Discrete and algebraic structures, winter semester 2015/16

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Part I

M. Kang

Chapter 1

Standard methods in enumerative combinatorics

recursions, generating functions, analytical methods

Chapter 2

Graph theory

matchings, durchlaufbarkeit, graph decomposition, planar graphs, stochastic aspects

Part II

K. Baur

Chapter 3

Multilinear algebra

[Lecture 1, 23.11.2015]
In this chapter, we discuss multilinear maps, tensor maps, tensor products and alternating maps. Multilinear maps can be viewed as a generalization of linear maps between vector spaces.

Some notations are explained at the very end of the document (just before the bibliography).

Unless mentioned otherwise, all vector spaces are finite dimensional vector spaces over some field K. Most of the time we will tacitly assume $K = \mathbb{C}$.

An important idea of this chapter is to convert a multilinear map on a product space into a linear map on the tensor product of its factor spaces and to study the relationship between these two maps.

this is from: classical analysis on normed spaces, Tsoy-Wo Ma. 15-1 Algebraic Tensor Products of Vector Spaces (ca. page 318)

3.1 Multilinear maps and tensor maps

Definition 3.1. Let $V_1, V_2, \dots, V_m$ and W be vector spaces over a field K. A map $\varphi : V_1 \times \cdots \times V_m \to W$ is *m-multilinear* or *multilinear* if it is linear in every argument, i.e. if

$$\varphi(v_1, \dots, v_i + \lambda v_i', \dots, v_m) = \varphi(v_1, \dots, v_i, \dots, v_m) + \lambda \varphi(v_1, \dots, v_i', \dots, v_m)$$

holds for all $i = 1, \dots, m$, and for all $\lambda \in K$.

A linear map $f \in \mathrm{Hom}(V, W)$ can be viewed as a *1-multilinear* map.

Remark. Linear map and multilinear maps are not the same!
Consider the linear map $f \in \mathrm{Hom}(V_1 \times V_2, W)$ and a multilinear map $\varphi : V_1 \times V_2 \to W$. f is linear and so,

$$f(v_1 + v_1', v_2 + v_2') = f(v_1, v_2) + f(v_1', v_2') = f(v_1, 0) + f(0, v_2) + f(v_1', 0) + f(0, v_2')$$

φ is multilinear, and so,

$$\varphi(v_1+v_1', v_2+v_2') = \varphi(v_1, v_2+v_2')+\varphi(v_1', v_2+v_2') = \varphi(v_1, v_2)+\varphi(v_1, v_2')+\varphi(v_1', v_2)+\varphi(v_1', v_2')$$

Note that $\varphi(v_1, 0) = 0 = \varphi(0, v_2)$ but that $f(v_1, 0)$ and $f(0, v_2)$ are not necessarily 0.

Example 3.2. The following maps are multilinear (please convince yourself of this!)

(a) $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $f(x, y) = xy$.

(b) $\varphi : V^* \times V \to \mathbb{C}$ defined by $\varphi(f, v) = f(v)$.

(c) $\varphi : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}$ defined by $\varphi(x, y) = x^T Ay$ for some matrix $A \in \mathbb{C}_{m\times n}$.

(d) $\otimes : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^{m\times n}$ defined by $\otimes(x, y) = xy^T$. This is a tensor map (see below).

(e) $\det : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ defined by $\det(x_1, \ldots, x_n) = \det A$ where A is $[x_1, \ldots, x_n]$ the $n \times n$-matrix formed by the column vectors x_i.

(f) $f : V_1 \times \cdots \times V_m \to \mathbb{C}$ defined by $f(v_1, \ldots, v_m) = \prod_{i=1}^m f_i(v_i)$ for given $f_i \in V_i^*$, $i = 1, \ldots, m$. ((We write $f = \prod_{i=1}^m f_i$. Then $\prod_{i=1}^m f_i(v_1, \ldots, v_m) = \prod_{i=1}^m f_i(v_i)$.))

(g) $g : V_1^* \times \cdots \times V_m^* \to W$ defined by $g(f_1, \ldots, f_m) = \prod_{i=1}^m f_i(v_i)$ for $v_i \in V_i$ given, $i = 1, \ldots, m$.

(h) Let $\varphi : V_1 \times\cdots\times V_m \to W$ and $\psi : V_1 \times\cdots\times V_m \to W$ be multilinear. Then $\alpha\varphi+\beta\psi$ is also multilinear (for all $\alpha, \beta \in K$). So the set $M(V_1, \ldots, V_m, W)$ of multilinear maps from $V_1 \times \cdots \times V_m$ to W is a vector space.

Let $\psi : V_1 \times \cdots \times V_m \to W$ be a multilinear map. If we describe how ψ acts on a basis of $V_1 \times \cdots \times V_m$, ψ is determined. So let $n_1, \ldots, n_m$ be the dimensions of V_i, $i = 1, \ldots, m$, let $E_i := \{e_{i1}, \ldots, e_{i,n_i}\}$ be a basis of V_i. So each $v_i \in V_i$ can be written as $v_i = \sum_{j=1}^{n_i} a_{ij}e_{ij}$ $(i = 1, \ldots, m)$.

To describe the basis of a product of m vector spaces of dimensions $n_1, \ldots, n_m$, we use the following notation. (from [5] chapter I.1, p.25,26)

$$\Gamma := \Gamma(n_1, \ldots, n_m) := \{\gamma : \gamma = (\gamma(1), \ldots, \gamma(m)) \mid 1 \le \gamma(i) \le n_i, i = 1, \ldots, m\}.$$

with $|\Gamma| = \prod_{i=1}^m n_i$. For an example, take $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2$, with

$$\Gamma = \Gamma(3, 2, 2) = \left\{ \begin{array}{l} (1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), \\ (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2), \\ (3, 1, 1), (3, 1, 2), (3, 2, 1), (3, 2, 2) \end{array} \right\}$$

We can order Γ according to the lexicographic order, as in the example. Moreover, we have (please check):

$$\prod_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} = \sum_{\gamma\in\Gamma} \prod_{i=1}^{m} a_{i\gamma(i)} \tag{3.1}$$

as will be used later.

From the set-up, we have

$$\begin{aligned}
\psi(v_1, \ldots, v_m) &= \psi(\sum_{j_1=1}^{n_1} a_{1,j_1} e_{1,j_1}, \ldots, \sum_{j_m}^{n_m} a_{m,j_m} e_{m,j_m}) \\
&= \sum_{j_1}^{n_1} \cdots \sum_{j_m}^{n_m} a_{1,j_1} \cdots a_{m,j_m} \psi(e_{1,j_1}, \ldots, e_{m,j_m}) \\
&= \sum_{\gamma \in \Gamma} a_{1,\gamma(1)} \cdots a_{m,\gamma(m)} \psi(e_{1,\gamma(1)}, \ldots, e_{m,\gamma(m)}) \\
&= \sum_{\gamma \in \Gamma} a_\gamma \psi(e_\gamma) \qquad (3.2)
\end{aligned}$$

where for $\gamma \in \Gamma$, we write

$$a_\gamma := \prod_{i=1}^{m} a_{i,\gamma(i)} \in \mathbb{C} \qquad (3.3)$$

$$e_\gamma := (e_{1,\gamma(1)}, \ldots, e_{m,\gamma(m)}) \in V_1 \times \cdots \times V_m \qquad (3.4)$$

so e_γ is the basis vector of $V_1 \times \cdots \times V_m$ with label γ.
Then the $\psi(e_\gamma)$ in (3.2) completely determine ψ.

