{n-1} + b_{n-2}, & n \geq 2 \\ b_0 = 0, b_1 = 1. \end{cases}$$

for the Fibonacci sequence. From Example 9 we have

$$b_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This yields

$$a_n = \frac{n!}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Exercise 2. Solve the recurrence

$$\begin{cases} n(n-1)a_n = (n-1)a_{n-1} + a_{n-2}, & n \geq 2 \\ a_0 = 0,\ a_1 = 1. \end{cases}$$

1.2.3 Non-linear recurrences

Example 12. *Consider the second order non-linear recurrence*

$$\begin{cases} a_n = \sqrt{a_{n-1} \cdot a_{n-2}}, & n \geq 2 \\ a_0 = 1,\ a_1 = 2. \end{cases}$$

We use the logarithm function, since

$$\log_a xy = \log_a x + \log_a y.$$

Let $b_n = \log a_n$. *We get*

$$\begin{cases} b_n = \frac{1}{2}\left(b_{n-1} + b_{n-2}\right), & n \geq 2 \\ b_0 = 0,\ b_1 = 1. \end{cases}$$

This can be solved similar to the previous linear recurrences.

Lecture 3 on 15.10.2018

1.3 Generating Functions

Let $\mathbb{K}$ denote a commutative ring, usually we take $\mathbb{K} = \mathbb{C}$.

1.3.1 Formal Power Series

Now let us return to Example 6.

Example 13. *Let a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0 \end{cases} \tag{1.3}$$

and let z be a formal indeterminate variable.

Multiplying both sides of (1.3) by z^{n+1} and summing over $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\sum_{n=0}^{\infty} a_{n+1} z^{n+1} \;=\; \sum_{n=0}^{\infty} (2a_n + 1)\, z^{n+1}.$$

If it were true that $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} \;=\; 2z\,\sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n$, we would obtain

$$\sum_{n=1}^{\infty} a_n z^n \;=\; 2z\sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n$$

Defining $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ we have

$$A(z) \;=\; 2zA(z) + z\sum_{n=0}^{\infty} z^n \quad \text{equiv.} \quad (1-2z)A(z) \;=\; z\sum_{n=0}^{\infty} z^n$$

because $a_0 = 0$. From this we we would further obtain

$$\begin{aligned} A(z) &\overset{?}{=} \frac{z}{1-2z}\sum_{n=0}^{\infty} z^n \overset{?}{=} \frac{z}{1-2z}\cdot\frac{1}{1-z} \;=\; 2z\,\frac{1}{1-2z} \;-\; z\,\frac{1}{1-z} \\ &\overset{?}{=} 2z\sum_{n=0}^{\infty}(2z)^n \;-\; z\sum_{n=0}^{\infty} z^n \\ &\overset{?}{=} \sum_{n=1}^{\infty}(2z)^n \;-\; \sum_{n=1}^{\infty} z^n \\ &\overset{?}{=} \sum_{n=1}^{\infty}(2^n-1)\, z^n \\ &= \sum_{n=0}^{\infty}(2^n-1)\, z^n, \end{aligned}$$

if the following were true:

$$(1-2z)A(z) = z\sum_{n=0}^{\infty} z^n \quad \overset{?}{\iff} \quad A(z) = \frac{z}{1-2z}\sum_{n=0}^{\infty} z^n \tag{1.4}$$

$$\sum_{n=0}^{\infty}(az)^n \overset{?}{=} \frac{1}{1-az}, \quad a \in \mathbb{K} \tag{1.5}$$

$$\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} \overset{?}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n. \tag{1.6}$$

Summing up, we would have $\sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty}(2^n-1)\, z^n$ *and therefore* $a_n = 2^n - 1$ *for every* $n \in \mathbb{N}\cup\{0\}$*, by comparing the coefficients, if* (1.4)–(1.6) *were true. When are they true?*

Definition 8 (Formal Power Series)**.** Given a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ with $a_n \in \mathbb{K}$ and a formal variable z, we call an infinite sum of the form

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{n\geq 0} a_n\, z^n$$

a *formal power series*. We call a_n the coefficient of z^n in $A(z)$ and use the notation

$$[z^n]\, A(z) := a_n.$$

The ring of formal power series is denoted by $\mathbb{K}[[z]]$ and endowed with the operations of addition and product:

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) + \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty}(a_n+b_n)\, z^n \tag{1.7}$$

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \tag{1.8}$$

We often skip $\cdot$ for the product of two formal power series. As a special case of (1.8) we have, for any $a \in \mathbb{K}$,

$$a\left(\sum_{n=0}^{\infty} b_n\, z^n\right) = \sum_{n=0}^{\infty} a\, b_n\, z^n. \tag{1.9}$$

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ we also define the following algebraic operations:

- Differentiation: $A'(z) := \sum_{n\geq 1} n a_n\, z^{n-1} = \sum_{n\geq 0}(n+1)a_{n+1}\, z^n$.
- Integration: $\int_0^z A(t)\,\mathrm{d}t := \sum_{n\geq 0} \frac{a_n}{n+1}\, z^{n+1} = \sum_{n\geq 1} \frac{a_{n-1}}{n}\, z^n$.

Definition 9. A formal power series $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ is called a *reciprocal* of a formal power series $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ (and vice versa) if $A(z)B(z) = B(z)A(z) = 1$.

Propostion 1. *A formal power series $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ has a reciprocal if and only if a_0 is a unit (i.e. invertible)*[1]*. In that case, the reciprocal is unique and so we denote the reciprocal of $A(z)$ by $\frac{1}{A(z)}$.*

Proof. Exercise! □

In Example 13 we wished to have

$$\begin{aligned} &(1-2z)A(z) = 1 \quad \overset{?}{\iff} \quad A(z) = \frac{1}{1-2z} \\ &\sum_{n=0}^{\infty} z^n \overset{?}{=} \frac{1}{1-z}, \\ &\sum_{n=0}^{\infty} (2a_n + 1)\, z^{n+1} \overset{?}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n. \end{aligned}$$

From definition, equality $(1-2z)A(z) = 1$ and Proposition 1, $A(z)$ is the (unique) reciprocal of $1-2z$ and vice versa, so

$$(1-2z)A(z) = 1 \quad \iff \quad A(z) = \frac{1}{1-2z}.$$

For the second equality $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, we shall show that the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} z^n$ is $B(z) := 1 - z$ and vice versa. Letting $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ with $a_n = 1$ for $n \geq 0$ and $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ with $b_0 = 1, b_1 = -1$ and $b_n = 0$ for $n \geq 2$, we have

$$A(z)B(z) \overset{(1.8)}{=} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n = a_0 b_0 + \sum_{n=1}^{\infty} (a_{n-1} b_1 + a_n b_0)\, z^n = 1.$$

By Proposition 1, $A(z)$ is the unique reciprocal of $B(z)$ and so $A(z) = \frac{1}{B(z)} = \frac{1}{1-z}$, that is, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

in the ring $\mathbb{K}[[z]]$.

Analogously one can show that for any $a \in \mathbb{K}$, the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} (az)^n$ is $B(z) := 1 - az$ and vice versa. In other words, for any $a \in \mathbb{K}$,

$$\sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az} \tag{1.10}$$

in the ring $\mathbb{K}[[z]]$.

[1] If $\mathbb{K}$ is a field, a_0 is a unit if and only if $a_0 \neq 0$

For the third equality $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} \;=\; 2z\,\sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n$, we use (1.7) and (1.8) to have

$$\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} \overset{(1.7)}{=} \quad \sum_{n=0}^{\infty} 2a_n\, z^{n+1} + \sum_{n=0}^{\infty} z^{n+1} \overset{(1.8)}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n.$$

That is, $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} = 2z\,\sum_{n=0}^{\infty} a_n z^n + z\,\sum_{n=0}^{\infty} z^n$ in the ring $\mathbb{K}[[z]]$.

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ and a constant $\beta \in \mathbb{K}$ we have the following properties:

- $A(\beta z) = \sum_{n\geq 0} a_n \beta^n\, z^n$, so $[z^n]\, A(\beta z) = a_n \beta^n = \beta^n a_n = \beta^n\, [z^n]\, A(z)$ (scaling)
- $(A(z)-a_0)/z = \sum_{n\geq 1} a_n\, z^{n-1} = \sum_{n\geq 0} a_{n+1}\, z^n$ (left shift)
- $z\, A(z) = \sum_{n\geq 0} a_n\, z^{n+1} = \sum_{n\geq 1} a_{n-1}\, z^n$ (right shift)
- $A(z)/(1-z) = A(z)\cdot \sum_{n\geq 0}\, z^n = \sum_{n\geq 0}\left(\sum_{0\leq k\leq n} a_k\right) z^n$ (partial sum)

Example 14. *Let us study the recursion for the Fibonacci sequence*

$$\begin{cases} a_0 & = 0 \\ a_1 & = 1 \\ a_{n+2} & = a_{n+1} + a_n, \quad n \geq 0, \end{cases}$$

using the generating function $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$. Multiplying both sides of the recursion by z^{n+2} and summing over $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}\, z^{n+2} &= \sum_{n=0}^{\infty}(a_{n+1}+a_n)\, z^{n+2} \\
\iff \quad \sum_{n=2}^{\infty} a_n z^n &= z \sum_{n=1}^{\infty} a_n\, z^n \;+\; z^2 \sum_{n=0}^{\infty} a_n\, z^n \\
\iff \quad A(z) - z &= z\, A(z) \;+\; z^2\, A(z) \\
\iff \quad (1-z-z^2)\, A(z) &= z \\
\iff \quad A(z) \;=\; \frac{z}{1-z-z^2} &= \frac{1}{\beta_1-\beta_2}\left(\frac{1}{1-\beta_1 z} - \frac{1}{1-\beta_2 z}\right) \\
&\overset{(1.10)}{=} \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}(\beta_1 z)^n - \sum_{n=0}^{\infty}(\beta_2 z)^n\right) \\
&\overset{(1.8)}{=} \left(\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\beta_1^n\, z^n - \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\beta_2^n\, z^n\right) \\
&\overset{(1.7)}{=} \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\beta_1^n - \beta_2^n\right)\, z^n,
\end{aligned}$$

where $\beta_1 = \frac{1+\sqrt{5}}{2}$ *and* $\beta_2 = \frac{1-\sqrt{5}}{2}$. *Therefore we have*

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \quad n \geq 0.$$

1.3.2 Ordinary and Exponential Generating Functions

Throughout the lecture we take $\mathbb{K} = \mathbb{C}$.

Definition 10 (Generating Functions)**.** The *ordinary generating function* (OGF) of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n.$$

The *exponential generating function* (EGF) of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!}.$$

Definition 11 (Combinatorial Class, Counting Sequence, and Generating Function)**.**

- A *combinatorial class* $\mathscr{A}$ is a finite or denumerable set on which a size function is defined such that the size $|\alpha|$ of an element α is a non-negative integer and the number of elements of any given size is finite.
- An element of size 1 in a combinatorial class is an *atom*.
- Given a combinatorial class $\mathscr{A}$ we denote by $\mathscr{A}_n$ the set of elements in $\mathscr{A}$ of size n for any $n \in \mathbb{N}\cup\{0\}$. ($\mathscr{A}_n$ is also a combinatorial class.) Let a_n denote the number of elements in $\mathscr{A}_n$, which is often denoted by $|\mathscr{A}_n|$.
- The *counting sequence of a combinatorial class* $\mathscr{A}$ is the sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ where $a_n = |\mathscr{A}_n|$.
- The *ordinary generating function* of a combinatorial class $\mathscr{A}$ is the ordinary generating function of the sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ with $a_n = |\mathscr{A}_n|$. Therefore we have

  $$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}.$$

  Notation: $[z^n]A(z) := a_n$.

- The *exponential generating function of a combinatorial class* $\mathscr{A}$ is the exponential generating function of the sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ with $a_n = |\mathscr{A}_n|$. Therefore we have

$$A(z) := \sum_{n=0}^{\infty} \frac{a_n}{n!}\, z^n = \sum_{\alpha\in\mathscr{A}} \frac{1}{|\alpha|!}\, z^{|\alpha|}.$$

Notation: $[z^n]A(z) := \frac{a_n}{n!}$.

We say the variable z *marks the size* in the generating function $A(z)$.

Two combinatorial classes are said to be *combinatorially equivalent* if their counting sequences are identical.