Theorem 3.3 (Multilinear extension)**.** *Let $E_i = \{e_{i,1}, \ldots, e_{i,n_i}\}$ be a basis of V_i, $i = 1 \ldots, m$. Let e_γ be as in (3.4). Let w_γ in W, for all $\gamma \in \Gamma = \Gamma(n_1, \ldots, n_m)$ be arbitrary. Then there exists a unique multilinear map $\varphi : V_1 \times \cdots \times V_m \to W$ such that $\varphi(e_\gamma) = w_\gamma$ for all γ.*

Proof. Since we want $\varphi(e_\gamma) = w_\gamma$ for all $\gamma \in \Gamma$, we need to define (by (3.2)

$$\varphi(v_1, \ldots, v_m) = \sum_{\gamma \in \Gamma} a_\gamma w_\gamma$$

for a_γ as in (3.3) and $v_i = \sum_{j=1}^{n_i} a_{ij} e_{ij}$. Let $v_i' = \sum_{j=1}^{n_i} a_{ij}' e_{ij}$, $i = 1, \ldots, m$. From the definition of φ, for $c \in K$,

$$\begin{aligned}
&\varphi(v_1, \ldots, v_i + c v_i', \ldots, v_m) \\
=& \sum_{\gamma \in \Gamma} a_{1,\gamma(1)} \cdots (a_{i,\gamma(i)} + c a_{i,\gamma(i)}') \cdots a_{m,\gamma(m)} w_\gamma \\
=& \sum_{\gamma \in \Gamma} a_{1,\gamma(1)} \cdots a_{i,\gamma(i)} \cdots a_{m,\gamma(m)} w_\gamma + c \sum_{\gamma \in \Gamma} a_{1,\gamma(1)} \cdots a_{i,\gamma(i)}' \cdots a_{m,\gamma(m)} w_\gamma \\
=& \varphi(v_1, \ldots, v_i, \ldots, v_m) + c\varphi(v_1, \ldots, v_i', \ldots, v_m)
\end{aligned}$$

i.e. φ is multilinear.

Now we show that $\varphi(e_\alpha) = w_\alpha$ for all $\alpha \in \Gamma$. [proof skipped in class]
For $\alpha \in \Gamma$, write

$$e_{i,\alpha(i)} = \sum_{j=1}^{n_j} \delta_{\alpha(i),j} e_{ij}.$$

From the definition of φ and $e_\alpha = (e_{1,\alpha(1)}, \ldots, e_{m,\alpha(m)})$, we have

$$\varphi(e_\alpha) = \varphi(e_{1,\alpha(1)}, \ldots, e_{m,\alpha(m)}) = \sum_{\gamma\in\Gamma} \delta_{\alpha(1),1} \ldots \delta_{\alpha(m),m} w_\gamma = \sum_{\gamma\in\Gamma} \delta_{\alpha,\gamma} w_\gamma = w_\alpha$$

(with $\delta_{\alpha,\gamma}$ the obvious generalization of the Kronecker delta). So we have established the existence.

Uniqueness: Suppose there is another multilinear map $\psi : V_1 \times \cdots \times V_m \to W$ such that $\psi(e_\gamma) = w_\gamma$ for all $\gamma \in \Gamma$. Then from (3.2) we have (for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$):

$$\psi(v_1, \ldots, v_m) = \sum_{\gamma\in\Gamma} a_\gamma \psi(e_\gamma) = \sum_{\gamma\in\Gamma} a_\gamma w_\gamma = \varphi(v_1, \ldots, v_m)$$

So $\psi = \varphi$. □

Let us point out some *differences between linear and multilinear maps*. When $T : V \to W$ is linear, T is completely determined by the $n := \dim V$ images $T(e_1), \ldots, T(e_n)$ where $E = \{e_1, \ldots, e_n\}$ is a basis of V. But to determine a multilinear map φ, we need $|\Gamma| = \prod_{i=1}^m \dim V_i$ images. In general, this is much more than $\dim(V_1 \times \cdots \times V_m) = \sum_{i=1}^m \dim V_i$.

Recall Example 3.2(d) with $m = n = 2$, i.e. $\otimes : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}_{2\times 2}$ defined by $\otimes(x, y) = xy^T$. Since $\operatorname{rk}(xy^T) \le \min\{\operatorname{rk} x, \operatorname{rk} y^T\} \le 1$, we have $\det(\otimes(x, y)) = 0$. But if $x_1 = (1, 0)^T$ and $x_2 := (0, 1)^T$, then

$$\det(\otimes(x_1, x_1) + \otimes(x_2, x_2)) \det I_2 = 1$$

(for I_2 the identity matrix in $\mathbb{C}_{2,2}$).
Hence $\otimes(x_1, x_1) + \otimes(x_2, x_2)$ is not in the image of $\otimes$ and $\operatorname{im} \otimes$ is not a subspace.
In general, the image $\operatorname{im} \varphi = \{\varphi(v_1, \ldots, v_m) \mid v_i \in V_i, i_1, \ldots, m\}$ is not necessarily a subspace of W. But we can consider the span $\langle \operatorname{im} \varphi\rangle$ of the image of φ. Clearly, $\dim\langle \operatorname{im} \varphi\rangle \le \prod_{i=1}^m \dim V_i$.

Definition 3.4. The *rank* of φ is defined to be

$$\operatorname{rk} \varphi = \operatorname{rank} \varphi = \dim\langle \operatorname{im} \varphi\rangle$$

The multilinear map φ is called a *tensor map* if $\operatorname{rk} \varphi = \prod_{i=1}^m \dim V_i$. In other words, a tensor map is a multilinear map with *maximal image span*.

Example 3.2(a) is a trivial tensor map. Example 3.2(c) is a tensor map.

Theorem 3.5. *The multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is a tensor map if and only if the set $\{\varphi(e_\gamma) \mid \gamma \in \Gamma\}$ is linearly independent (for e_γ as in (3.4)).*

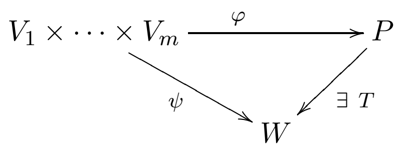
Proof. From (3.2), $\langle \varphi(e_\gamma) \mid \gamma \in \Gamma \rangle = \langle \operatorname{im} \varphi \rangle$ and $|\Gamma| = \prod_{i=1}^m \dim V_i$. □

Theorem 3.6. *Tensor maps exist, i.e. for $V_1, \ldots, V_m$ there exist W and $\varphi : V_1 \times \cdots \times V_m \to W$ such that φ is a tensor map.*

Proof. By Theorem 3.3, pick W a vector space with $\dim W = \prod_{i=1}^m \dim V_i$ and let $\{w_\gamma : \gamma \in \Gamma\}$ be a basis so that the w_γ ($\gamma \in \Gamma$) determine the multilinear map φ which is obviously a tensor map. □

Clearly, tensor maps on $V_1 \times \cdots \times V_m$ are not unique. The study of multilinear maps is reduced to the study of some linear map (not unique) via a tensor map.

Definition 3.7. A multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is said to have the *universal factorization property* if for any multilinear map $\psi : V_1 \times \cdots \times V_m \to W$, there is $T \in \operatorname{Hom}(P, W)$ such that $\psi = T \circ \varphi$. [1]



[Lecture 2, 24.11.2015]

Theorem 3.8. *The multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is a tensor map if and only if φ has universal factorization property.*

Proof. Suppose that φ is a tensor map. Then $\{\varphi(e_\gamma) \mid \gamma \in \Gamma\}$ is a basis of $\langle \operatorname{im} \varphi \rangle$. There is a unique[2] $T \in \operatorname{Hom}(\langle \operatorname{im} \varphi \rangle, W)$ such that $T\varphi(e_\gamma) = \psi(e_\gamma)$ for all $\gamma \in \Gamma$. Since $T\varphi$ and ψ are multilinear maps on $V_1 \times \cdots \times V_m$ (Exercise 2 below or Problem 7.1 (a)), Theorem 3.3 implies $T\varphi = \psi$.

Conversely, suppose that φ has the universal factorization property. In particular consider a tensor map ψ on $V_1 \times \cdots \times V_m$, i.e. , $\dim \langle \operatorname{im} \psi \rangle = \prod \dim V_i$. (Such a map ψ exists by Theorem 3.6).

Then $T\varphi = \psi$ for some linear map T. Thus, $T(\langle \operatorname{im} \varphi \rangle) = \langle \operatorname{im} \psi \rangle$. Hence

$$\prod_i \dim V_i = \dim \langle \operatorname{im} \psi \rangle \leq \dim \langle \operatorname{im} \varphi \rangle \leq \prod_i \dim V_i$$

So $\operatorname{rk} \varphi = \prod \dim V_i$ and φ is a tensor map. □

Definition 3.9. A multilinear map $\varphi : V_1 \times \cdots \times V_r \to W$ with $W = K$ is called a *multilinear form* or an *r-form*. If $V_1 = \cdots = V_r =: V$ (and $W = K$), then φ is called an *r-form on V* or a *multilinear form on V*.