Lecture 4 on 16.10.2018

Propostion 2. *For the product of two exponential generating functions we have*

$$\left(\sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} b_n\, \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}\, a_k b_{n-k}\right) \frac{z^n}{n!}.$$

Proof. Exercise! □

Example 15 (Triangulations of Convex Polygon)**.** *Let $\mathscr{T}$ denote the class of all triangulation of convex polygons, in which one edge of each convex polygon is distinguished. The size of a triangulation in $\mathscr{T}$ is defined as the number of triangles it is composed of. Then $\mathscr{T}$ is a combinatorial class. For $n \geq 1$ we let $\mathscr{T}_n$ denote the class of all triangulation of convex polygons of size n and let $t_n := |\mathscr{T}_n|$. Set $t_0 := 1$. That is, $\mathscr{T}_n$ is the set of all triangulations of convex $(n+2)$-gons (i.e. with n triangles) with one distinguished edge for $n \geq 0$.*

The sequence $(t_n)_{n\in\mathbb{N}\cup\{0\}}$ begins with $1, 1, 2, 5, 14, 42, \ldots$

By deleting the triangle incident to the distinguished edge we obtain the recursion

$$t_n = \sum_{k=0}^{n-1} t_k\, t_{n-1-k}, \quad n \geq 1.$$

Let $T(z) := \sum_{n=0}^{\infty} t_n\, z^n$ be the ordinary generating function of $\mathscr{T}$. Multiplying both sides of $t_n = \sum_{k=0}^{n-1} t_k\, t_{n-1-k}$ by z^n and summing over $n \geq 1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} t_n\, z^n &= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} t_k\, t_{n-1-k}\right) z^n \\
\iff \quad T(z) - t_0 &= z \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} t_k\, t_{n-1-k}\right) z^{n-1} \\
\iff \quad T(z) - 1 &= z \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} t_k\, t_{n-k}\right) z^n \\
T(z) - 1 &\overset{(1.9)}{=} z\, T(z)^2.
\end{aligned}$$

Therefore, $T(z)$ satisfies the quadratic equation

$$z\,T(z)^2 - T(z) + 1 \;=\; 0.$$

Now we consider $T(z)$ as a function defined on $\mathscr{C}$. Then the correct solution is

$$T(z) \;=\; \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative and so $T(z)$ is increasing along the positive real-axis. Now we use the generalized binomial theorem 3 to get

$$\sqrt{1-4z} \;=\; (1-4z)^{1/2} = \sum_{k\geq 0}\binom{\frac{1}{2}}{k}(-4z)^k = 1 + \sum_{k\geq 1}\binom{\frac{1}{2}}{k}(-4z)^k$$

and plug this in the above equation to get

$$T(z) \;=\; \frac{1-\sqrt{1-4z}}{2z} \;=\; \sum_{k\geq 1}\frac{-1}{2}\binom{\frac{1}{2}}{k}(-4z)^k z^{k-1} \;=\; \sum_{k\geq 0}\frac{-1}{2}\binom{\frac{1}{2}}{k+1}(-4)^{k+1}z^k$$

Then use Example 3 to replace $\binom{1/2}{k+1}$ and get

$$\begin{aligned} T(z) = & \sum_{k\geq 1}\frac{-1}{2}\frac{(-1)^k(-1)^{k+1}4^{k+1}}{4^{k+1}(2k+1)}\binom{2(k+1)}{k+1}z^k \;=\; \sum_{k\geq 0}\frac{1}{2}\binom{2(k+1)}{k+1}\frac{1}{2k+1}z^k \\ = & \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}z^k \end{aligned}$$

(last equality easy to check).

From this, we get for $n \geq 1$:

$$t_n = [z^n]T(z) = \frac{1}{n+1}\binom{2n}{n}$$

The number $\frac{1}{n+1}\binom{2n}{n}$ is known as the Catalan number.

Using Stirling's formula

$$n! = \left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of triangulations of size n (i.e. of a convex $n+2$-gon)

$$t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!} = (1+o(1))\frac{1}{\sqrt{\pi}}4^n n^{-3/2}.$$

1.4 Symbolic method

1.4.1 Unlabelled combinatorial objects

For a given combinatorial class $\mathscr{A}$, denote by $\mathscr{A}_n$ the set of elements of size n in $\mathscr{A}$, and let $a_n = |\mathscr{A}_n|$. We say the OGF

$$A(z) = \sum_{n\geq 0} a_n z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}$$

enumerates $\mathscr{A}$.

Basic constructions and OGF's

(1) $\mathscr{E}$ is the *neutral class* that consists of a single element of size 0.
The OGF of $\mathscr{E}$ is 1.

(2) $\mathscr{Z}$ is the *atomic class* that consists of a single element of size 1.
The OGF of $\mathscr{Z}$ is z.

(3) The *combinatorial sum (disjoint union)* $\mathscr{A}+\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A}\cap\mathscr{B}=\emptyset$ is the set of objects consisting of two disjoint copies of $\mathscr{A}$ and $\mathscr{B}$, in which the size of an element $\alpha\in\mathscr{A}+\mathscr{B}$ is defined as $|\alpha|_{\mathscr{A}}$ if $\alpha\in\mathscr{A}$ and $|\alpha|_{\mathscr{B}}$ if $\alpha\in\mathscr{B}$ (i.e. the size of an element in $\alpha\in\mathscr{A}+\mathscr{B}$ is inherited from its size in its class of origin). In order to formalise $\mathscr{A}+\mathscr{B}$ we introduce red marker to $\mathscr{A}$ and blue marker to $\mathscr{B}$. The combinatorial sum $\mathscr{A}+\mathscr{B}$ is a well-defined combinatorial class. Its OGF satisfies

$$\sum_{\alpha\in\mathscr{A}+\mathscr{B}} z^{|\alpha|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|} + \sum_{\alpha\in\mathscr{B}} z^{|\alpha|}.$$

(4) *The Cartesian product* $\mathscr{A}\times\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A}\times\mathscr{B} = \{(\alpha,\beta) \mid \alpha\in\mathscr{A},\beta\in\mathscr{B}\},$$

in which the size of a pair (α,β) is defined as $|\alpha|+|\beta|$. The Cartesian product $\mathscr{A}\times\mathscr{B}$ is a well-defined combinatorial class. Its OGF is

$$\sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|\alpha|+|\beta|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}\cdot\sum_{\beta\in\mathscr{B}} z^{|\beta|} = A(z)\cdot B(z).$$

For any $n\geq 1$, we define $\mathscr{A}^n = \mathscr{A}^{n-1}\times\mathscr{A} = \mathscr{A}\times\cdots\times\mathscr{A}$ recursively.

(5) The *sequence* $\mathrm{SEQ}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0=\emptyset$ that consists of sequences of elements from $\mathscr{A}$ is the infinite sum

$$\mathscr{E}+\mathscr{A}+\mathscr{A}^2++\mathscr{A}^3+\ldots.$$

Note that the condition $\mathscr{A}_0 = \emptyset$ (equiv. $a_0 = 0$) guarantees the finiteness condition for sizes, and therefore the sequence construction $\text{SEQ}(\mathscr{A})$ is a well-defined combinatorial class. Its OGF is

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)},$$

where the latter equality is because $[z^0]A(z) = 0$.

(6) The *multiset construction* $\text{MSET}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite multisets (i.e. repetition allowed) of elements from $\mathscr{A}$, more precisely, we define

$$\text{MSET}(\mathscr{A}) := \text{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is the equivalence class of sequences defined by

$$(\alpha_1, \ldots, \alpha_n) \sim_{\text{R}} (\beta_1, \ldots, \beta_n)$$

iff there is a permutation σ of $[n]$ such that for all $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$.

We determine its generating sequence.

Case $\mathscr{A}$ finite

For a finite set $\mathscr{A}$, we let $\alpha_1, \alpha_2, \ldots$ be a canonical listing of the elements of $\mathscr{A}$. Then any multiset can be sorted in such a way that it can be viewed as formed by a sequence of repeated elements of α_1, followed by a sequence of repeated elements of α_2, and so on. It follows that

$$\text{MSET}(\mathscr{A}) = \prod_{\alpha \in \mathscr{A}} \text{SEQ}(\alpha)$$

(a finite product). Therefore, the OGF of $\mathscr{C} = \text{MSET}(\mathscr{A})$ is

$$C(z) = \prod_{\alpha \in \mathscr{A}} \frac{1}{1 - z^{|\alpha|}} = \prod_{n \geq 1} \left(\frac{1}{1 - z^n} \right)^{a_n}.$$

Consider the formal power series

$$\exp(z) := \sum_{n \geq 0} \frac{z^n}{n!}, \quad \ln(z) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (z-1)^n.$$

Note that $\exp(\ln(z)) = z$, $\ln(z^n) = n \cdot \ln(z)$ and $\ln\left(\frac{1}{1-z}\right) = \sum_{n \geq 1} \frac{z^n}{n}$.

Applying the exp-ln transformation to $C(z)$ we have

$$\begin{aligned}
C(z) &= \exp(\ln(C(z))) \\
&= \exp\Big(\ln\Big(\prod_{n\geq 1}\Big(\frac{1}{1-z^n}\Big)^{a_n}\Big)\Big) \\
&= \exp\Big(\sum_{n\geq 1} a_n \ln\Big(\frac{1}{1-z^n}\Big)\Big) \\
&= \exp\Big(\sum_{n\geq 1} a_n \sum_{k\geq 1}\frac{(z^n)^k}{k}\Big) \\
&= \exp\Big(\sum_{k\geq 1}\frac{1}{k}\sum_{n\geq 1} a_n (z^k)^n\Big) \\
&= \exp\Big(\sum_{k\geq 1}\frac{1}{k}A(z^k)\Big) \\
&= \exp\left(A(z)+\frac{A(z^2)}{2}+\frac{A(z^3)}{3}+\dots\right).
\end{aligned}$$

Infinite class $\mathscr{A}$: The case of infinite class $\mathscr{A}$ follows by a limit argument.

(7) The *power set construction* $\mathrm{PS\small{ET}}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite subsets of $\mathscr{A}$ (without repetition).

For a finite set $\mathscr{A}$, we have

$$\mathrm{PSET}(\mathscr{A}) = \prod_{\alpha\in\mathscr{A}}(\mathscr{E}+\{\alpha\}),$$

because distributing the products in all possible ways forms all possible combinations of elements of $\mathscr{A}$ (i.e. the sets of elements from $\mathscr{A}$ without repetition). So, the OGF of $\mathscr{C} = \mathrm{PSET}(\mathscr{A})$ is

$$\begin{aligned}
C(z) &= \prod_{\alpha\in\mathscr{A}}(1+z^{|\alpha|}) = \prod_{n\geq 1}(1+z^n)^{a_n} \\
&= \exp\Big(\ln\Big(\prod_{n\geq 1}(1+z^n)^{a_n}\Big)\Big) \\
&= \exp\Big(\sum_{n\geq 1} a_n \ln(1+z^n)\Big) \\
&= \exp\Big(\sum_{n\geq 1} a_n \sum_{k\geq 1}(-1)^{k-1}\frac{z^{nk}}{k}\Big) \\
&= \exp\Big(\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}A(z^k)\Big) \\
&= \exp\left(A(z)-\frac{A(z^2)}{2}+\frac{A(z^3)}{3}\pm\dots\right).
\end{aligned}$$

(8) A further construction is the *substitution*. Let $\mathscr{B}$ and $\mathscr{C}$ be combinatorial classes. Then the substitution of $\mathscr{C}$ into $\mathscr{B}$ (or the *composition of $\mathscr{B}$ and $\mathscr{C}$*), denoted by $\mathscr{B} \circ \mathscr{C}$ or $\mathscr{B}[\mathscr{C}]$ is defined as

$$\mathscr{B} \circ \mathscr{C} = \mathscr{B}[\mathscr{C}] = \sum_{k \geq 0} \mathscr{B}_k \times \mathrm{SEQ}_k(\mathscr{C}).$$

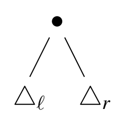
$\mathscr{B}[\mathscr{C}]$ means "substitute elements of $\mathscr{C}$ for atoms of $\mathscr{B}$".

Lecture 4 on 10.10.2017

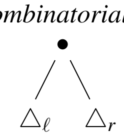
Applications

Example 16. *A* binary tree *is a combinatorial structure that is recursively defined such that*

- *it is either a single external node $\circ$, or*
- *it consists of an internal node (the root $\bullet$) and two binary trees attached to the root (left tree $\triangle_\ell$ and right tree $\triangle_r$),*

$\circ$ *or*  .

Let $\mathscr{B}$ denote the class of all binary trees, in which the size of a binary tree is defined as the number of internal nodes. Then $\mathscr{B}$ is a combinatorial class. For $n \geq 0$ we let $\mathscr{B}_n$ denote the class of all binary trees of size n and let $b_n := |\mathscr{B}_n|$. (Note that if a binary tree has n internal nodes, then it has $n+1$ external nodes. Thus b_n counts the number of binary trees with n internal nodes and $n+1$ external nodes.)

The sequence $(b_n)_{n \in \mathbb{N} \cup \{0\}}$ begins with $1, 1, 2, 5, 14, 42, \ldots$ Let $B(z) := \sum_{n=0}^{\infty} b_n z^n$ be the ordinary generating function of the combinatorial class $\mathscr{B}$. We have $b_0 = 1$, since the only (binary) tree is $\circ$ and for $n \geq 1$,  .

Thus its OGF satisfies

$$B(z) = 1 + zB(z)^2.$$

The solution for the quadratic equation

$$zB(z)^2 - B(z) + 1 = 0$$

is

$$B(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Since $b_n \geq 0$, $B(z)$ increases along the real axis and therefore it increases. Thus the correct solution of $B(z)$ is

$$B(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

As in Example 15 we get for $n \geq 1$:

$$b_n = [z^n]B(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 9 (Number of binary trees)**.** *The number b_n of binary trees with n internal nodes (equiv. $n+1$ external nodes) is given by the so-called Catalan number*

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

Remark 2. Using Stirling's formula

$$n! = \left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of binary trees

$$b_n = \frac{1}{n+1}\binom{2n}{n} = (1+o(1))\frac{1}{\sqrt{\pi}}4^n n^{-3/2}.$$

Example 17. *Let us return to triangulations of convex polygons we saw in Example 15.*

Let $\mathscr{T}_n$ be the set of all triangulations of convex $(n+2)$-gons (n triangles) with one edge distinguished. Using the basic constructions described above we have

$$\mathscr{T} = \biguplus_{n\geq 0}\mathscr{T}_n = \sum_{n\geq 0}\mathscr{T}_n, \qquad \mathscr{T}_0 = \mathscr{E}, \quad \mathscr{T}_1 = \mathscr{Z},$$

where $\mathscr{T}_0$ contains only one edge and $\mathscr{T}_1$ one triangle. By removing one edge of the convex $(n+2)$-gon, we end up with two separate convex triangulations (sharing one node), so

$$\mathscr{T} \;=\; \mathscr{E} + \mathscr{T}\times\mathscr{Z}\times\mathscr{T}$$

$$T(z) \;=\; 1 + zT^2(z),$$

among whose two solutions we choose

$$T(z) \;=\; \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative.

As in Example 15,

$$t_n = [z^n]T(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \ldots = \frac{1}{n+1}\binom{2n}{n}.$$

This suggests a bijection between binary trees and triangulations, where each internal node corresponds to one triangle. Alternatively, the root node of a binary tree B corresponds to the distinguished edge of a triangulation T, and each internal node of B to a diagonal edge of T, and each external node to the external edges of T except the distinguished edge of T.

Example 18. *Let $\mathscr{S}$ be the set of binary strings with no two consecutive 0 bits; for example,* $\emptyset, 0, 1, 01, 10, 11, 010, 011, \ldots$.

$$\mathscr{S} = \mathscr{E} + \{0\} + \{1\} \times \mathscr{S} + \{01\} \times \mathscr{S}.$$

From (3) and (4) in Section 1.4.1 we get

$$S(z) = 1 + z + zS(z) + z^2 S(z)$$

and thus

$$S(z) = \frac{1+z}{1-z-z^2}.$$

Exercise 3. Let $S(z)$ be as above.

- Find the closed solution form of $s_n = |\mathscr{S}_n|$.
- Find a recurrence of s_n that leads to $S(z)$.

1.4.2 Labelled combinatorial objects

A labelled combinatorial class $\mathscr{A}$ is a combinatorial class, where each atom carries an integer label in such a way that the labels of atoms occurring in an object are distinct, and the collection of the labels of atoms occurring in an object of size n is the complete integer interval $[n]$.

Given a labelled combinatorial class $\mathscr{A}$, we consider the EGF

$$A(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n = \sum_{\alpha \in \mathscr{A}} \frac{1}{|\alpha|!} z^{|\alpha|}.$$

Basic constructions and EGFs

(1) Neutral class $\mathscr{E}$. The EGF of $\mathscr{E}$ is 1.

(2) Atomic class $\mathscr{Z}$. The EGF of $\mathscr{Z}$ is z.

(3) Disjoint union $\mathscr{A} + \mathscr{B}$ of two labelled combinatorial classes $\mathscr{A}$ and $\mathscr{B}$. The EGF of $\mathscr{A} + \mathscr{B}$ is $A(z) + B(z)$.

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(4) Labelled product $\mathscr{A} * \mathscr{B}$ of two labelled combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A} * \mathscr{B} := \sum_{\alpha \in \mathscr{A}, \beta \in \mathscr{B}} \alpha * \beta.$$

Given $\alpha \in \mathscr{A}, \beta \in \mathscr{B}$, let $\alpha * \beta$ denote the set of all pairs (α', β') where the atoms of (α', β') get distinct labels from $[n]$ where $n = |\alpha| + |\beta|$, such that the labelling preserves the relative order of labels of α, β. There are $\binom{|\alpha|+|\beta|}{|\alpha|}$ possibilities for such labelling (choose which labels go to the first substructure, then the previous relative order determines which label belongs to which element). In other words, if $\mathscr{C} = \mathscr{A} * \mathscr{B}$, $C(z) = A(z) \cdot B(z)$.

(5) Labelled sequence $\mathrm{SEQ}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\mathrm{SEQ}(\mathscr{A}) = \mathscr{E} + \mathscr{A} + \mathscr{A} * \mathscr{A} + \ldots = \sum_{k \geq 0} \mathrm{SEQ}_k(\mathscr{A}),$$

where $\mathrm{SEQ}_k(\mathscr{A}) = \mathscr{A} * \cdots * \mathscr{A}$ is the labelled product of k copies of $\mathscr{A}$. The EGF of $\mathrm{SEQ}(\mathscr{A})$ is

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)}.$$

(6) Set $\mathrm{SET}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\mathrm{SET}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is an equivalence relation such that $(\alpha_1, \ldots, \alpha_n) \sim_{\mathscr{R}} (\beta_1, \ldots, \beta_n)$ if there exists a permutation σ of $[n]$ such that for each $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\mathrm{SET}(\mathscr{A})$ is

$$1 + A(z) + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \ldots = \exp(A(z)).$$

(7) Cylces $\mathrm{CYC}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

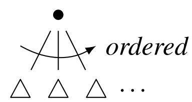
$$\mathrm{CYC}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{S},$$

where $\mathscr{S}$ is the equivalence relation such that $(\alpha_1, \ldots, \alpha_n) \sim_{\mathscr{R}} (\beta_1, \ldots, \beta_n)$ if there exists a cyclic permutation σ of $[n]$ such that for each $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\mathrm{CYC}(\mathscr{A})$ is

$$1 + A(z) + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \ldots = \log \frac{1}{1 - A(z)}.$$

Recall that a tree is a connected graph without cycles.

Example 19 (Plane trees, ordered trees)**.** Plane trees *are trees embedded in the plane, so that subtrees attached to each vertex are ordered, say from left to right. There is a root vertex, implicitly defined; to a root vertex, subtrees are attached in a specified order.*



So,

$$\mathscr{P} = \mathscr{Z} * \textsc{Seq}(\mathscr{P}).$$

The generating function is then

$$P(z) = z \cdot \frac{1}{1 - P(z)}.$$

This yields a quadratic equation $P(z)^2 - P(z) + z = 0$. As previously, since $P(z)$ increases along the real axis, the correct solution is

$$P(z) = \frac{1 - \sqrt{1-4z}}{2} = z \cdot B(z),$$

where $B(z)$ is the OGF of binary trees.

Let p_n be the number of plane trees on n vertices, and b_n the number of binary trees on $n+1$ external nodes, so $p_{n+1} = b_n$. $P(z) = zB(z)$ suggests that there is a combinatorial bijection between plane trees on $n+1$ vertices and binary trees on $n+1$ external nodes.

Exercise 4. Find such a bijection between plane trees and binary trees.

Example 20 (2-regular graphs)**.** *Let $\mathscr{R}$ be the set of all 2-regular labelled graphs, i.e. each vertex has exactly 2 neighbours. Note that connected 2-regular graphs are undirected cycles of length ≥ 3.*

$$\mathscr{R} = \textsc{Set}(\textit{connected 2-regular graphs}) = \textsc{Set}(\textsc{UCyc}_{\geq 3}(\mathscr{Z})),$$

where UCYC *is the undirected cyclic construction. Then, the generating function is*

$$R(z) = \exp\left(\textsc{UCyc}_{\geq 3}(z)\right) = \frac{e^{-z/2 - z^2/4}}{\sqrt{1-z}},$$

since the generating function for $\textsc{UCyc}_{\geq 3}(\mathscr{Z})$ *is* $\frac{1}{2}\left(\log(\frac{1}{1-z}) - z - \frac{z^2}{2}\right)$.

How can we derive $[z^n]R(z)$?

1.5 Analytic Methods

1.5.1 Analytic functions

Definition 12. Let $\Omega \subset \mathbb{C}$ be a region (i.e. open and connected). A function $f : \mathbb{C} \to \mathbb{C}$ is called *complex differentiable* or *holomorphic at* $z_0 \in \Omega$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. It is called *holomorphic in* Ω if f is holomorphic for every point in Ω.

Definition 13. Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \mathbb{C} \to \mathbb{C}$ is called *analytic at* $z_0 \in \Omega$ if $\exists\, \varepsilon > 0$ such that $\forall z \in B_\varepsilon(z_0)$, $f(z)$ is representable by a convergent power series expansion around z_0

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n, \quad \text{for some } c_n \in \mathbb{C}.$$

It is called *analytic in* Ω if f is analytic for every point in Ω.

Theorem 10. *Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in Ω iff it is analytic in Ω.*

Definition 14. Let $\Omega \subset \mathbb{C}$ be a region and $f : \mathbb{C} \to \mathbb{C}$ be holomorphic at $z_0 \in \Omega$. Set $f^{(0)}(z_0) := f(z_0)$. Then the power series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is called the *Taylor series expansion of $f(z)$ around z_0*.

Exercise 5. Find the maximum regions, in which the functions $z \mapsto e^z$, $z \mapsto \frac{1}{1-z}$, $z \mapsto (1+z)^m$ (for fixed $m \in \mathbb{N}$) are analytic. Check that their Taylor series expansions around the origin are given by

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}, \qquad \frac{1}{1-z} = \sum_{n \geq 0} z^n, \qquad (1+z)^m = \sum_{n \geq 0} \binom{m}{n} z^n.$$

Exercise 6. Find the Taylor series expansions of

$$\frac{1}{\sqrt{1-4z}}, \quad \sin z, \quad \cos z, \quad z \cdot e^z, \quad \ln z, \quad \frac{1}{1-z} \ln \frac{1}{1-z}, \quad \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}$$

around the origin and find the maximum regions where these power series converge.

Exercise 7. Prove

$$\sum_{n \geq m} \binom{n}{m} z^n = \frac{z^m}{(1-z)^{m+1}}.$$

Exercise 8. Prove
$$\sum_{n\geq 0} H_n z^n = \frac{1}{1-z}\ln\frac{1}{1-z}$$
where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the n-th *harmonic number*.

Exercise 9. Find OGF for each of the following sequences:
$$\{2^{n+1}\}_{n\geq 0}, \qquad \{n2^{n+1}\}_{n\geq 0}, \qquad \{nH_n\}_{n\geq 1}, \qquad \{n^3\}_{n\geq 2}.$$

1.5.2 Cauchy's coefficient formula

Theorem 11 (Cauchy's coefficient formula, 1st version)**.** *Let $\Omega \subset \mathbb{C}$ be a region, f be analytic in Ω and γ be a simple loop in Ω. Then*
$$\int_\gamma f = 0.$$

Theorem 12 (Cauchy's coefficient formula, 2nd version)**.** *Let $\Omega \subset \mathbb{C}$ be a region and f be analytic in Ω. Let $z_0 \in \Omega$ and γ be a simple loop in Ω encircling z_0 exactly once. Then*
$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$
In general, for $n \geq 0$
$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

1.5.3 Lagrange Inversion Theorem

Theorem 13 (Langrange Inversion, 1st version)**.** *Let $\phi(u) = \sum_{k\geq 0}\phi_k u^k$ be a power series in $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then the equation*
$$A = z\phi(A)$$
admits a unique solution in $\mathbb{C}[[z]]$ and the coefficients of this solution
$$A(z) = \sum_{n\geq 0} a_n z^n$$
are given by $a_0 = 0$ and, for $n \geq 1$,
$$a_n = \frac{1}{n}[u^{n-1}]\phi(u)^n.$$
Furthermore, for any holomorphic function H,
$$[z^n]H(A(z)) = \frac{1}{n}[u^{n-1}]\left(H'(u)\phi(u)^n\right).$$
In particular, for $k \geq 1$,
$$[z^n]A(z)^k = \frac{k}{n}[u^{n-k}]\phi(u)^n.$$

Theorem 14 (Langrange Inversion, 2nd version)**.** *Let the generating function*

$$A(z) = \sum_{n\geq 0} a_n z^n$$

satisfy the functional equation

$$z = \psi(A(z)),$$

where $\psi(0) = 0$ *but* $\psi'(0) \neq 0$ *(i.e.* ψ *is the compositional inverse of* A*).*

If

$$\psi(A) = \frac{A}{\phi(A)},$$

i.e. $A = \psi(A) \cdot \phi(A) = z\phi(A)$*, then*

$$a_n = \frac{1}{n}[z^{n-1}]\phi(z)^n = \frac{1}{n}[z^{n-1}]\left(\frac{z}{\psi(z)}\right)^n.$$

Application of Lagrange Inversion to Binary Trees

Let $\tilde{\mathscr{B}}$ denote the class of all binary trees, in which the size of a binary tree is defined as the total number of all vertices (internal nodes and leaves alike). For $n \geq 1$ we let $\tilde{\mathscr{B}}_n$ denote the class of all binary trees of size n and let $\tilde{b}_n := |\tilde{\mathscr{B}}_n|$ and set $\tilde{b}_0 = 0$.

Let $\tilde{B}(z)$ be the ordinary generating function of the combinatorial class $\tilde{\mathscr{B}}$. Then

$$\tilde{B}(z) \;=\; z \;+\; z\,\tilde{B}(z)^2 \tag{1.11}$$

Let $\tilde{B} = \tilde{B}(z)$ and $\phi(u) := 1 + u^2$. Then (1.11) can be rewritten as

$$\tilde{B} \;=\; z\,\phi(\tilde{B}) \tag{1.12}$$

Note that $\phi(u) = 1 + u^2 = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.12) admits a unique solution $\tilde{B}(z) := \sum_{n=0}^{\infty} \tilde{b}_n\, z^n$ in the ring $\mathbb{C}[[z]]$ where the coefficients are given by

$$\begin{aligned}
\tilde{b}_n \;&=\; \frac{1}{n}\;[u^{n-1}]\;\phi(u)^n \\
&=\; \frac{1}{n}\;[u^{n-1}]\;(1+u^2)^n \\
&=\; \frac{1}{n}\;[u^{n-1}]\;\left(\sum_{k=0}^{n}\binom{n}{k}\,u^{2k}\right) \\
&=\; \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n}\,\binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

In other words, we have, for any $n \in \mathbb{N}$,

$$\tilde{b}_{2n+1} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

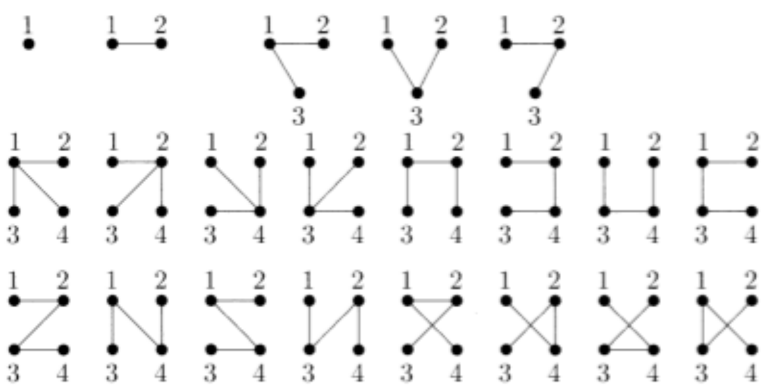
Remark 3. Recall that b_n was defined in Theorem 9 as the number of binary trees of size n, in which the size of a binary tree is defined as the number of internal nodes. If a binary tree has n internal nodes, then it has $n+1$ external nodes and therefore $2n+1$ vertices in total. Thus we have $\tilde{b}_{2n+1} = b_n$.