[1] note that T may depend on ψ.

[2] This is a fact from linear algebra: if V and W are K-vector spaces with a basis $\{v_1, \ldots, v_n\}$ of V and if $w_1, \ldots, w_n$ are vectors in W. Then there exists a unique homomorphism $T : V \to W$ such that $T(v_i) = w_i$ for all i.

Exercises for Section 3.1

1. Let $V_1, \ldots, V_m$, $W_1, \ldots, W_m$ and W be $\mathbb{C}$-vector spaces.
   Let $\varphi : W_1 \times \cdots \times W_m \to W$ be multilinear and $T_i : V_i \to W_i$ linear for all i.
   Define $\psi : V_1 \times \cdots \times V_m \to W$ by $\psi(v_1, \ldots, v_m) = \varphi(T_1 v_1, \ldots, T_m v_m)$.
   Show that ψ is multilinear.

2. Let $V_1, \ldots, V_m$, W and W' be $\mathbb{C}$-vector spaces.
   Prove that if $\varphi : V_1 \times \cdots \times V_m \to W$ is multilinear and $T : W \to W'$ is linear, then $T \circ \varphi$ is multilinear.

3. Show that for $n > 1$, the determinant function $\det : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ is not a tensor map.

4. Suppose that the multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ has the universal factorization property. Show that the linear map T (from the defn of the universal factorization property) is unique if and only if $\langle \operatorname{im} \varphi \rangle = P$.

3.2 Tensor products and unique factorization

Let P be a vector space. If there is a tensor map $\otimes : V_1 \times \cdots \times V_m \to P$ such that $\langle \operatorname{im} \otimes \rangle = P$, then P is said to be a *tensor product* of $V_1, \ldots, V_m$ or a *tensor space*. It is written as $\otimes_{i=1}^m V_i$ or as $V_1 \otimes \cdots \otimes V_m$. If $V_1 = \cdots = V_m =: V$, we also write $V^{\otimes m}$. We will see below that all tensor product spaces of $V_1, \ldots, V_m$ are isomorphic and hence we will just say that $\otimes_{i=1}^m V_i$ is *the* tensor product of $V_1, \ldots, V_m$. We have

$$\dim(\otimes_{i=1}^m V_i) = \dim \langle \operatorname{im} \otimes \rangle = \prod_{i=1}^m \dim V_i$$

The elements of $\otimes_{i=1}^m V_i$ are *tensors*. The tensors of the form

$$\otimes(v_1, \ldots, v_m) =: v_1 \otimes \cdots \otimes v_m$$

are the *decomposable tensors* (or pure tensors), i.e., tensors in $\operatorname{im} \otimes$ are decomposable. The decomposable tensors span $\otimes_{i=1}^m V_i$, so we can find a basis of decomposable tensors for $\otimes_{i=1}^m V_i$. The tensors which are not decomposable, i.e., the tensors in $\langle \operatorname{im} \otimes \rangle \setminus \operatorname{im} \otimes$ are called *indecomposable tensors*. From Theorem 3.6, we get the following.

Theorem 3.10. *Let P be a vector space with $\dim P = \prod_{i=1}^m \dim V_i$. Then there exists $\otimes : V_1 \times \cdots \times V_m \to P$, so that P is the tensor space.*

Theorem 3.11. *Tensor spaces of $V_1, \ldots, V_m$ are isomorphic:*
If $P = \otimes_{i=1}^m V_i$ and $Q = \boxtimes_{i=1}^m V_i$ are tensor products of $V_1, \ldots, V_m$, then there exists an invertible $T \in \operatorname{Hom}(P, Q)$ such that $T \circ \otimes = \boxtimes$.

Proof. Since P and Q are tensor spaces of $V_1, \dots, V_m$, $\dim P = \dim Q$. The sets

$$\{\otimes e_\gamma \mid \gamma \in \Gamma\}, \quad \{\boxtimes e_\gamma \mid \gamma \in \Gamma\}$$

are bases of P and Q. We get T by sending basis elements to basis elements. [rest of proof skipped in class]
By Theorem 3.8 (universal factorization property) there exists $T \in \operatorname{Hom}(P, Q)$ such that $T \otimes (e_\gamma) = \boxtimes e_\gamma$, for all $\gamma \in \Gamma$. Thus T is invertible and $T\otimes = \boxtimes$. □

Theorem 3.12 (Unique factorization property). *Let $\psi : V_1 \times \cdots \times V_m \to W$ be multilinear. Then there exist a unique linear map $T : \otimes_{i=1}^m V_i \to W$ such that $\psi = T \circ \otimes$, i.e.,*

$$\psi(v_1, \dots, v_m) = T \otimes (v_1, \dots, v_m) = T(v_1 \otimes \cdots \otimes v_m).$$

Proof. Use Exercise 3.1.4 or Problem 8.2 (a) □

In other words, the tensor map on $\otimes_{i=1}^m V_i$ has the *unique factorization property.*

Corollary 3.13. *Let $\varphi : V_1 \times \cdots \times V_m \to P$ be multilinear. Then φ is a tensor map and $\langle \operatorname{im} \varphi \rangle = P$ if and only if φ has the unique universal factorization property.*

Exercises for Section 3.2

1. Show that if some $v_i = 0$, then $v_1 \otimes \cdots \otimes v_m = 0$.

2. Let $z \in U \otimes V$ so that z can be represented as $z = \sum_{i=1}^k u_i \otimes v_i$. Prove that if k is the smallest number among all such representations, then $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets.

3. Suppose that $e_1, e_2 \in V$ are linearly independent. Prove that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes V$ is indecomposable.

4. Let $P = \otimes_{i=1}^m V_i$ with tensor map φ and let $T \in \operatorname{Hom}(P, Q)$ be invertible. Prove that $\psi = T \circ \varphi$ is also a tensor map and that Q is the tensor space with tensor map ψ.

3.3 Basic properties of tensors and induced inner products

As before, let $V_1, \dots, V_m$ be vector spaces over $K = \mathbb{C}$.

Theorem 3.14. *Consider the tensor product $\otimes_{i=1}^m V_i$ and tensors $u_1 \otimes \cdots \otimes u_m, \dots, w_1 \otimes \cdots \otimes w_m \in \otimes_i V_i$.*
If $u_1 \otimes \cdots \otimes u_m + \cdots + w_1 \otimes \cdots \otimes w_m = 0$ then $\varphi(u_1, \dots, u_m) + \cdots + \varphi(w_1, \dots, w_m) = 0$ for any multilinear map $\varphi : V_1 \times \cdots \times V_m \to W$.

Proof. By Theorem 3.12 we can write $\varphi = T \circ \otimes$ for some $T \in \mathrm{Hom}(\otimes_{i=1}^m V_i, W)$. So

$$\varphi(u_1, \dots, u_m) = T \otimes (u_1, \dots, u_m) = T(u_1 \otimes \dots \otimes u_m)$$

and similarly for all decomposable tensors in the sum. Hence

$$\begin{aligned} &\varphi(u_1, \dots, u_m) + \dots + \varphi(w_1, \dots, w_m) \\ &= T(u_1 \otimes \dots \otimes u_m) + \dots + T(w_1 \otimes \dots \otimes w_m) \\ &= T(\underbrace{u_1 \otimes \dots \otimes u_m + \dots + w_1 \otimes \dots \otimes w_m}_{=0}) \\ &= 0 \end{aligned}$$

□

[Lecture 3, 30.11.2015]

Theorem 3.15. *Let $v_i \in V_i$, $i = 1, \dots, m$. Then $v_1 \otimes \dots \otimes v_m = 0$ if and only if there exists i with $v_i = 0$.*

Proof. $\Longleftarrow$ is clear.
$\Longrightarrow$: Suppose $v_1 \otimes \dots \otimes v_m = 0$ but $v_i \neq 0$ for all i. Then for each i there exists $f_i \in V_i^*$ such that $f_i(v_i) = 1$. Define

$$\varphi := \prod_{i=1}^m f_i : V_1 \times \dots \times V_m \to \mathbb{C}$$

This is a multilinear map (cf. Example 3.2 (f)). Then

$$\varphi(v_1, \dots, v_m) = \prod_{i=1}^m f_i(v_i) = 1$$

But by Theorem 3.14, $\varphi(v_1, \dots, v_m) = 0$, a contradiction. □

Theorem 3.16. *Let $u_1 \otimes \dots \otimes u_m$ and $v_1 \otimes \dots \otimes v_m$ be in $V_1 \otimes \dots \otimes V_m$.*
$u_1 \otimes \dots \otimes u_m = v_1 \otimes \dots \otimes v_m \neq 0 \iff v_i = c_i u_i$ for all $i = 1, \dots, m$ and $\prod_{i=1}^m c_i = 1$.