Application of Lagrange Inversion to Labelled Trees

Definition 15.

- A *labelled tree* (also called *Cayley tree*) is a tree whose vertices are distinctly labelled by numbers in $\mathbb{N}$: The vertices of a Cayley tree on n vertices are labelled by distinct numbers from $[n]$.
- A *rooted labelled tree* on n vertices is a labelled tree on n vertices, in which one vertex is distinguished by a mark from the other vertices.

Let $\mathscr{C}$ denote the class of all labelled trees, in which the size of a tree is defined as the number of vertices, and let $\mathscr{C}_n$ denote the set set of all Cayley tree on n vertices. Then $\mathscr{C}$ is a (labelled) combinatorial class, and $\mathscr{C}_n$ is also a (labelled) class – it is a *subclass* of $\mathscr{C}$. For $n \geq 1$ we let $c_n := |\mathscr{C}_n|$ and set $c_0 = 0$. Let $C(z) := \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial class $\mathscr{C}$. The sequence $(c_n)_{n\geq 0}$ begins with $0, 1, 1, 3, 16, 125\ldots$

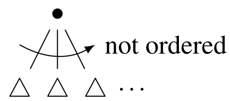


Theorem 15 (Cayley's formula (Cayley 1889))**.**

$$c_n \;=\; n^{n-2}, \quad n \geq 2.$$

Proof. Let $\mathscr{T}$ denote the (labelled combinatorial) class of all rooted labelled trees (i.e. the set of all Cayley trees in $\mathscr{C}$, in which one vertex is distinguished by a mark from the other vertices) and let $\mathscr{T}_n$ denote the class of all rooted labelled tree on n vertices. For $n \geq 1$ we let $t_n := |\mathscr{T}_n|$ and set $t_0 = 0$. Because there are n ways to choose a root vertex of a labelled tree on n vertices, the number of Cayley trees on n vertices is equal to $t_n = n\, c_n$.

Let $T(z) := \sum_{n=0}^{\infty} t_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial labelled class $\mathscr{T}$.



$$\mathscr{T} = \mathscr{Z} * \textsc{Set}(\mathscr{T}).$$

Its EGF

$$T(z) \;=\; z\,\exp(T(z)). \tag{1.13}$$

So $T(z)$ is implicitly defined.

Let $T = T(z)$ and $\phi(u) := \exp(u) = \sum_{k\geq 0} \frac{u^k}{k!}$. Then (1.13) can be rewritten as

$$T \;=\; z\,\phi(T) \tag{1.14}$$

Note that $\phi(u) := \exp(u) = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.14) admits a unique solution in the ring $\mathbb{C}[[u]]$ whose coefficients are given by $T(z) = \sum_{n=0}^{\infty} t_n\, z^n$, where

$$\begin{aligned}\frac{t_n}{n!} \;&=\; \frac{1}{n}\,[u^{n-1}]\;\phi(u)^n \;=\; \frac{1}{n}\,[u^{n-1}]\;(\exp(u))^n \;=\; \frac{1}{n}\,[u^{n-1}]\;\exp(un)\\ &=\; \frac{1}{n}\,[u^{n-1}]\left(\sum_{k=0}^{\infty}\frac{(un)^k}{k!}\right) \;=\; \frac{1}{n}\,\frac{n^{n-1}}{(n-1)!} \;=\; \frac{n^{n-1}}{n!}.\end{aligned}$$

In other words, for any $n \in \mathbb{N}$,

$$t_n = n^{n-1}.$$

Therefore, we have

$$c_n = \frac{t_n}{n} = n^{n-2}.$$

□

Remark 4. There are several interesting proofs for this in "Proofs from the Book" by M. Aigner and G. M. Ziegler. We shall see one proof in Section 2.2.2 (Chapter 2).

Lecture 6 on 23.10.2018

1.5.4 Singularities

Definition 16. Let f be an analytic function in a region Ω and z_0 be a point on the boundary of Ω.

We say f is *analytically continuable at* z_0 if there is an analytic function g defined in a region Ω' containing z_0 such that $g(z) = f(z)$ in $\Omega \cap \Omega'$.

Example 21. $f(z) = \frac{1}{1-z}$ *is analytic for* $|z| < 1$*, and is analytically continuable except for* $z_0 = 1$.

Definition 17. A function f is said to be *singular at* $z_0 \in \overline{\Omega}$ or z_0 is called *a singularity of* f if f is not analytically continuable at z_0.

Theorem 16 (Boundary singularities)**.** *Let f be analytic at the origin and let R be the finite radius of convergence of a power series expansion of f at the origin. Then f has necessarily a singularity on the boundary of the disc of convergence.*

Definition 18. If f is analytic at the origin with radius R of convergence, then a *dominant singularity* is a singularity on the circle around 0 with radius $R > 0$.

Theorem 17 (Pringsheim's Theorem)**.** *Let f be analytic at the origin and $f(z) = \sum_{n\geq 0} f_n z^n$ be its convergent power series expansion at the origin with the radius of convergence $R > 0$. If $f_n \geq 0, \forall n \geq 0$, then the point $z = R$ is a singularity of f; in this case, $z = R$ is a dominant singularity of f.*

The singularity $z = R$ of Theorem 17 is often called *the* dominant singularity of f.

Remark 5. The radius R of convergence of $\sum_{n\geq 0} f_n z^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |f_n|^{\frac{1}{n}}},$$

so we have

$$f_n = R^{-n}\theta(n),$$

where

$$\limsup_{n\to\infty} |\theta(n)|^{\frac{1}{n}} = 1.$$

1.5.5 Meromorphic functions

Definition 19. A function $f(z)$ is *meromorphic at* z_0 if $\forall z$ in a neighbourhood $B_\varepsilon(z_0)$ of z_0 with $z \neq z_0$, it can be represented as

$$f(z) = \frac{h(z)}{g(z)},$$

where $h(z)$ and $g(z)$ are analytic at z_0 and $g(z) \not\equiv 0$. In this case $f(z)$ admits an expansion of the form

$$f(z) = \sum_{n\geq -M} f_n (z - z_0)^n$$

for $z \in B_\varepsilon(z_0)$, for some $M \in \mathbb{Z}$. If $f_{-M} \neq 0$, $M \geq 1$, we say $f(z)$ has a *pole of order M at* z_0. In this case, we have

$$\begin{aligned} f(z) &= f_{-M}(z - z_0)^{-M} + O((z - z_0)^{-M+1}) \\ &= f_{-M}(-z_0)^{-M}\left(1 - \frac{z}{z_0}\right)^{-M} + O\left(\left(1 - \frac{z}{z_0}\right)^{-M+1}\right). \end{aligned}$$

Remark 6. Scaling rule: If $f(z)$ analytic around 0 and singular at z_0, then $g(z) = f(z_0 z)$ is singular at 1. If $z_0 > 0$ is a dominant singularity of $f(z)$, then $f(z)$ admits a convergent series expansion (around 0):

$$f(z) = \sum_{n \geq 0} f_n z^n, \quad |z| < z_0.$$

Therefore, $g(z)$ admits a convergent series expansion

$$g(z) = \sum_{n \geq 0} g_n z^n, \quad |z| < 1, \quad g_n = f_n z_0^n.$$

1.5.6 Newton's generalised binomial theorem

For any $\alpha \in \mathbb{C}$,

$$[z^n](1-z)^{-\alpha} = \binom{-\alpha}{n} \stackrel{?}{=} \binom{n+\alpha-1}{\alpha-1} \stackrel{?}{=} \Theta(n^{\alpha-1}).$$

Theorem 18. *Let* $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

$$[z^n](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right),$$

where the Γ*-function is defined as*

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, \mathrm{d}t.$$

Properties of the Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \, \mathrm{d}t = \sqrt{\pi}.$$

$$\begin{cases} \Gamma(t+1) = t\,\Gamma(t) \\ \Gamma(1) = 1. \end{cases}$$

For $n \in \mathbb{N}$,

$$\Gamma(n+1) = n\Gamma(n) = \ldots = n!\,.$$

1.5.7 Transfer theorem

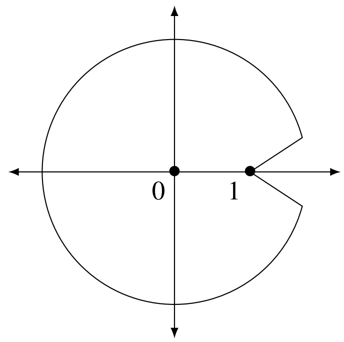
Transfer theorems allow to extract the asymptotic order of coefficients of error terms in singular expansions. For details see chapter VI in [2].

Definition 20. Given two numbers R, ϕ with $R > 1$ and $0 < \phi < \frac{\pi}{2}$, then the set

$$\Delta = \Delta(R, \phi) = \{z \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi\}$$

is called Δ-domain at 1.

A function f is called Δ-analytic at 1 if it is analytic at some Δ-domain at 1.



Theorem 19 (Transfer theorem)**.** *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and assume that $f(z)$ is Δ-analytic at 1.*

If $f(z) = O((1-z)^{-\alpha})$, then

$$[z^n]f(z) = O(n^{\alpha-1}).$$

If $f(z) = o((1-z)^{-\alpha})$, then

$$[z^n]f(z) = o\left(n^{\alpha-1}\right).$$

Therefore, if $f(z) = (1-z)^{-\alpha} + o((1-z)^{-\alpha})$, then

$$[z^n]f(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right) + o(n^{\alpha-1}).$$

Example 22 (2-regular graphs)**.** *Consider the EGF of 2-regular graphs,*

$$\mathscr{G} = \textsc{Set}(\textsc{UCyc}_{\geq 3}\mathscr{Z}).$$

Then

$$G(z) = \exp\left(\frac{1}{2}\left(\log\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2}\right)\right) = \frac{1}{\sqrt{1-z}}e^{-\frac{z}{2}-\frac{z^2}{4}}.$$

$\frac{1}{\sqrt{1-z}}$ *is analytic in $\mathbb{C} \setminus [1,\infty)$. $G(z)$ is in particular Δ-analytic at 1.*

To derive the singular expansion of G (an expansion of the singularity of G) near 1, we use the Taylor expansion of $e^{-\frac{z}{2}-\frac{z^2}{4}}$ at 1.

$$\begin{aligned} h(z) = e^{-\frac{z}{2}-\frac{z^2}{4}} &= h(1) + h'(1)(z-1) + \frac{h''(1)}{2}(z-1)^2 + \ldots \\ &= e^{-\frac{3}{4}} + e^{-\frac{3}{4}}(1-z) + \frac{1}{4}e^{-\frac{3}{4}}(z-1)^2 + \ldots \end{aligned}$$

The singular expansion of $G(z)$ near 1 is

$$\begin{aligned} G(z) &= \frac{e^{-\frac{3}{4}}}{\sqrt{1-z}} + e^{-\frac{3}{4}}(1-z)^{\frac{1}{2}} + e^{-\frac{3}{4}}(1-z)^{\frac{3}{2}} + \ldots \\ &= e^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}} + O\left((1-z)^{\frac{1}{2}}\right). \qquad (z \to 1) \end{aligned}$$

Using the transfer theorem, we get

$$[z^n]G(z) = e^{-\frac{3}{4}} \frac{n^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{n}\right)\right) + O\left(n^{-\frac{3}{2}}\right) = \frac{e^{-\frac{3}{4}}}{\sqrt{n\pi}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

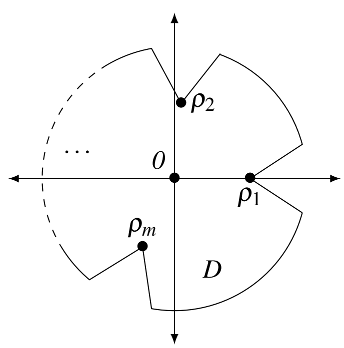
The first term corresponds to $h(1)$, the second is from the theorem.

1.5.8 Multiple singularities

Theorem 20. *Let $f(z)$ be analytic in $|z| < \rho$. Suppose that f has a finite number of singularities on the circle $|z| = \rho$, at points $\rho_j = \rho \cdot e^{i\theta_j}$ for $j = 1, \ldots, m$. Assume there exists a Δ-domain Δ_0 at 1 such that $f(z)$ is analytic in the region*

$$D = \bigcap_{j=1}^{m} \rho_j \Delta_0,$$

where $\rho_j \Delta_0$ is the dilation (rotation) of the Δ-domain Δ_0 for each ρ_j.



Let

$$S = \{(1 - z)^{-\alpha} \mid \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\}.$$

Assume there exist m functions $g_1(z), \ldots, g_m(z)$, each a linear combination of the functions in S, and there exists a function $h(z) = (1 - z)^{-\alpha_0}$ such that

$$f(z) = g_j\left(\frac{z}{\rho_j}\right) + O\left(h\left(\frac{z}{\rho_j}\right)\right) \qquad \text{as } z \to \rho_j \text{ in } D.$$

Then

$$\begin{aligned}[z^n]f(z) &= \sum_{j=1}^{m} \left([z^n]g_j\left(\frac{z}{\rho_j}\right) + O\left([z^n]h\left(\frac{z}{\rho_j}\right)\right)\right) \\ &= \sum_{j=1}^{m} \rho_j^{-n}[z^n]g_j(z) + O\left(\rho_j^{-n} n^{\alpha_0 - 1}\right).\end{aligned}$$

Remark 7.

$$[z^n]g_j(z) = [z^n] \sum_{k=1}^{\ell} c_{jk}(1 - z)^{-\alpha_{j,k}} = \sum_{k=1}^{\ell} c_{jk} \frac{n^{\alpha_{j,k} - 1}}{\Gamma(\alpha_{j,k})}.$$

Remark 8 (Recipe for singularity analysis)**.**

- Find the grammar that determines the combinatorial class and the corresponding generating function
- Locate the singularities
- Check the Δ-analyticity of the generating function
- Do the singular expansion at each singularity separately
- Extract and sum the asymptotics using the Transfer Theorem and basic scaling.