Proof. $\Longleftarrow$ is clear.
$\Longrightarrow$: Suppose $u_1 \otimes \dots \otimes u_m = v_1 \otimes \dots \otimes v_m \neq 0$. From Theorem 3.15, all u_i and v_i are non-zero. From Theorem 3.14 we know that

$$\varphi(u_1, \dots, u_m) = \varphi(v_1, \dots, v_m).$$

holds for any multilinear map φ from $V_1 \times \dots \times V_m$ to some $\mathbb{C}$-vector space W.
Suppose that u_k and v_k are not linearly dependent for some k (i.e. assume that there is an index k such that u_k is not a (non-zero) multiple of v_k). Then there is $f_k \in V_k^*$ such

that $f_k(v_k) = 1$ and $f_k(u_k) = 0$. For $i \neq k$ choose $f_i \in V_i^*$ such that $f_i(v_i) = 1$. (This is multilinear, cf. Example 3.2 (f)). Set $\varphi := \prod_{i=1}^m f_i$. Then

$$\varphi(v_1, \dots, v_m) = \prod_{i=1}^m f_i(v_i) = 1$$

and since $f_k(u_k) = 0$

$$\varphi(u_1, \dots, u_m) = \prod_{i=1}^m f_i(u_i) = 0,$$

contradicting $\varphi(u_1, \dots, u_m) = \varphi(v_1, \dots, v_m)$. Hence for all k there exists $c_k \neq 0$ such that $v_k = c_k u_k$. From

$$0 \neq u_1 \otimes \cdots \otimes u_m = v_1 \otimes \cdots \otimes v_m = (\prod_{i=1}^m c_i) u_1 \otimes \cdots \otimes u_m$$

we have $\prod_{i=1}^m c_i = 1$. □

Since the decomposable elements span the tensor space $\otimes_{i=1}^m V_i$, each $z \in \otimes_{i=1}^m V_i$ is a linear combination of decomposable tensors. Let k be the smallest number of decomposable tensors in all such linear combinations for z. We call k the *rank* or the *smallest length* of z. The rank of $z \neq 0$ is one if and only if z is decomposable.

Theorem 3.17. *Assume that $z \in U \otimes V$ can be represented as $z = \sum_{i=1}^r u_i \otimes v_i$ for some $r \geq 1$. Then the smallest length of z is k if and only if $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets, respectively.*

Proof. The implication $\Longrightarrow$ is Exercise 2 of Section 3.2 or Problem 8.1 (b).
$\Longleftarrow$: Assume that $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets. Let $z = \sum_{j=1}^r x_j \otimes y_j$, we show that $k \leq r$. Let $l \in \{1, \dots, m\}$ be arbitrary. Since the v_i are linearly independent, there is $g \in V^*$ such that $g(v_l) = 1$ and $g(v_j) = 0$ for $j \neq l$. Let $f \in U^*$ be arbitrary. Then the map $\varphi := fg : U \times V \to \mathbb{C}$ is bilinear (cf. Example 3.2 (f)). By assumption,

$$\sum_{i=1}^k u_i \otimes v_i = \sum_{j=1}^r x_j \otimes y_j$$

Using Theorem 3.14 for $\varphi = fg$, we have (first equality by the choice of g)

$$f(u_l) = \sum_{i=1}^k f(u_i) g(v_i) = \sum_{j=1}^r f(x_j) g(y_j) = f(\sum_{j=1}^r g(y_i) x_j)$$

Since f is arbitrary, we get $u_l = \sum_{j=1}^r g(y_i) x_j$, so u_l is in $\langle x_1, \dots, x_r \rangle$ for all $l = 1, \dots, k$. (choosing other linear maps g when l varies). Since $\{u_1, \dots, u_k\}$ is linearly independent, we have $k \leq r$. □

We now consider an induced inner product (a reminder on inner products is at the end of the chapter, cf. Subsection 3.7.1) of $\otimes_{i=1}^m V_i$. Suppose that $(\cdot,\cdot)_i$ is an inner product on V_i and $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ is an orthonormal basis of V_i for all $i = 1, \ldots, m$. We know that

$$E := \{e_\gamma^\otimes := e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)} : \gamma \in \Gamma\}$$

is a basis of $\otimes_{i=1}^m V_i$, where $\Gamma = \Gamma(n_1, \ldots, n_m)$. We would like to have an inner product $(\cdot,\cdot)$ on the tensor product $\otimes_{i=1}^m V_i$ such that E is an orthonormal basis, i.e.

$$(e_\alpha^\otimes, e_\beta^\otimes) = (e_{1\alpha(1)} \otimes \cdots \otimes e_{m\alpha(m)}, e_{1\beta(1)} \otimes \cdots \otimes e_{m\beta(m)}) = \delta_{\alpha,\beta}$$

($\delta_{\alpha,\beta}$ is the generalization of the Kronecker delta to m-tuples).
Such an inner product is unique (Section 3.7.1).

We define

$$(u, v) := \sum_{\gamma\in\Gamma} a_\gamma \overline{b_\gamma} \tag{3.5}$$

where $u = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma^\otimes$, $v = \sum_{\gamma\in\Gamma} b_\gamma e_\gamma^\otimes \in \otimes_{i=1}^m V_i$. With this defintion, E becomes an ONB (cf. Problem 8.4 (a)). The definition appears to depend on the choice of the basis, but it does not (cf. Problem 7.3).

Theorem 3.18. *Let $V_1, \ldots, V_m$ be inner product spaces with orthonormal bases $E_i = \{e_{i1}, \ldots, e_{i,n_i}\}$, $i = 1, \ldots, m$. The inner product obtained from (3.5) satisfies*

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^m (u_i, v_i)_i \tag{3.6}$$

(where $u_i, v_i \in V_i$ for all i and $(\cdot,\cdot)_i$ is the inner product on V_i).

Proof. We write u_i and v_i in terms of the basis E_i,

$$u_i = \sum_{j=1}^{n_i} a_{ij} e_{ij}, \quad v_i = \sum_{j=1}^{n_i} b_{ij} e_{ij}, \quad i = 1, \ldots, m.$$

Since $\otimes$ is multilinear, from (3.2) we get

$$u^\otimes := u_1 \otimes \cdots \otimes u_m = \otimes(u_1, \ldots, u_m) = \sum_{\gamma\in\Gamma} \Big(\overbrace{\prod_{i=1}^m a_{i\gamma(i)}}^{a_\gamma}\Big) e_\gamma^\otimes$$

$$v^\otimes := v_1 \otimes \cdots \otimes v_m = \otimes(v_1, \ldots, v_m) = \sum_{\gamma\in\Gamma} \Big(\underbrace{\prod_{i=1}^m b_{i\gamma(i)}}_{b_\gamma}\Big) e_\gamma^\otimes$$

By the definition (3.5) of the inner product and since $\prod_{i=1}^m \sum_{j=1}^{n_i} c_{ij} \overset{(3.1)}{=} \sum_{\gamma\in\Gamma} \prod_{i=1}^m c_{i\gamma(i)}$,

$$\begin{aligned}(u^\otimes, v^\otimes) &= \sum_{\gamma\in\Gamma}\prod_{i=1}^m a_{i\gamma(i)}\overline{b_{i\gamma(i)}}\\ &\overset{(3.1)}{=} \prod_{i=1}^m \sum_{j=1}^{n_i} a_{ij}\overline{b_{ij}}\\ &= \prod_{i=1}^m (\sum_{j=1}^{n_i} a_{ij}e_{ij}, \sum_{j=1}^{n_i} b_{ij}e_{ij})_i\\ &= \prod_{i=1}^m (u_i, v_i)_i.\end{aligned}$$

(where $(\cdot,\cdot)_i$ denotes the inner product on V_i). □

There are many bilinear maps from $(\otimes_{i=1}^k V_i) \times (\otimes_{i=k+1}^m V_i)$ to $\otimes_{i=1}^m V_i$ since

$$\dim \otimes_{i=1}^m V_i = \prod_{i=1}^m n_i = \prod_{i=1}^k n_i \prod_{i=k+1}^m n_i = \dim(\otimes_{i=1}^k V_i)\dim(\otimes_{i=k+1}^m V_i). \tag{3.7}$$

What we like is one that maps the pair $(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m)$ to $v_1 \otimes \cdots \otimes v_m$.

Theorem 3.19. *There is a unique multilinear map* $\boxtimes : (\otimes_{i=1}^k V_i) \times (\otimes_{i=k+1}^m V_i) \to \otimes_{i=1}^m V_i$ *such that*

$$\boxtimes(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_m \tag{3.8}$$

and

$$(V_1 \otimes \cdots \otimes V_k) \boxtimes (V_{k+1} \otimes \cdots \otimes V_m) = V_1 \otimes \cdots \otimes V_m \tag{3.9}$$

Proof. By (3.7), the dimensions are ok, so the tensor map $\boxtimes$ satisfying (3.8) exists (Theorem 3.10) and is unique (Theorem 3.12). From $\langle \operatorname{im} \boxtimes \rangle = \langle v_1 \otimes \cdots \otimes v_m \mid v_i \in V_i \rangle = \otimes_{i=1}^m V_i$, (3.9) follows. (See Exercise no. 4 in 3.3 for details.) □

[Lecture 4, 1.12. 2015]

We also write $\otimes$ for $\boxtimes$ in Theorem 3.18. So we can write (3.9) as

$$(V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_m) = V_1 \otimes \cdots \otimes V_m$$

and (3.8) can be written as

$$(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_m.$$

Exercises for Section 3.3

1. Suppose that $v_1, \ldots, v_k \in V$ are linearly independent and let $u_1, \ldots, u_k \in U$. Prove that $\sum_{i=1}^k u_i \otimes v_i = 0$ if and only if $u_1 = \cdots = u_k = 0$.

2. Let $v_1, \ldots, v_k \in V$ and $A \in \mathbb{C}_{k\times k}$. Suppose $AA^T = I_k$ and $u_j = \sum_{i=1}^k a_{ij} v_i$, $j = 1, \ldots, k$. Prove that $\sum_{i=1}^k u_i \otimes u_i = \sum_{i=1}^k v_i \otimes v_i$. (Problem 8.1. (b)).

3. Define $\otimes : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}_{k\times n}$ by $x \otimes y := xy^T$. Let $\mathbb{C}^k$ and $\mathbb{C}^n$ be equipped with the standard inner products. Prove that for any $A, B \in \mathbb{C}_{k\times n} = \mathbb{C}^k \otimes \mathbb{C}^n$, the induced inner product is given by $(A, B) = \operatorname{tr}(B^*A)$. (notation: $\operatorname{tr}(B^*A)$ is the trace of the square matrix B^*A)
   Cf. Problem 8.4 (b).

4. Let $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ be a basis of V_i, $i = 1, \ldots, m$. Define

$$\varphi : (V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_m) \to V_1 \otimes \cdots \otimes V_m$$

   by $\varphi(e_{1i_1} \otimes \cdots \otimes e_{ki_k}, e_{k+1 i_{k+1}} \otimes \cdots \otimes e_{mi_m}) = e_{1i_1} \otimes \cdots \otimes e_{mi_m}$ (with bilinear extension). Show that φ is the tensor map satisfying

$$\varphi(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \ldots \otimes v_m$$

   (Problem 9.1 (a))

5. Let $z = \sum_{i=1}^k u_i \otimes v_i \otimes w_i \in U \otimes V \otimes W$. Prove that if $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are linearly independent and $w_i \neq 0$ for all i, then k is the smallest length of z.

3.4 Induced maps

In this section, we study $\operatorname{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$. Let $T_i \in \operatorname{Hom}(V_i, W_i)$ for all i. Then we define a multilinear map from $V_1 \times \cdots \times V_m$ to $\otimes_{i=1}^m W_i$ by

$$\varphi(v_1, \ldots, v_m) := T_1 v_1 \otimes \cdots \otimes T_m v_m$$

By Theorem 3.12 (with $\otimes_{i=1}^m W_i$ in the role of W) there is a unique $T \in \operatorname{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$ such that ($\varphi = T \circ \otimes$, i.e.)

$$T(v_1 \otimes \cdots \otimes v_m) = T_1 v_1 \otimes \cdots \otimes T_m v_m$$

We denote this T by $T_1 \otimes \cdots \otimes T_m$ and call it the *induced map* of $T_1, \ldots, T_m$, i.e.

$$(\otimes_{i=1}^m T_i)(v_1 \otimes \cdots \otimes v_m) = T_1 v_1 \otimes \cdots \otimes T_m v_m.$$

(Drawing the corresponding diagram with vector spaces might be helpful).

One can show that $T_1 \otimes \cdots \otimes T_m$ is a tensor (an element) of the tensor product $\otimes_{i=1}^m \operatorname{Hom}(V_i, W_i)$.

if time: this is in section 3.7, thm 3.7.2

The main focus of this section is to study the map $T_1 \otimes \cdots \otimes T_m \in \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$ as a linear map. We will see how it behaves with respect to composition, how to determine its rank and its adjoint.

Theorem 3.20. *Let $S_i \in \mathrm{Hom}(W_i, U_i)$, $T_i \in \mathrm{Hom}(V_i, W_i)$, $i = 1, \ldots, m$. Then*

$$(\otimes_{i=1}^m S_i) \circ (\otimes_{i=1}^m T_i) = \otimes_{i=1}^m (S_i \circ T_i).$$

(It might be helpful to draw the diagrams with the vector spaces to illustrate the statement).

Proof. It is enough to show the claim on decomposable elements, since $\otimes_{i=1}^m V_i$ is spanned by them. This is what we do.

$$\begin{aligned}(\otimes_{i=1}^m S_i)(\otimes_{i=1}^m T_i)(v_1 \otimes \cdots \otimes v_m) =& (\otimes_{i=1}^m S_i)(T_1 v_1 \otimes \cdots \otimes T_m v_m) \\ =& S_1 T_1 v_1 \otimes \cdots \otimes S_m T_m v_m \\ =& \otimes_{i=1}^m (S_i T_i)(v_1 \otimes \cdots \otimes v_m).\end{aligned}$$

□

Theorem 3.21. *Let $T_i \in \mathrm{Hom}(V_i, W_i)$, $i = 1, \ldots, m$. Then*

$$\mathrm{rk}\,(T_1 \otimes \cdots \otimes T_m) = \prod_{i=1}^m \mathrm{rk}\, T_i$$

Proof. Let $\mathrm{rk}\, T_i = k_i$ for all i. So there is a basis $\{e_{i1}, \ldots, e_{ik_i}, e_{ik_i+1}, \ldots, e_{in_i}\}$ for V_i such that $T_i e_{i1}, \ldots, T_i e_{ik_i}$ are linearly independent in W_i and $T_i e_{ik_i+1} = \cdots = T_i e_{in_i} = 0$, $i = 1, \ldots, m$. The set $\{e_\gamma^\otimes : \gamma \in \Gamma(n_1, \ldots, n_m)\}$ is a basis for $\otimes_{i=1}^m V_i$.

Moreover,

$$(\otimes_{i=1}^m T_i) e_\gamma^\otimes = T_1 e_{1\gamma(1)} \otimes \cdots \otimes T_m e_{m\gamma(m)}$$

so that if $\gamma \notin \Gamma(k_1, \ldots, k_m)$, then $(\otimes_{i=1}^m T_i) e_\gamma^\otimes = 0$ (as for some i, $\gamma(i) > k_i$). Since $Te_{i1}, \ldots, Te_{i,k_i}$ are linearly independent in W_i for all i, the vectors

$$(\otimes_{i=1}^m T_i) e_\gamma^\otimes, \quad \gamma \in \Gamma(k_1, \ldots, k_m)$$

are linearly independent in $\otimes_{i=1}^m W_i$ (why?), hence

$$\mathrm{rk} \otimes_{i=1}^m T_i = |\Gamma(k_1, \ldots, k_m)| = \prod_{i=1}^m k_i = \prod_{i=1}^m \mathrm{rk}\, T_i.$$

□

The next result describes the adjoint of $T_1 \otimes \cdots \otimes T_m$ viewed as a linear map. For notation/reminder: see Subsection 3.7.2 at the end of this chapter.