Example 23. *Let $\mathscr{P}$ be the class of permutations with cycles of odd length. Then*

$$\mathscr{P} = \textsc{Set}(\mathscr{A}),$$
$$\mathscr{A}(\mathscr{Z}) = \textsc{Cyc}_{odd}(\mathscr{Z}).$$

For the generating functions, this means

$$A(z) = \sum_{k=odd} \frac{z^k}{k} = \frac{1}{2}\log\frac{1+z}{1-z},$$
$$P(z) = e^{A(z)} = \exp\left(\frac{1}{2}\log\frac{1+z}{1-z}\right) = \sqrt{\frac{1+z}{1-z}}.$$

The dominating singularities of this function are $\{\pm 1\}$. Let Δ_0 be a Δ-domain at 1.

$P(z)$ is analytic in the region $D = \Delta_0 \cap (-1)\Delta_0$ since $\sqrt{1-z}$ is analytic in $\mathbb{C}\setminus[1,\infty)$ and $\sqrt{1+z}$ is analytic in $\mathbb{C}\setminus(-\infty,-1]$.

Singular expansions, with $g(z) = \sqrt{1+z}$ and writing $P^+(z)$ for $P(z)$ as $z \to +1$ in D, $P^-(z)$ for $P(z)$ as $z \to -1$ in D:

$$P^+(z) = \frac{g(1)}{\sqrt{1-z}} + \frac{g'(1)(z-1)}{\sqrt{1-z}} + \dots \qquad \text{(as } z \to +1 \text{ in } D)$$
$$= \frac{\sqrt{2}}{\sqrt{1-z}} - 2^{-\frac{3}{2}}\sqrt{1-z} + O\left((1-z)^{\frac{3}{2}}\right)$$
$$P^-(z) = \frac{1}{\sqrt{2}}\sqrt{1+z} + O\left((1+z)^{\frac{3}{2}}\right). \qquad \text{(as } z \to -1 \text{ in } D)$$

Extract asymptotics:

$$[z^n]P(z) = \frac{2^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} n^{-\frac{1}{2}} - \frac{2^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)} n^{-\frac{3}{2}} + O\left(n^{-\frac{5}{2}}\right)$$
$$+ \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} n^{-\frac{3}{2}} + O\left(n^{-\frac{5}{2}}\right)$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

Chapter 2

Graph Theory

Lecture 7 on 29.10.2018

Suggested reading for Chapter 2: [1] and [4].

2.1 Matchings, Eulerian Tours, and Hamiltonian Cycles

2.1.1 Basic Terminologies

Notation 1 (Subsets)**.** *Given a set X and $k \in \mathbb{N}$, we denote by $\binom{X}{k}$ the set of all k-element subsets of X.*

Definition 21 (Graphs)**.** A graph is a pair (V,E) where $E \subseteq \binom{V}{2}$.
The elements of V are called *vertices (or nodes, points)* and the elements of E are called *edges*. Two vertices v,w are called *adjacent or neighbours* if $\{v,w\} \in E$. A vertex v is said to be *incident* with an edge $e \in E$ if $v \in e$. We call v an *end (vertex)* of e. Edges are called *adjacent* if they have a common end vertex.

Note that all graphs considered will be simple (by the above definition) and finite (finitely many vertices, thus finitely many edges).
If G is a graph, we will write $V(G)$ to denote the set of vertices of G and $E(G)$ to denote the set of edges of G.

Definition 22 (Walk, Paths, Cycles, Distance)**.** Let $G = (V,E)$ be a graph.

(1) A *walk* in G is an alternating sequence

$$W = v_0 e_0 v_1 e_1 v_2 \dots v_{k-1} e_{k-1} v_k$$

of vertices and edges of G such that $e_i = \{v_i, v_{i+1}\}$ for each $0 \le i \le k-1$. It is said to be *closed* if $v_0 = v_k$. The *length of* W is k, i.e. the number of its edges.

(2) A walk $v_0 e_0 v_1 e_1 v_2 \dots v_{k-1} e_{k-1} v_k$ in G is called a *path* if $v_0, v_1, \dots, v_k$ are all distinct. We write $P = v_0 v_1 v_2 \dots v_{k-1} v_k$ and call it a v_0–v_k path.

(3) A *cycle* in G is a closed walk $C = v_0 e_0 v_1 e_1 v_2 \dots v_{k-1} e_{k-1} v_k$ such that $v_0, v_1, \dots, v_{k-1}$ are distinct.

(4) The *distance of two vertices* v,w in G is the length of a shortest v-w path in G.

Definition 23 (Subgraphs, induced subgraphs, spanning subgraphs)**.** Given two graphs $G = (V,E)$ and $G' = (V',E')$, we set

$$G \cup G' := (V \cup V', E \cup E')$$
$$G \cap G' := (V \cap V', E \cap E').$$

(1) If $G \cap G = \emptyset$, then we say that G and G' are *disjoint*.

(2) If $V \subset V'$ and $E \subset E'$, then G is a *subgraph* of G'. We write $G \subseteq G'$.

(3) If $G' \subset G$ and $G' \neq G$, then G' is a *proper subgraph* of G. We write $G' \subsetneq G$.

(4) If $G' \subseteq G$ and G' contains all edges $\{v,w\} \in E$ with $v,w \in V'$, then G' is called an *induced subgraph* of G. We write $G' = G[V']$.

(5) If $G' \subseteq G$ and $V = V'$, then G' is called a *spanning subgraph* of G.

Definition 24 (Components, connectedness)**.** Let $G = (V,E)$ be a graph.

(1) G is called *connected* if G contains a v–w path between any two vertices $v,w \in V$.

(2) A maximal connected subgraph of G is called a *component* of G.

(3) For $k \in \mathbb{N}$, G is called *k-connected* if $|V| > k$ and $G[V \setminus X]$ is connected for every set $X \subseteq V$ with $|X| < k$.

Definition 25 (Degrees)**.** Let $G = (V,E)$ be a graph.

(1) For $v \in V$,
$$d(v) := |\{w \in V : \{v,w\} \in E\}|$$
is called the *degree* of the vertex v. A vertex of degree 0 is called *isolated*.

(2) The number
$$\delta(G) := \min |\{d(v) : v \in V\}|$$
is called *minimum degree* of G, the number
$$\Delta(G) := \max |\{d(v) : v \in V\}|$$
is called *maximum degree* of G, and the number
$$d(G) := \frac{1}{|V|} \sum_{v \in V} d(v)$$
is called *average degree* of G.

(3) If all the vertices have the same degree d, then G is called *d-regular*.

A 3-regular graph is also called *cubic*.

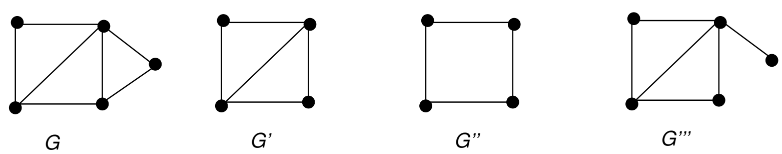
Remark 9. Clearly we have
$$|V| d(G) = \sum_{v \in V} d(v) = 2|E|$$
$$\delta(G) \quad \leq \quad d(G) \quad \leq \quad \Delta(G)$$

Definition 26 (Graph isomorphism)**.** We say that two graphs $G = (V,E)$ and $G' = (V',E')$ are *isomorphic* and write $G \simeq G'$, if there is a bijection $\phi : V \to V'$ with
$$\{v,w\} \in E \quad \Leftrightarrow \quad \{\phi(v), \phi(w)\} \in E', \forall v,w \in V.$$

Definition 27 (Graph property)**.** A class of graphs that is closed under isomorphism is called a *graph property*.

An example of a graph property is the class of graphs containing a triangle. A graph property only depends on the abstract structure, not on a representation of a graph (like a drawing).

Example 24.



(1) Determine whether G', G'', G''' are induced subgraphs of G or not

(2) Determine whether G', G'', G''' are spanning subgraphs of G or not

2.1.2 Matchings

Definition 28 (Independent set, matchings, vertex-cover)**.** Let $G = (V, E)$ be a graph.

(1) A set of vertices or a set of edges of G is called *independent* if no two of its elements are adjacent. An independent set is also called a *stable set*.

(2) A set M of independent edges in G is called a *matching* of G. In other words, a matching is a set of pairwise non-adjacent edges.

A vertex is called *matched (or saturated)* if it is an endpoint of an edge in the matching.

(3) A d-regular spanning subgraph of G is called a d-factor of G.

A 1-factor (indeed, its edge set) is called a *perfect matching*.

(4) A set $U \subseteq V$ is called a *vertex cover* of G if every edge of G is incident with a vertex in U.

(5) A *maximal matching* is a matching which is maximal with respect to inclusion (of edges). A *maximum matching* is a matching that contains the largest possible number of edges.

Note that every maximum matching is maximal but not every maximal matching is a maximum matching. As an example take the diagonal edge in the graph G' of Example 24. This gives a maximal matching but choosing the two horizontal edges gives a matching with two edges, a maximum matching in this case.

Notation 2 (Complete graph)**.** *By K_n, we denote the* complete graph *on n vertices, i.e.* $|V(K_n)| = n$ *and* $E(K_n) = \binom{V(K_n)}{2}$.

Definition 29 (Independence number, clique number)**.** Let $G = (V,E)$ be a graph.

(1) The *independence number of* G, denoted by, $\alpha(G)$, is the number of vertices in a maximum independent set (of vertices) in G.

(2) The *clique number of* G, denoted by $\omega(G)$, is the number of vertices in a maximum clique (=complete graph) in G.

Example 25. *In Example 24,* $\alpha(G) =$?? *and* $\omega(G) =$??.

Remark 10 (Perfect matching)**.** Let $H \subseteq G$ be a subgraph of $G = (V,E)$.

$H \subseteq G$ is a 1-factor
$\iff$ $E(H)$ is a perfect matching of G
$\iff$ Every vertex in G is incident to exactly one edge of H

Definition 30 (Bipartite graph)**.** A graph $G = (V,E)$ is called *bipartite* if V admits a partition into two sets, $V = A \dot{\cup} B$, such that every edge $e \in E$ has its ends in different classes.

Lemma 21. *A graph* $G = (V,E)$ *is bipartite if and only if it contains no odd cycle*

Proof. Exercise! □

Notation 3 (Complete bipartite graph)**.** *By $K_{s,t}$, we denote the* complete bipartite graph *with* $V(K_{s,t}) = A \dot{\cup} B$, $|A| = s$, $|B| = t$, *and* $E(K_{s,t}) = \big\{\{a,b\} \colon a \in A, b \in B\big\}$.

Problem 1 (Maximum bipartite matching)**.**

Input: A bipartite graph $G = (A \dot{\cup} B, E)$
Output: A matching in G with as many edges as possible

Definition 31 (Alternating paths, augmenting paths)**.** Let $G = (A \dot{\cup} B, E)$ be a bipartite graph and M be an arbitrary matching in G.

(1) A path which begins with an unmatched vertex and contains alternatingly edges from $E \setminus M$ and from M is called an *alternating path* (with respect to M).

(2) An alternating path that ends also at an unmatched vertex is called an *augmenting path*.

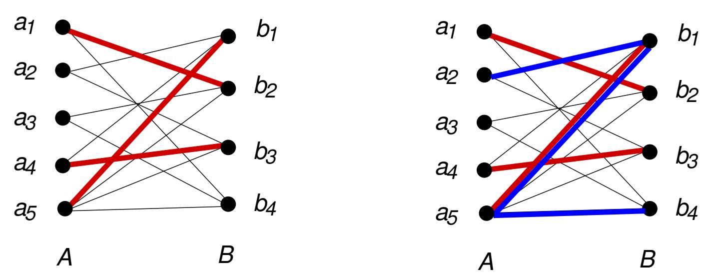
Remark 11. An augmenting path P can be used to turn M into a larger matching

$$M \triangle E(P) \; := \; (M \setminus E(P)) \; \cup \; (E(P) \setminus M)$$

It has one edge more than M and it covers all the vertices covered by M, plus the ends of P.

Example 26.

Let $G = (A \dot{\cup} B, E)$ be a bipartite graph given below.



- $M = \{\{a_1, b_2\}, \{a_4, b_3\}, \{a_5, b_1\}\}$ is a matching
- $P = a_2\ \{a_2, b_1\}\ b_1\ \{b_1, a_5\}\ a_5\ \{a_5, b_4\}\ b_4$ is an augmenting path (with respect to M)
- $M \triangle E(P) = \{\{a_1, b_2\}, \{a_4, b_3\}, \{a_2, b_1\}, \{a_5, b_4\}\}$ is also a matching

Exercise 1. *Let M be a matching in a bipartite graph $G = (A \dot{\cup} B, E)$.*

- *Show that if M contains fewer edges than some other matching in G (i.e. M is suboptimal), then G contains an augmenting path with respect to M.*
- *Describe an algorithm that finds as efficiently as possible a matching of maximum cardinality.*

Theorem 22 (König 1931)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph.*

The number of edges in a maximum matching in G equals the number of vertices in a minimum vertex cover of G.

Proof. Exercise! □

Theorem 23 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph.*

G contains a matching that saturates all vertices in A if and only if

$$|N(S)| \geq |S|, \quad \forall S \subseteq A$$

where $N(S) = \{v \in B : \{u, v\} \in E \text{ for some } u \in S\}$ is the neighborhood of S in B.

Proof. Exercise! □

2.1.3 Eulerian Tours and Hamiltorian Cycles

Definition 32 (Eulerian tour)**.** A closed walk in a graph is called *Eulerian tour* if it traverses (= visits) every edge of the graph exactly once. A graph is called *Eulerian* if it contains an Eulerian tour.