Theorem 3.22. *Let $T_i \in \mathrm{Hom}(V_i, W_i)$, where V_i and W_i are inner product spaces for $i = 1, \dots, m$. Then*

$$(\otimes_{i=1}^m T_i)^* = \otimes_{i=1}^m T_i^*.$$

Proof. We use the notation $(\cdot, \cdot)$ for the inner products on all the spaces V_i, W_i.

$$\begin{aligned}
((\otimes_{i=1}^m T_i)v^{\otimes}, w^{\otimes}) &= (T_1 v_1 \otimes \cdots \otimes T_m v_m, w_1 \otimes \cdots \otimes w_m) && \text{(inner product on } \otimes_i W_i) \\
&= \prod_{i=1}^m (T_i v_i, w_i) && \text{(defin.: inner prod on } \otimes_i W_i \text{ in terms of inner prod on } W_i\text{'s)} \\
&= \prod_{i=1}^m (v_i, T_i^* w_i) && \text{(adjoints of } T_i^*\text{, now on } V_i\text{'s)} \\
&= (v_1 \otimes \cdots \otimes v_m, T_1^* w_1 \otimes \cdots \otimes T_m^* w_m) && \text{(defin. of inner prod on } \otimes_i V_i) \\
&= (v^{\otimes}, \otimes_{i=1}^m T_i^* w^{\otimes})
\end{aligned}$$

Since $\otimes_{i=1}^m V_i$ is spanned by decomposable tensors, we have the desired result. □

Exercises for Section 3.4

1. Prove that
   (a) Prove that $T_1 \otimes \cdots \otimes T_m = 0$ if and only if some $T_i = 0$,
   (b) $T_1 \otimes \cdots \otimes T_m$ is invertible if and only if all T_i are invertible.
2. Let $S_i, T_i \in \mathrm{Hom}(V_i, W_i)$ for $i = 1, \dots, m$. Prove that $\otimes_{i=1}^m T_i = \otimes_{i=1}^m S_i \neq 0$ if and only if $T_i = c_i S_i \neq 0$ for $i = 1, \dots, m$ and $\prod_{i=1}^m c_i = 1$.
3. Let $T_i \in \mathrm{End}\, V_i$ for $i = 1, \dots, m$. Prove that $\otimes_{i=1}^m T_i$ is invertible if and only if T_i is invertible for every i. In this case, $(\otimes_{i=1}^m T_i)^{-1} = \otimes_{i=1}^m T_i^{-1}$.
4. Let $T_i \in \mathrm{Hom}(V_i, W_i)$ for $i = 1, \dots, m$. Define

   $$\varphi : \mathrm{Hom}(V_1, W_1) \times \cdots \times \mathrm{Hom}(V_m, W_m) \to \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$$

   by $\varphi(T_1, \dots, T_m) = T_1 \otimes \cdots \otimes T_m$. Prove that φ is multilinear.

Exercise 4 can be used to show that $T_1 \otimes \cdots \otimes T_m$ is an element of a tensor product, namely of $\mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$.

3.5 Some models of tensor products

[Lecture 5, 8.12. 2015]

We now consider the special case $V_1 = \cdots = V_m$ where all the spaces V_i are equal and where the target space is the field $\mathbb{C}$.

Recall that $M(V,\dots,V;\mathbb{C})$ denotes the space of all m-multilinear maps $f : V^m := \times^m(V) := V\times\cdots\times V \to \mathbb{C}$.

The goal is to see that the tensor product $\overbrace{V\otimes\cdots\otimes V}^{m \text{ copies}}$ can be viewed as $M(\overbrace{V^*,\dots,V^*}^{m\text{ copies}};\mathbb{C})$ or as $M(V,\dots,V;\mathbb{C})$ and that the tensor product $V^*\otimes\cdots\otimes V^*$ can be understood as $M(\underbrace{V,\dots,V}_{m\text{ copies}};\mathbb{C})$.

Let $E = \{e_1,\dots,e_n\}$ be a basis of V and let $E^* = \{f_1,\dots,f_n\}$ be the dual basis of V^*, i.e.,

$$f_i(e_j) = \delta_{ij}, \quad i,j = 1,\dots,n.$$

In this situation we abbreviate the set $\Gamma(\overbrace{n,\dots,n}^{m\text{ times}})$ as

$$\Gamma(n:m) = \{\gamma : \gamma = (\gamma(1),\dots,\gamma(m)), 1\le \gamma(i)\le n, i = 1,\dots,m\}.$$

We write $V^{\otimes m}$ or also $\otimes^m(V)$ for the m-fold tensor product $V\otimes\cdots\otimes V$.

Theorem 3.23. *1. The set $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha\in\Gamma(n:m)\}$ is a basis of $M(V,\dots,V;\mathbb{C})$.*

2. $M(V,\dots,V;\mathbb{C}) = (V^*)^{\otimes m}$, in other words: there is a tensor map $\otimes : V^*\times\cdots\times V^* \to M(V,\dots,V;\mathbb{C})$ and $\dim M(V,\dots,V;\mathbb{C}) = n^m$.

3. $M(V^*,\dots,V^*;\mathbb{C}) = V^{\otimes m}$, in other words: there is a tensor map $\otimes : V\times\cdots\times V \to M(V^*,\dots,V^*;\mathbb{C})$ and $\dim M(V^*,\dots,V^*;\mathbb{C}) = n^m$.

Proof. (1) We first show that the set $S := \{\prod_{i=1}^m f_{\alpha(i)} : \alpha\in\Gamma(n:m)\}$ spans $M(V,\dots,V;\mathbb{C})$. For this, first observe that for each $e_\beta = (e_{\beta(1)},\dots,e_{\beta(m)}) \in V\times\cdots\times V$ and $\beta\in\Gamma(n:m)$,

$$(\prod_{i=1}^m f_{\alpha(i)})e_\beta = \prod_{i=1}^m f_{\alpha(i)}(e_{\beta(i)}) = \delta_{\alpha,\beta} \tag{3.10}$$

Let $f\in M(V,\dots,V;\mathbb{C})$. Then we claim we can write f as follows

$$f = \sum_{\alpha\in\Gamma(n:m)} f(e_\alpha)\prod_{i=1}^m f_{\alpha(i)}$$

where $e_\alpha = (e_{\alpha(1)},\dots,e_{\alpha(m)})$. This works since by (3.10),

$$\left(\sum_{\alpha\in\Gamma(n:m)} f(e_\alpha)\prod_{i=1}^m f_{\alpha(i)}\right)(e_\beta) = \sum_{\alpha\in\Gamma(n:m)} f(e_\alpha)\delta_{\alpha,\beta} = f(e_\beta), \qquad \beta\in\Gamma(n:m)$$

So f can be written as a linear combination of elements of S.

It remains to show that S is a linearly independent set. Assume

$$\sum_{\alpha\in\Gamma(n:m)} c_\alpha \prod_{i=1}^m f_{\alpha(i)} = 0.$$

Then

$$0 = \left(\sum_{\alpha\in\Gamma(n:m)} c_\alpha \prod_{i=1}^m f_{\alpha(i)}\right)(e_\beta) = \Big(\sum_{\alpha\in\Gamma(n:m)} c_\alpha \underbrace{\prod_{i=1}^m f_{\alpha(i)}(e_\beta)}_{\delta_{\alpha,\beta}}\Big) = c_\beta \quad \text{for every } \beta\in\Gamma(n:m)$$

(2) It is easy to see that the map $\otimes : (V^*)^m = \times^m(V^*) \to M(V,\dots,V;\mathbb{C})$ defined by

$$\otimes(g_1,\dots,g_m) = \prod_{i=1}^m g_i$$

is multilinear. From part (1), $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha\in\Gamma(n:m)\}$ is a basis of $M(V,\dots,V;\mathbb{C})$. So

$$\dim M(V,\dots,V;\mathbb{C}) = |\Gamma(n:m)| = n^m = (\dim V)^m = (\dim V^*)^m.$$

Therefore, $\otimes$ is a tensor map and $\langle \operatorname{im}\otimes\rangle = M(V,\dots,V;\mathbb{C})$, i.e. $M(V,\dots,V;\mathbb{C}) = \otimes^m V^*$.

(3) Analoguous to the proof of (2): We can define $\otimes : V^m \to M(V^*,\dots,V^*;\mathbb{C})$ by $\otimes(v_1,\dots,v_m) = \prod_{i=1}^m v_i$ where the latter is defined as follows:

$$\left(\prod_{i=1}^m v_i\right)(g_1,\dots,g_m) := \prod_{i=1}^m g_i(v_i)$$

□

By Theorem 3.23 (2) and (3), we can say that $M(V,\dots,V;\mathbb{C})$ *is a model for* $(V^*)^{\otimes m}$ and that $M(V^*,\dots,V^*,\mathbb{C})$ is a model for $V^{\otimes m}$.
Another model for $V^{\otimes m}$ is $M(V,\dots,V;\mathbb{C})^*$, the dual space of $M(V,\dots,V;\mathbb{C})$, as we will see now.

Theorem 3.24. *$M(V,\dots,V;\mathbb{C})^*$ is a model for $\otimes^m V$, i.e., there is a tensor map $\otimes : V\times\cdots\times V \to M(V,\dots,V;\mathbb{C})^*$ and $\dim M(V,\dots,V;\mathbb{C})^* = n^m$.*

Proof. For $v_1,\dots,v_m\in V$ define $\prod_{i=1}^m v_i \in M(V,\dots,V;\mathbb{C})^*$ by

$$(\prod_{i=1}^m v_i)f = f(v_1,\dots,v_m), \quad f\in M(V,\dots,V;\mathbb{C}).$$

From Theorem 3.23 (1), the set $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha\in\Gamma_{m,n}\}$ is a basis of $M(V,\dots,V;\mathbb{C})$. Now from (3.10), $\{\prod_{i=1}^m e_{\alpha(i)} : \alpha\in\Gamma(n:m)\}$ is the dual basis of $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha\in\Gamma(n:m)\}$ and thus is a basis of $M(V,\dots,V;\mathbb{C})^*$. Then define $\otimes : V\times\cdots\times V\to M(V,\dots,V;\mathbb{C})^*$ by $\otimes(v_1,\dots,v_m) = \prod_{i=1}^m v_i$. □

Elements of the vector space $M(V,\dots,V;\mathbb{C})$ are called *contra-variant tensors*; elements of $M(V^*,\dots,V^*,\mathbb{C})$ are called *covariant tensors*.[3]

The tensor space

$$V_q^p := \overbrace{V\otimes\cdots\otimes V}^{p}\otimes\overbrace{V^*\otimes\cdots\otimes V^*}^{q}$$

is called a *tensor space of type* (p,q) (with *covariant type of degree* p *and with contra-variant type of degree* q). Analoguous to the previous treatment, under some tensor map, $M(V^*,\dots,V^*,V,\dots,V;\mathbb{C})$ (p copies of V and q copies of V^*, compare this with Theorem 3.23) is a model of V_q^p as we will indicate now:

Let $E=\{e_1,\dots,e_n\}$ be a basis of V and let $E^*=\{f_1,\dots,f_n\}$ be the dual basis of V^*. Then

$$\{\prod_{i=1}^p e_{\alpha(i)}\prod_{j=1}^q f_{\beta(j)} : \alpha\in\Gamma(n:p),\beta\in\Gamma(n:q)\}$$

is a basis for

$$M(V^*,\dots,V^*,V,\dots,V;\mathbb{C}) \quad (p \text{ copies of } V \text{ and } q \text{ copies of } V^*)$$

Define $\otimes : V\times\cdots\times V\times V^*\times\cdots\times V^*\to M(V^*,\dots,V^*,V,\dots,V;\mathbb{C})$ by

$$\otimes(e_{\alpha(1)},\dots,e_{\alpha(p)},f_{\beta(1)},\dots,f_{\beta(q)})=\prod_{i=1}^p e_{\alpha(i)}\prod_{j=1}^q f_{\beta(j)}$$

Then

$$\begin{aligned}&\{e_{\alpha(1)}\otimes\cdots\otimes e_{\alpha(p)}\otimes f_{\beta(1)}\otimes\cdots\otimes f_{\beta(q)} : \alpha\in\Gamma_{p,n},\beta\in\Gamma_{q,n}\}\\ =&\{e_\alpha^\otimes\otimes f_\beta^\otimes : \alpha\in\Gamma_{p,n},\beta\in\Gamma_{q,n}\}\end{aligned}$$

is a basis of V_q^p.

Exercises for Section 3.5

1. Define a simple tensor map $\boxtimes : (V^*)^m\to(\otimes^m V)^*$ such that $\boxtimes^m V^*=(\otimes^m V)^*$.

2. Let $M(V_1,\dots,V_m;W)$ be the set of all multilinear maps from $V_1\times\cdots\times V_m$ to W. Prove that $\dim M(V_1,\dots,V_m;W)=\dim W\cdot\prod_{i=1}^m\dim V_i$.

[3] contravariant vectors: under a base change using the invertible matrix A, the entries of a contravariant vector change with the inverse A^{-1} of the matrix of the base change. Under this base change, covariant vectors change with A.

3.6 Exterior spaces

now chapter 5 (page 127) in the book? maybe not.
or waffle, pages 7-8 and Lerman, pages 710 -and the notes.
First following Waffle: Then Serge Lang, XIX, §1
And book by Gallier (diff geo)

In this section, we consider some specific examples of multilinear maps. The starting point is the space of multilinear maps (forms) $M(V, \dots, V; K)$ from m copies of the K-vector space V to the field K.

We start by considering a specific notion of multiplication, namely one that relates to area and volume. The basic idea is as follows: Given two vectors v and w, we can form the parallelogram that they span, and write $v \wedge w$ for something as the "area" of the parallelogram. This is not quite the usual notion of area, however, because we want to think of it as not just a single number (in general) but also as having a "two-dimensional direction" (the same way a single vector v both has a size and a direction). That is, if we had a parallelogram pointing in a "different direction", i.e. in a different plane, we would think of it as different.

What would the properties of $v \wedge w$ be. Scaling the entry v or the entry w scales the parallelogram, so it should scale its area. So we expect, for scalars c, to have $(cv) \wedge w = c(v \wedge w) = v \wedge (cw)$. So the operation $\wedge$ should be bilinear. Another property of $\wedge$ would be, that for any vector v, $v \wedge v$ should be 0 - if the vectors point in the same direction, the "parallelogram" they span is just a line segment and has no area. These are the only two properties we really need.

As before, vector spaces are finite dimensional and defined are over some field K - mostly over $\mathbb{C}$ or over the real numbers $\mathbb{R}$.

Definition 3.25. Let V be a vector space. Then the *exterior square (product)* $\bigwedge^2(V)$ of V is the quotient of $V \otimes V$ by the subspace U spanned by the elements $v \otimes v$ for all $v \in V$. We write $v \wedge w$ for the image of $v \otimes w$ under the quotient map $V \otimes V \to \bigwedge^2(V)$.

What does $\bigwedge^2(V)$ look like? First observation (consequence of $v \wedge v = 0$):

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v.$$

So for any v and w, $v \wedge w = -w \wedge v$. The operation $\wedge$ is thus anti-commutative or *alternating*.

[Lecture 6, 15.12. 2015]

In terms of a basis of V, say $\{e_i\}_i$, we have that the set $\{e_i \otimes e_j\}_{ij}$ is a basis of $V \otimes V$. In $\bigwedge^2(V)$, we have $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. So $\bigwedge^2(V)$ can be spanned by the elements $e_i \wedge e_j$ for $i < j$. We can show that they are linearly independent:

Theorem 3.26. *Suppose $\{e_i\}_{i=1,\dots,n}$ is a basis for V. Then $\{e_i \wedge e_j\}_{1 \le i < j \le n}$ is a basis for $\bigwedge^2(V)$. In particular,* $\dim \bigwedge^2(V) = \binom{n}{2}$.

Proof. The idea behind the proof is that $\bigwedge^2(V)$ is the "free" (or "universal") vector space in which you can multiply two elements of V in an anti-commutative way, so to show that the $e_i \wedge e_j$ (for $i < j$) are linearly independent, you have to construct some vector space with such a multiplication in which they are linearly independent.