Theorem 24 (Euler 1736)**.** *A connected graph is Eulerian if and only if every vertex has even degree.*

Proof. Exercise! □

Definition 33 (Hamiltonian cycle)**.** A *Hamiltonian cycle* in a graph G is a closed walk that contains every vertex in G exactly once.

If $|V(G)| \geq 3$, any such a walk forms a cycle. Therefore, such a walk is called a Hamiltonian *cycle*.

Problem 2.
When does a graph contain a Hamiltonian cycle?

Theorem 25 (Dirac 1952)**.** *Every graph $G = (V,E)$ with $|V| \geq 3$ and minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle.*

Proof. Exercise! (cf. Ore's theorem) □

Definition 34 (Graph power)**.**

Given a graph $G = (V,E)$ and $d \in \mathbb{N}$, we denote by G^d (e.g. G^2 is called the square of G) the graph with vertex set V in which two vertices are adjacent if they have distance at most d in G.

Theorem 26 (Fleischner 1974)**.** *If G is a 2-connected graph, then G^2 has a Hamiltonian cycle.*

Remark 12. Problem 2 is still studied in the contemporary research.

Lecture 8 on 30.10.2018

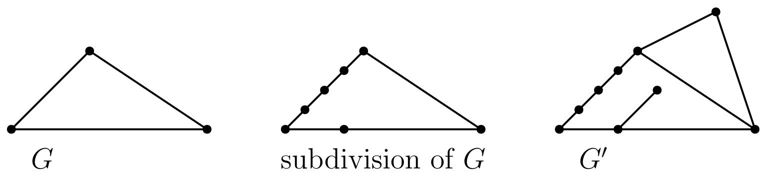
2.2 Planar Graphs and Trees

2.2.1 Planar and Plane Graphs

Definition 35 (Subdivision, topological minor, minor)**.** Let $G = (V,E)$ and $G' = (V',E')$ be graphs.

(1) G' is a *subdivision of* G if G' is obtained from G by inserting new vertices on some or all of its edges (i.e. we replace some of the edges of G by paths of positive (larger) length).

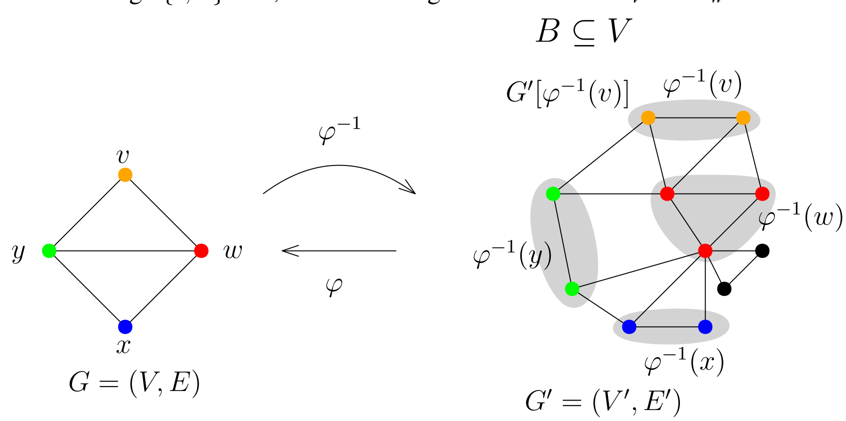
(2) G is a *topological minor of* G' if G' contains a subdivision of G as a subgraph.



(3) G is a *minor of* G' if there is a map $\varphi : B \subseteq V' \to V$ such that

- for every vertex $v \in V$, its pre-image $\varphi^{-1}(v)$ is non-empty and connected[1] in G' and
- for every edge $\{u,w\} \in E$ there is an edge in G' between $\varphi^{-1}(v)$ and $\varphi^{-1}(w)$.

In other words, for every $v \in V$, there is a non-empty subset U_v of V' such that these sets are vertices of pairwise disjoint and connected subgraphs in G', and for each edge $\{v,w\} \in E$, there is an edge in G' between U_v and U_w.



Definition 36 (Planar graph)**.**

(1) A graph is called *planar* if it can be embedded/drawn in the plane without crossing edges.

(2) A planar graph with a given planar embedding is called *plane graph*.

(3) A *face* of a plane graph G is a component of $\mathbb{R}^2 \setminus G$. In particular, there is always precisely one unbounded face, the "outside face".

[1] More precisely, the graph $[G(\varphi^{-1}(v))]$ induced by the pre-image is connected in G'.

Example 1. *K_5 and $K_{3,3}$ are not planar and neither are topological minors of them.*

Theorem 27 (Kuratowski 1930, Wagner 1937)**.**

- *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a topological minor.*
- *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

Theorem 28 (Euler characteristics)**.** *In a connected plane graph $G = (V, E)$, the number of faces equals $|E| - |V| + 2$.*

Proof. Induction □

Remark 13 (Number of faces)**.** Note that the number of faces is independent of the plane embedding.

Corollary 29.

- *A plane graph with $n \geq 3$ vertices hast at most $3n - 6$ edges.*
- *Every plane triangulation with $n \geq 3$ vertices (i.e. every face is bounded by a triangle) has exactly $3n - 6$ edges.*

Proof. Exercise!
Use double counting argument (each face contains (at most) 3 edges, on the other hand each edge contains exactly 2 faces) and then use Euler characteristics. □

2.2.2 Trees

Definition 37 (Forest, tree, leaf)**.**

(1) An acyclic graph (a graph without cycles) is called a *forest*.
(2) A connected forest is called a *tree*.
(3) The vertices of degree one in a forest are called *leaves*.

Remark 14. A graph is acyclic if and only if it does not contain a K_3 as a minor.

Lemma 30. *Every tree with at least two vertices (or equivalently at least one edge) contains at least two leaves.*

Proof. Exercise! □

Lemma 31 (Tree growing lemma)**.** *Let $G = (V, E)$ be a graph and $d(v) = 1$ for some $v \in V$. Then G is a tree if and only if $G[V \setminus \{v\}]$ is a tree.*

Proof. Exercise! □

Lemma 32. *Let $G = (V, E)$ be a graph. Then the following statements are equivalent.*

1. *G is a tree;*

2. *G is minimally connected, that is, G is connected, but $G' = (V, E \setminus \{e\})$ is disconnected for every edge $e \in E$;*

3. *G is maximally acyclic, that is, G contains no cycle, but $G'' = (V, E \cup \{v, w\})$ does for any two non-adjacent vertices $v, w \in V$;*

4. *G is connected and $|E| = |V| - 1$.*

Proof. Exercise! □

Theorem 33 (Cayley's formula)**.** *Let $n \geq 2$ be an integer and let $T(K_n)$ denote the number of all spanning trees of the complete graph K_n. Then $T(K_n) = n^{n-2}$.*

For an illustration see the pictures for Theorem 15 in Section 1.5.

Proof (Prüfer Code). Let $\mathscr{T}_n$ be the set of all spanning trees of K_n, labelled with $[n]$ and let

$$\mathscr{S}_{n-2} := \{(s_1, s_2, \ldots, s_{n-2}) : 1 \leq s_i \leq n\} = [n]^{n-2}$$

be the set of all strings of length $n-2$ over the alphabet $[n]$. The strategy is to find a bijection

$$\begin{aligned} f : \mathscr{T}_n &\to \mathscr{S}_{n-2} \\ T &\mapsto f(T) := s = (s_1, s_2, \ldots, s_{n-2}) \end{aligned}$$

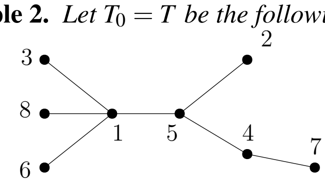
This is known as the *Prüfer code*. It is defined successively as follows: We start with a labelled tree $T_0 = T$ and construct a sequence of trees of (nested) subtrees $T_0, T_1, \ldots, T_{n-2}$ from which we obtain the numbers $s_1, s_2, \ldots, s_{n-2}$.

Suppose the tree T_{i-1} has already been constructed for $i \geq 1$. We

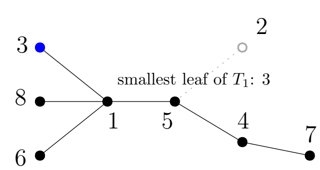
- take the smallest (labelled) leaf in T_{i-1} and remove it from T_{i-1} together with the edge incident to it;
- call the resulting graph T_i;
- define the i-th term s_i of the sequence s as the label of the (unique) neighbour of the leaf that is removed from T_{i-1}.

Doing this successively for $i = 1, 2, \ldots, n-2$ we obtain the sequence $s = (s_1, s_2, \ldots, s_{n-2})$.

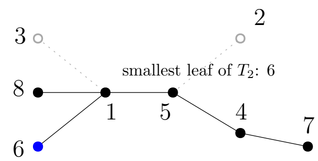
Example 2. *Let $T_0 = T$ be the following three with 8 vertices:*

T_0 

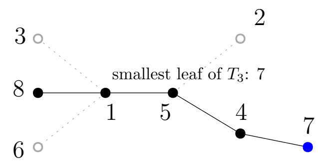
5 *(smallest of the leaves of T_0: 2)*

T_1 

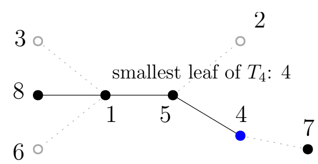
$s_1 = 5$*, neighbour of* 2

T_2 

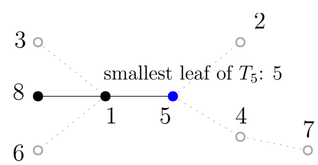
$s_2 = 1$, *neighbour of* 3

T_3 

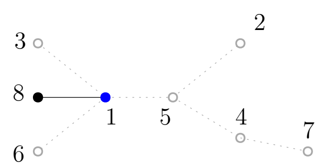
$s_3 = 1$, *neighbour of* 6

T_4 

$s_4 = 4$, *neighbour of* 7

T_5 

$s_5 = 5$, *neighbour of* 4

T_6 

$s_6 = 1$, *neighbour of* 5

$s = (5,1,1,4,5,1)$.

Conversely, we shall show how to construct the original tree $T = f^{-1}(s)$ from the sequence $s = (s_1,\ldots,s_{n-2})$ (that arose from some $T \in \mathscr{T}_n$).

Let $s = (s_1,\ldots,s_{n-2}) \in [n]^{n-2}$ be given and suppose that $s = f(T)$. For each $i = 1,\ldots,n-2$, denote by l_i the leaf of T_{i-1} that was removed in order to obtain T_i. Then the edge removed in that step was $e_i := \{l_i, s_i\}$. Finally, denote by e_{n-1} the (unique) edge of T_{n-2}.

Recall that a vertex v occurs as an entry of s if a leaf adjacent to v is removed at some step of the "removal" process. Because the graph T_{n-2} at the end of the process does not contain any vertices of degree at least two, all vertices that have degree at least two in T appear in s. Vice versa, leafs of T will not appear in s. Thus, the leafs of T are precisely the vertices in $[n] \setminus \{s_1,\ldots,s_{n-2}\}$. Therefore, we have

$$l_1 = \min([n] \setminus \{s_1,s_2,\ldots,s_{n-2}\})$$

Analogously, for each $i = 2, \ldots, n-2$, the leafs of T_{i-1} are precisely the vertices in $[n] \setminus \{s_i, s_{i+1}, \ldots, s_{n-2}, l_1, \ldots, l_{i-1}\}$, because $l_1, \ldots, l_{i-1}$ have already been removed in earlier steps, while $s_i, \ldots, s_{n-2}$ have degree at least two in T_{i-1}. Therefore,

$$l_i = \min\left([n] \setminus \{s_i, s_{i+1}, \ldots, s_{n-2}, l_1, \ldots, l_{i-1}\}\right).$$

This way, we can determine the leafs $l_1, \ldots, l_{n-2}$ that are deleted during the "removal" process. In terms of edges, this means that $e_i = \{l_i, s_i\}$ for $i = 1, \ldots, n-2$, and e_{n-1} is the edge between the two vertices in $[n] \setminus \{l_1, \ldots, l_{n-2}\}$. Thus, given the sequence s, the tree T with $s = f(T)$ can be uniquely determined. In other words, the function f is injective.

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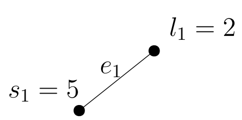
To complete the proof, we need to prove that f is surjective. To prove this, observe that we can define a graph G from *any* sequence $s \in [n]^{n-2}$, even if we do not know whether $s = f(T)$ for some tree T. We aim to show that G is a tree and that the Prüfer code "encoding" applied to G yields the original sequence s. Surjectivity of f then follows directly from these two properties.

Let $E_i = \{e_i, e_{i+1}, \ldots, e_{n-1}\}$ for $1 \le i \le n-1$ and where for $i < n-1$, $e_i = \{l_i, s_i\}$, $l_i = \min([n] \setminus \{s_i, s_{i+1}, \ldots, l_1, \ldots, l_{i-1}\}$ and where e_{n-1} is the edge between the two vertices in $[n] \setminus \{l_1, \ldots, l_{n-2}\}$. Then let $G_i = ([n], E_i)$. To show that G is a tree, observe first that G_{n-1} is a tree (consisting of two vertices and the edge e_{n-1}) plus isolated vertices. Note that none of the edges $e_{i+1}, \ldots, e_{n-1}$ can be incident to l_i and therefore l_i is a leaf of G_i. By Lemma 31, it follows inductively for $i = n-1, n-2, \ldots, 1$ that G_i is a tree plus isolated vertices. In particular, $G_1 = G$ is a tree.

To show that the encoding applied to G yields the original sequence s, it suffices to verify that l_i is the smallest leaf of G_i for $1 \le i \le n-2$. Observe that by the definition of $l_i = \min\left([n] \setminus \{s_i, s_{i+1}, \ldots, s_{n-2}, l_1, \ldots, l_{i-1}\}\right)$, a smaller leaf of G_i can only occur among $l_1, \ldots, l_{i-1}$ or $s_i, s_{i+1}, \ldots, s_{n-2}$. Because $l_1, l_2, \ldots, l_{i-1}$ are isolated vertices in G_i, the first case does not occur. So consider s_k for some $i \le k \le n-2$. In G_k, s_k is a neighbour of l_k and has another neighbour, so it is not a leaf of G_k. Because $G_k \subset G_i$, s_k is not a leaf of G_i. Therefore, none of $s_i, s_{i+1}, \ldots, s_{n-2}$ is a leaf of G_i. $\square$

Example (Example 2 cont.). $s = (s_1, s_2, \ldots, s_6) = (5, 1, 1, 4, 5, 1)$. We show how to get a tree from this. (Try the same on an arbitrary word, say in $\mathscr{S}_5$ to illustrate the surjectivity of f)

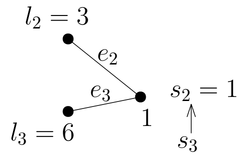
Step 1



$$\begin{aligned} l_1 &= \text{min of } [8] \setminus \{1,4,5\} \\ &= \min\{1,3,6,7,8\} \\ &= 2 \end{aligned}$$

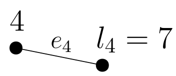
Step 2

Step 3



$$
\begin{aligned}
l_2 &= \text{min of } [n] \setminus \{s_2, \ldots, s_6, l_1\} \\
&= \min\{3, 6, 7, 8\} \\
&= 3 \\
l_3 &= \text{min of } [n] \setminus \{s_3, \ldots, s_6, l_1, l_2\} \\
&= \min\{6, 7, 8\} \\
&= 6
\end{aligned}
$$

Step 4

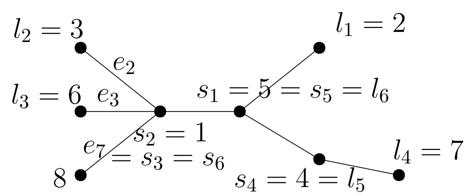


$$
\begin{aligned}
l_4 &= \text{min of } [n] \setminus \{s_4, \ldots, s_6, l_1, l_2, l_3\} \\
&= \min\{7, 8\} \\
&= 7
\end{aligned}
$$

Step 5

Step 6

Step 7



$$
\begin{aligned}
l_5 &= \text{min of } [n] \setminus \{s_5, s_6, l_1, l_2, l_3, l_4\} \\
&= \min\{4, 8\} \\
&= 4 \\
l_6 &= \text{min of } [n] \setminus \{s_6, l_1, \ldots, l_5\} \\
&= \min\{5, 8\} = 5 \\
e_7 &= \text{edge between the two vertices } [n] \setminus \{l_1, \ldots l_6\} = \{1, 8\}
\end{aligned}
$$

Chapter 3

Stochastic Aspects – Random Graphs

Suggested reading for Chapter 3: [3] and [4].

3.1 Basics

Lemma 34 (Union bound)**.**

For a finite or countable set of events $A_1, A_2, \ldots$,

$$\mathbb{P}\left[\bigcup A_i\right] \leq \sum \mathbb{P}[A_i].$$

In words, "the probability that at least one of the events $A_1, A_2, \ldots$ happens is no greater than the sume of the probabilities of individual events".

Proof. Exercise!

- For the finite case induction on n
- For the countable case σ-subadditivity of (probabilistic) measure

□

Lemma 35 (Markov's inequality)**.**

For any non-negative random variable X and any constant $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Proof. Exercise! □

Remark 15 (First moment method)**.** For a non-negative integral random variable X and $t = 1$, Markov's inequality implies that

$$\mathbb{P}[X \neq 0] = \mathbb{P}[X \geq 1] \leq \mathbb{E}[X].$$

In particular, if $\mathbb{E}[X] \to 0$, then $\mathbb{P}[X \neq 0] \to 0$. Therefore it is useful to prove the non-existence of certain events.

Remark 16 (Variant of Markov's inequality)**.**

Markov's inequality in terms of deviation from the expectation:

$$\mathbb{P}[X \geq t\mathbb{E}[X]] \leq \frac{1}{t}$$

Definition 38 (Variance and covariance)**.** The *variance* of a random variable X is defined as

$$\mathrm{Var}[X] := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X,Y] := \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Lemma 36 (Chebyshev's inequality)**.** *For any random variable X and any constant $t > 0$,*

$$\mathbb{P}\left[\,|X - \mathbb{E}[X]| \geq t\,\right] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

Proof. Apply Markov's inequality! □

Remark 17 (Second moment method)**.** Let X be a random variable with $\mathbb{E}[X] > 0$. Then Chebyshev's inequality implies

$$\mathbb{P}[X = 0] \leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} - 1.$$

Therefore, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, equivalently $\mathbb{E}[X^2] = (1 + o(1))\mathbb{E}[X]^2$, then $\mathbb{P}[X = 0] \to 0$, in other words, $\mathbb{P}[X > 0] \to 1$. Therefore it is useful to prove the existence of certain events.

Remark 18 (Application of Chebyshev's inequality)**.** Lemma 36 implies that for any $\varepsilon > 0$,

$$\mathbb{P}\left[\,|X - \mathbb{E}[X]| \geq \varepsilon\mathbb{E}[X]\,\right] \leq \frac{\mathrm{Var}[X]}{\varepsilon^2\mathbb{E}[X]^2}.$$

Thus, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, then $\mathbb{P}\left[\,|X - \mathbb{E}[X]| \geq \varepsilon\mathbb{E}[X]\,\right] \to 0$, equivalently, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, then $\mathbb{P}\left[\,|X - \mathbb{E}[X]| \leq \varepsilon\mathbb{E}[X]\,\right] \to 1$.

Let $X = X_1 + X_2 + \ldots$ be a sum of indicator random variables[1]. Then

$$\begin{aligned}
\mathrm{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \mathbb{E}\left[\left(\textstyle\sum_i X_i\right)\left(\textstyle\sum_i X_i\right)\right] - \mathbb{E}\left[\textstyle\sum_i X_i\right]\mathbb{E}\left[\textstyle\sum_i X_i\right] \\
&= \textstyle\sum_i \left(\mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2\right) + \textstyle\sum_{i \neq j}\left(\mathbb{E}\left[X_i X_j\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right]\right) \\
&= \textstyle\sum_i \mathrm{Var}\left[X_i\right] + \textstyle\sum_{i \neq j} \mathrm{Cov}\left[X_i, X_j\right]
\end{aligned}$$

Since the X_i's are indicator variables, we have $\mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[X_i\right]$, and therefore

$$\begin{aligned}
\mathrm{Var}\left[X_i\right] &= \mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2 \leq \mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[X_i\right] \\
\textstyle\sum_i \mathrm{Var}\left[X_i\right] &\leq \textstyle\sum_i \mathbb{E}\left[X_i\right] = \mathbb{E}\left[\textstyle\sum_i X_i\right] = \mathbb{E}[X]
\end{aligned}$$

and

$$\mathrm{Cov}\left[X_i, X_j\right] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \leq \mathbb{E}[X_i X_j].$$

Note that if X_i and X_j are independent, then

$$\mathrm{Cov}\left[X_i, X_j\right] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = 0.$$

[1] A random variable associated with the occurence of an event; X_A has value 1 if A occurs and 0 otherwise.

Hence,

$$\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j] \leq \mathbb{E}[X] + \sum_{i \neq j,\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j]. \quad (3.1)$$

Remark 19 (Sum of indicator random variables)**.** Let $X = X_1 + X_2 + \ldots$ be a sum of indicator random variables with $\mathbb{E}[X] > 0$. Lemma 36 and (3.1) imply that

$$\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{i \neq j} \mathbb{E}[X_i X_j]}{\mathbb{E}[X]^2}$$

and Remark 18 and (3.1) imply that

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]] \leq \frac{\mathrm{Var}[X]}{\varepsilon^2 \mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{i \neq j\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j]}{\varepsilon^2 \mathbb{E}[X]^2}.$$

Therefore, if $\mathbb{E}[X] \to \infty$ and $\sum_{i \neq j\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j] = o(\mathbb{E}[X]^2)$, then

$$\mathbb{P}[X \geq 1] \to 1 \quad \text{and} \quad \mathbb{P}[|X - \mathbb{E}[X]| \leq \varepsilon \mathbb{E}[X]] \to 1.$$

(Cf. Remark 18).

Definition 39 (Erdős–Rényi random graph)**.** Let $p \in [0,1]$ be a constant independent of n or let $p = p(n) \in [0,1]$ be a function in n.

The *binomial random graph* $G(n,p)$, also known as the *Erdős–Rényi random graph*, is a graph with vertex set $[n] := \{1, 2, \ldots, n\}$, in which each pair of vertices is joined by an edge with probability p, independently of each other. Equivalently, given $p \in [0,1]$, let Ω be the set of all graphs with vertex set $[n]$ and define, for each $G \in \Omega$,

$$\mathbb{P}[G] := p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|},$$

where $E(G)$ is the edge set of G.

Example 3 (Expected number of edges in $G(n,p)$)**.**

Let $G = G(n,p)$. Let X be the number of edges in G. This can be written as the sum of indicator random variables

$$X := |E(G)| = \sum X_{\{u,v\}},$$

where

$$X_{\{u,v\}} = \begin{cases} 1 & \text{if } \{u,v\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Using the linearity of expectation we obtain

$$\mathbb{E}[X] = \sum \mathbb{E}\left[X_{\{u,v\}}\right] = \sum \mathbb{P}\left[\{u,v\} \in E(G)\right] = \sum p = \binom{n}{2} p.$$

Therefore,

$$\mathbb{E}[X] = \binom{n}{2} p \rightarrow \begin{cases} 0 & \text{if } p = o\left(n^{-2}\right), \\ \infty & \text{if } p = \omega\left(n^{-2}\right). \end{cases}$$

By Markov's inequality (cf. Remark 15), if $p \ll n^{-2}$, then

$$\mathbb{P}[X \neq 0] = \mathbb{P}[X \geq 1] \leq \mathbb{E}[X] \rightarrow 0.$$

3.2 Ramsey Number and First Moment Method

Definition 40 (Ramsey number)**.** The Ramsey number $R(k,l)$ is defined as the smallest integer n such that any graph G on n vertices contains either a clique of order k or an independent set of order l, i.e.

$$R(k,l) \;:=\; \min\{n \in \mathbb{N} : \forall\, G \text{ on } n \text{ vertices } \omega(G) \geq k \vee \alpha(G) \geq l\}.$$

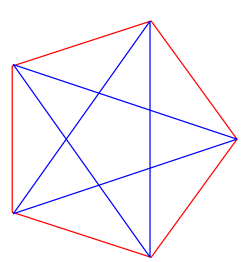
Equivalently, $R(k,l)$ is the smallest integer n such that in any two-coloring of the edges of the complete graph K_n by red and blue, there exists either a red K_k or a blue K_l.

Remark 20 (Ramsey number)**.**

$$\begin{aligned} &(0) \quad R(k,1) = R(1,k) = 1 \\ &(1) \quad R(k,l) = R(l,k) \\ &(2) \quad R(k,2) = R(2,k) = k \\ &(3) \quad R(k,l) \leq R(k-1,l) + R(k,l-1) \\ &(4) \quad R(k,l) \leq \binom{k+l-2}{k-1} \end{aligned}$$

Proof. Exercise! □

Example. $R(3,3) = 6$: one can show that $R(3,3) \leq 6$ (exercise). To see that $R(3,3) > 5$, consider the colouring of K_5 where all the outer edges are of one colour, all the crossing edges of the inner star of the other colour:



Theorem 37 (Erdős–Szekeres 1935).

$$R(k,k) \le (1+o(1))\frac{4^{k-1}}{\sqrt{\pi k}}$$

Proof. Using Remark 20 (4) and applying Stirling's formula we have

$$\begin{aligned}
R(k,k) \le \binom{2k-2}{k-1} &= \frac{(2k-2)!}{((k-1)!)^2} \\
&= \frac{\left(1+O\left(\frac{1}{2(k-1)}\right)\right)\sqrt{4\pi(k-1)}\left(\frac{2(k-1)}{e}\right)^{2(k-1)}}{\left(\left(1+O\left(\frac{1}{k-1}\right)\right)\sqrt{2\pi(k-1)}\left(\frac{k-1}{e}\right)^{k-1}\right)^2} \\
&= (1+o(1))\frac{2^{2(k-1)}}{\sqrt{\pi(k-1)}} \\
&= (1+o(1))\frac{4^{k-1}}{\sqrt{\pi k}}
\end{aligned}$$

□

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Theorem 38 (Erdős 1947).

$$R(k,k) \ge (1+o(1))\frac{k}{e\sqrt{2}}\sqrt{2}^k$$

Proof. We shall apply the first moment method.

Consider a random graph $G = G\left(n, \frac{1}{2}\right)$. For each subset $S \subseteq [n]$ of size k,

$$\mathbb{P}[S \text{ forms a clique}] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}}.$$

So, using the union bound (Lemma 34) we have

$$\begin{aligned}
\mathbb{P}[\omega(G) \ge k] &= \mathbb{P}[G \text{ contains a clique of size } k] \\
&= \mathbb{P}\left[\bigcup_{S\in\binom{[n]}{k}}[S \text{ forms a clique}]\right] \\
&\le \sum_{S\in\binom{[n]}{k}}\mathbb{P}[S \text{ forms a clique}] \\
&= \binom{n}{k}2^{-\binom{k}{2}}.
\end{aligned}$$

Similarly, for each subset $S \subseteq [n]$ of size k,

$$\mathbb{P}[S \text{ is an independent set}] = \left(1 - \frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}},$$

and thus we have

$$\begin{aligned}\mathbb{P}[\alpha(G) \geq k] &= \mathbb{P}\left[\bigcup_{S \in \binom{[n]}{k}} [S \text{ is an independent set}]\right] \\ &\leq \sum_{S \in \binom{[n]}{k}} \mathbb{P}[S \text{ is an independent set}] \\ &= \binom{n}{k} 2^{-\binom{k}{2}}.\end{aligned}$$

Summing up, we have

$$\mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] \leq \mathbb{P}[\omega(G) \geq k] + \mathbb{P}[\alpha(G) \geq k] \leq \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Therefore, if

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1,$$

then

$$\mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] < 1.$$

In other words,

$$\mathbb{P}[\omega(G) < k \wedge \alpha(G) < k] = 1 - \mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] > 0.$$

This means that there *exists* a graph G on n vertices with

$$\omega(G) < k \wedge \alpha(G) < k.$$

This implies

$$R(k,k) > n.$$

Using the binomial coefficient approximation

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

we deduce that

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-\frac{k(k-1)}{2}} = \left(\frac{en}{k} \cdot 2^{\frac{1}{k}} \cdot 2^{-\frac{k-1}{2}}\right)^k = \left(n \cdot \frac{e\sqrt{2}}{k\sqrt{2}^k} 2^{\frac{1}{k}}\right)^k,$$

which will be smaller than 1 if

$$n < \left(\frac{e\sqrt{2}}{k\sqrt{2}^k} \cdot 2^{\frac{1}{k}}\right)^{-1} = \frac{k\sqrt{2}^k}{e\sqrt{2}} \underbrace{2^{-\frac{1}{k}}}_{\to 1 \text{ as } k \to \infty} = (1 + o(1)) \frac{k}{e\sqrt{2}} \sqrt{2}^k.$$

$\square$

3.3 Independence Number and Deletion Method

Theorem 39 (Weak Turán Theorem)**.** *For any graph G with vertex set $[n]$ and average degree $d(G) \geq 1$,*

$$\alpha(G) \;\geq\; \frac{n}{2d(G)}\,.$$

Proof. We shall apply the so-called *Deletion Method*.

Let $S \subset [n]$ be a random set defined by

$$\mathbb{P}[v \in S] \;:=\; \frac{1}{d(G)},$$

the events $v \in S$ being mutually independent. Observe that $d(G)$ is necessary for this to be a probability.

Let $X = |S|$ and Y be the number of edges in the induced subgraph $G[S]$. For each $\{u,v\} \in E(G)$, let

$$Y_{\{u,v\}} = \begin{cases} 1 & \text{if } \{u,v\} \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously we have

$$Y = \sum_{\{u,v\} \in E(G)} Y_{\{u,v\}}$$

and using the linearity of expectation we have

$$\mathbb{E}[Y] = \sum_{\{u,v\} \in E(G)} \mathbb{E}[Y_{\{u,v\}}] = |E(G)| \left(\frac{1}{d(G)}\right)^2 = \frac{nd(G)}{2d(G)^2} = \frac{n}{2d(G)},$$

where the penultimate equality follows from $d(G) := \frac{1}{n} \sum_{v \in [n]} d(v) = \frac{2|E(G)|}{n}$. On the other hand, we have

$$\mathbb{E}[X] = \mathbb{E}[|S|] = \frac{n}{d(G)}.$$

By the linearity of expectation we have

$$\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] = \frac{n}{d(G)} - \frac{n}{2d(G)} = \frac{n}{2d(G)}. \tag{3.2}$$

This implies that there exists a set $S_0 \subset [n]$, for which the number of vertices in the induced subgraph $G[S_0]$ minus the number of edges in $G[S_0]$ satisfies

$$|V(G[S]) - E(G[S])| \geq \frac{n}{2d(G)},$$

for otherwise (i.e. if for all $S \subset [n]$, $|V(G[S]) - E(G[S]) < n/2d(G))$

$$\mathbb{E}[X - Y] < \frac{n}{2d(G)},$$

which contradicts (3.2).

Now select one vertex from each edge of $G[S_0]$ and delete it from S_0. This results in a set S^* which contains at least $\frac{n}{2d(G)}$ vertices and no edges, in other words, S^* is an independent set of size at least $\frac{n}{2d(G)}$, which implies that $\alpha(G) \geq \frac{n}{2d(G)}$ as desired. □

3.4 Subgraphs and Second Moment Method

Definition 41 (Monotone family)**.**

Let Γ be the set of all 2-element subsets of $[n]$ and 2^Γ be the set of all subsets of Γ. A family $\mathscr{F} \subseteq 2^\Gamma$ (i.e. a family of subsets of Γ) is called *increasing* if

$$A \subseteq B \subseteq \Gamma \wedge A \in \mathscr{F} \quad \Rightarrow \quad B \in \mathscr{F}.$$

It is called *decreasing* if the family of complements in Γ is increasing (or, eqivalently, if $2^\Gamma \setminus \mathscr{F}$ is increasing). It is called *monotone* if it is increasing or decreasing.

Remark 21 (Monotone property)**.** We identify a property of subsets of Γ with the corresponding family of all subsets having that property.

In case of graph properties, a property is increasing (resp. decreasing) if adding (resp. deleting) edges does not violate the property.

Example 4 (Monotone properties)**.** *(1) Being connected (increasing property)*

(2) Containing a triangle (increasing property)

(3) Having an isolated vertex (decreasing property)

(4) Being planar (decreasing property)

Theorem 40 (Increasing property)**.** *For any increasing property $\mathscr{F} \subseteq 2^\Gamma$ and $0 \leq p_1 \leq p_2 \leq 1$,*

$$\mathbb{P}\left[E\left(G(n,p_1)\right) \in \mathscr{F}\right] \leq \mathbb{P}\left[E\left(G(n,p_2)\right) \in \mathscr{F}\right].$$

Proof. Define $p_0 \in [0,1]$ by

$$p_1 + (1-p_1)p_0 = p_2.$$

Then

$$G(n,p_0) \cup G(n,p_1) = G(n,p_2)$$

(”two-round exposure”). Because $\mathscr{F}$ is increasing, we have

$$E\left(G\left(n,p_1\right)\right)\in\mathscr{F}\quad\Rightarrow\quad E\left(G(n,p_0)\cup G(n,p_1)\right)\in\mathscr{F}$$

and therefore

$$\mathbb{P}\left[E\left(G\left(n,p_1\right)\right)\in\mathscr{F}\right]\;\le\;\mathbb{P}\left[E\left(G(n,p_0)\cup G\left(n,p_1\right)\right)\in\mathscr{F}\right]\;=\;\mathbb{P}\left[E\left(G\left(n,p_2\right)\right)\in\mathscr{F}\right].$$

□

Exercise 2. *Is it true that for any decreasing property $\mathscr{F}\subseteq 2^{\Gamma}$ and $0\le p_1\le p_2\le 1$,*

$$\mathbb{P}\left[E\left(G\left(n,p_1\right)\right)\in\mathscr{F}\right]\;\ge\;\mathbb{P}\left[E\left(G\left(n,p_2\right)\right)\in\mathscr{F}\right]?$$

Definition 42 (Threshold)**.**

For an increasing property $\mathscr{F}\subseteq 2^{\Gamma}$, a sequence $\left(f\left(n\right)\right)_{n\ge 1}$ is called a *threshold* if

$$\mathbb{P}\left[E\left(G\left(n,p\right)\right)\in\mathscr{F}\right]\overset{n\to\infty}{\longrightarrow}\begin{cases}0 & \text{if } p=o\left(f\left(n\right)\right),\\ 1 & \text{if } p=\omega\left(f\left(n\right)\right).\end{cases}$$

For decreasing families, thresholds are defined analogously with the cases $p=o(f(n))$ and $p=\omega(f(n))$ exchanged. Note that if $f(n)$ is a threshold, then every function $g(n)$ with $g(n)=\Theta(f(n))$ is also a threshold.

Theorem 41 (Bollobás-Thomason 1987)**.** *Every monotone property has a threshold.*

Theorem 42 (Triangle threshold)**.**

$$\mathbb{P}\left[G\left(n,p\right)\text{contains a triangle}\right]\overset{n\to\infty}{\longrightarrow}\begin{cases}0 & \textit{if } p=o\left(\frac{1}{n}\right),\\ 1 & \textit{if } p=\omega\left(\frac{1}{n}\right).\end{cases}$$

In other words, $f\left(n\right)=\frac{1}{n}$ is a threshold for the property of containing a triangle.

Proof. Let X denote the number of triangles contained in $G\left(n,p\right)$. Define indicator variables

$$X_S=\begin{cases}1 & \text{if } S \text{ forms a triangle},\\ 0 & \text{otherwise}.\end{cases}$$

Then

$$X=\textstyle\sum_{S\in\binom{[n]}{3}}X_S$$

and

$$\mathbb{E}[X]=\sum_{S\in\binom{[n]}{3}}\mathbb{E}[X_S]=\sum_{S\in\binom{[n]}{3}}\mathbb{P}\left[S\text{ forms a triangle}\right]=\binom{n}{3}p^3=\Theta\left(n^3p^3\right).$$

For the first statement of the theorem, we apply Markov's inequality (the first moment method) to obtain

$$\mathbb{P}\left[G\left(n,p\right)\text{contains a triangle}\right] \;=\; \mathbb{P}\left[X \geq 1\right] \;\leq\; \mathbb{E}[X] \to 0, \quad \text{if } p \ll \frac{1}{n}.$$

In order to prove the second part we observe that

$$\mathbb{E}[X] = \Theta\left(n^3 p^3\right) \to \infty, \quad \text{if } p \gg \frac{1}{n}.$$

But does it imply what we want? Not necessarily! We shall use the second moment method to prove

$$P\left[X = 0\right] \to 0, \quad \text{if } p \gg \frac{1}{n}.$$

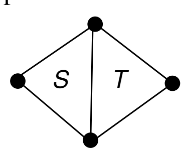
Assume that $p \gg \frac{1}{n}$ (p fixed). Then $\mathbb{E}[X] \to \infty$ and by Chebyshev's inequality, more precisely Remark 19, we have

$$\mathbb{P}\left[X = 0\right] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{S \neq T \;\in \binom{[n]}{3}\; X_S, X_T \text{ dependent}} \mathbb{E}\left[X_S X_T\right]}{\mathbb{E}[X]^2}.$$

Thus it suffices to show

$$\sum_{S \neq T \;\in \binom{[n]}{3}\; X_S, X_T \text{ dependent}} \mathbb{E}\left[X_S X_T\right] = o(\mathbb{E}[X]^2).$$

Let S and T be two distinct 3-element subsets of $[n]$. Then they may share exactly two vertices or at most one vertex. In the latter case, S and T can not share any edge in common in $G(n,p)$, and so X_S and X_T are independent. So we consider only pairs of $S \neq T \;\in \binom{[n]}{3}$ that share two vertices. If S and T form triangles, then $S \cup T$ forms a graph with two triangles sharing one edge in common (as in the following picture).



In this case we have

$$\mathbb{E}\left[X_S X_T\right] = p^5.$$

Summing up we have

$$\sum_{S \neq T \;\in \binom{[n]}{3}\; X_S, X_T \text{ dependent}} \mathbb{E}\left[X_S X_T\right] \;=\; \Theta(n^4 p^5) \;\ll\; \Theta(n^6 p^6) = \mathbb{E}[X]^2,$$

as desired. □

Remark 22 (Subgraph threshold)**.** Given a fixed graph H with $|V(H)| = v_h$ and $|E(H)| = e_H$, what is the threshold for the property that $G(n,p)$ contains a subgraph isomorphic to H (a copy of H)?

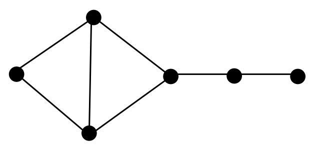
To answer this question we let X denote the number of copies of H. Then

$$\mathbb{E}[X] = \Theta\left(\binom{n}{v_H} p^{e_H}\right) = \Theta\left(n^{v_h} p^{e_H}\right) \to \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{v_H}{e_H}}\right), \\ \infty & \text{if } p = \omega\left(n^{-\frac{v_H}{e_H}}\right). \end{cases}$$

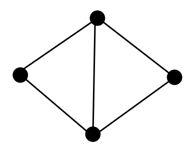
Is it true that

$$\mathbb{P}[X > 0] \to \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{v_H}{e_H}}\right), \\ 1 & \text{if } p = \omega\left(n^{-\frac{v_H}{e_H}}\right). \end{cases}$$

This is indeed not true in general. To see this let us consider the kite graph H and the diamond graph D defined as follows.



H



D

Let X_H denote the number of kite graphs contained in $G(n,p)$ and X_D denote the number of diamond graphs contained in $G(n,p)$. Then we have

$$\mathbb{P}[X_D \geq 1] \leq \mathbb{E}[X_D] = O\left(n^4 p^5\right) \to 0 \quad \text{if } p = o\left(n^{-\frac{4}{5}}\right),$$
$$\mathbb{P}[X_H \geq 1] \leq \mathbb{E}[X_H] = O\left(n^6 p^7\right) \to 0 \quad \text{if } p = o\left(n^{-\frac{6}{7}}\right).$$

On one hand we have

$$n^{-\frac{v_H}{e_H}} = n^{-\frac{6}{7}} \ll n^{-\frac{4}{5}} = n^{-\frac{v_D}{e_D}},$$

but on the other hand we $\mathbb{P}[X_H \geq 1] \leq \mathbb{P}[X_D \geq 1]$, because D is a subgraph of H.

What's wrong here?

Definition 43 (Maximum density)**.** Let

$$m_H := \max\left\{\frac{e_{H'}}{v_{H'}} : H' \subseteq H \quad \text{with} \quad v_{H'} > 0\right\}$$

Theorem 43 (Bollobás 1981)**.**

For any graph H with $e_H, v_H > 0$,

$$\mathbb{P}[G(n,p) \text{ contains a copy of } H] \to \begin{cases} 0 & \text{if } p = o\left(n^{-\frac{1}{m_H}}\right), \\ 1 & \text{if } p = \omega\left(n^{-\frac{1}{m_H}}\right). \end{cases}$$

Proof. Follow the lines of the proof of Theorem 42. □

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