We define a vector space E as follows: an element of E is a formal linear combination of symbols e_{ij} for $i < j$. So the set $\{e_{ij}\}_{i<j}$ is a basis of E.
We define a map $T : V \otimes V \to E$ by $T(e_i \otimes e_j) = e_{ij}$ if $i < j$, $T(e_i \otimes e_j) = -e_{ji}$ if $i > j$, and $T(e_i \otimes e_i) = 0$. We want to show that T gives a map $S : \bigwedge^2(V) \to E$; it suffices to show that $T(v \otimes v) = 0$ for all $v \in V$. Let $v = \sum c_i e_i$; then

$$v \otimes v = \sum_{ij} c_i c_j e_i \otimes e_j = \sum_i c_i^2 e_i \otimes e_i + \sum_{i<j} c_i c_j (e_i \otimes e_j + e_j \otimes e_i).$$

We thus see that $T(v \otimes v) = 0$. Hence T gives a map $S : \bigwedge^2(V) \to E$ which sends $e_i \wedge e_j$ to e_{ij}. Since the e_{ij} (for $i < j$) are linearly independent in E by construction, this implies that the $e_i \wedge e_j$ (for $i < j$) are linearly independent, and hence a basis. □

One thing to note about $\bigwedge^2(V)$ (as we know about $\bigotimes^2(V)$) is that *not* every element is of the form $v \wedge w$. (That is, not every "area vector" is just an area in some plane; it can also be a sum of areas in different planes.) For example, if $\{e_i\}$ is a basis of V, $\dim V \geq 4$, then $e_1 \wedge e_2 + e_3 \wedge e_4$ cannot be simplified to a single $v \wedge w$.

Definition 3.27. Let $r \geq 0$ be an integer, and V be a vector space. Then the *rth exterior power of* V, $\bigwedge^r(V)$ is the quotient of $V \otimes \cdots \otimes V$ (with r factors) by the subspace U spanned by all tensors $v_1 \otimes \cdots \otimes v_r$ for which two of the v_i are equal.
Elements of $\bigwedge^r(V)$ are *alternating r-tensors* or *alternating tensors of degree r*. Elements of the form $u_1 \wedge \cdots \wedge u_r$ with $u_i \in V$ are called *simple* or *decomposable alternating r-tensors*.
The *exterior algebra* $\bigwedge(V)$ is the direct sum $\bigoplus_{r\geq 0} \bigwedge^r(V)$. It is also called the *Grassmann algebra* or the *alternating algebra*.

Remark 3.28. (1) The exterior algebra is an algebra over the field K. This means, $\bigwedge(V)$ is a K-vector space with a bilinear product, it is a set with multiplication, addition and scalar multiplication by elements of the field. (Note that it is finite dimensional, since we assumed that V is finite dimensional, cf. Theorem 3.31).

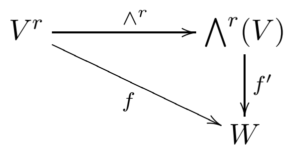
(2) We could have worked with modules over a commutative ring instead (and this is also done when introducing tensor products, exterior product, etc.), so replacing V and K with a module E over a commutative ring R and considered r-multilinear maps from E^r to the field K. The tensor product $E^{\otimes m}$ and the exterior product $\bigwedge^r(E)$ are also modules for R.

By definition, there is an r-multilinear map $V^r \to \bigwedge^r(V)$ (called canonical) obtained from the composition

$$V^r \longrightarrow V^{\otimes r} \longrightarrow V^{\otimes r}/U = \bigwedge^r(V). \tag{3.11}$$

The map is alternating (should be clear). We also have:

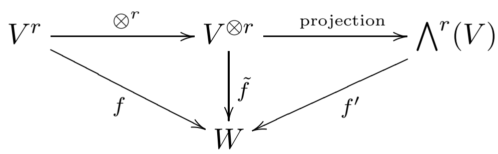
Theorem 3.29 (Universal property of the rth exterior power of a vector space)**.** *Let U and V be vector spaces. If $f : V^r \to W$ is r-multilinear and alternating, there exists a unique linear map $f' : \bigwedge^r(V) \to W$ such that $f = f' \circ \wedge^r$, i.e. making the diagram*



commutative.

The proof of this works analoguously as the proofs for the statements about tensor products (Theorems 3.8 and 3.12), so we can omit it.

Proof. By the the universal property of tensor maps of Theorem 3.8 and by Theorem 3.12, there is a unique linear map $\tilde{f} : V^{\otimes r} \to W$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_r) = f(v_1, \ldots, f_r)$. Since f is alternating, $\tilde{f}$ vanishes on the subspace U from Definition 3.27. Hence it induces $f' : V^{\otimes r}/U \to W$ is as desired.



This induced map vanishes on the subspace U (spanned by all the tensors with repeated entry), hence it induces f'.

□

Corollary 3.30. *The space of r-linear alternating maps* $\mathrm{Alt}^r(V;W) = \{f : V^r \to W :$ *f is alternating$\}$ is isomorphic to* $\mathrm{Hom}(\wedge^r(V), W)$*. In particular, if $W = K$, this yields an isomorphism* $(\wedge^r(V))^* \cong \mathrm{Alt}^r(V;K)$.

We think of an element of $\bigwedge^r(V)$ as some sort of "r-dimensional volume vector". Similarly as for $r = 2$, one can show the following:

Theorem 3.31. *Let $\{e_i\}_{i=1,\ldots,n}$ be a basis for V and $r \geq 0$. Then $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}\}_{i_1 < \cdots < i_r}$ is a basis for $\bigwedge^r(V)$. In particular,* $\dim \bigwedge^r(V) = \binom{n}{r}$.

Proof. The claims can be shown analoguosly as the statements of Theorem 3.26. □

Example 3.32. Let V be a vector space with basis $\{v_1, v_2, v_3, v_4\}$. Then

	basis
$\bigwedge^0(V)$	$\{1\}$
$\bigwedge^1(V)$	$\{v_1, v_2, v_3, v_4\}$
$\bigwedge^2(V)$	$\{v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4, v_3 \wedge v_4\}$
$\bigwedge^3(V)$	$\{v_1 \wedge v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_4, v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4\}$
$\bigwedge^4(V)$	$\{v_1 \wedge v_2 \wedge v_3 \wedge v_4\}$

We now consider special values of r more generally (using Theorem 3.31):

- $\bigwedge^0(V) = K$, since the "empty" tensor product is K.
- $\bigwedge^1(V) = V$.
- $\bigwedge^n(V)$ is 1-dimensional, spanned by $e_1 \wedge \cdots \wedge e_n$ for any basis $\{e_i\}_i$ of V.
- $\dim \bigwedge^r(V) = 0$ if $r > n$.

Remark 3.33. Note that given any linear map $T : V \to W$ between two vector spaces, we get a linear map $\bigwedge^r T : \bigwedge^r(V) \to \bigwedge^r(W)$ by setting $\bigwedge^r T(v_1 \wedge \cdots \wedge v_r) = T(v_1) \wedge \cdots \wedge T(v_r)$ (and extending linearly to $\bigwedge^r(V)$). To see that this is well-defined, we note that this map is multilinear and vanishes if there exist $i \neq j$ with $v_i = v_j$. (Intuitively, the idea is that given a linear map, it also gives us a way to turn r-dimensional volumes into r-dimensional volumes.) This can be extended to a map $\bigwedge(T) : \bigwedge(V) \to \bigwedge(W)$, by setting $\bigwedge(T)(x_1 \wedge \cdots \wedge x_t) = T(x_1) \wedge \cdots \wedge T(x_t)$ for any $x_1, \dots, x_t$ in V (any $t \geq 0$). (The map $\bigwedge(T)$ is a homomorphism of graded K-algebras.)

Now in particular, we can consider the case $W = V$ and $r = n = \dim V$. In this case we have a map T from V to itself. What is T doing to (n-dimensional) volume in V? The space $\bigwedge^n(V)$ is 1-dimensional, so $\bigwedge^n(T)$ is a linear map from a 1-dimensional space to itself. Any such map is multiplication by some scalar, and this is independent of the choice of basis of V. This scalar is what T multiplies volumes by, from a geometric point of view.

Definition 3.34. Let $T : V \to V$ be a linear map and $n = \dim V$. Then the *determinant* $\det(T)$ is the scalar such that $\bigwedge^n T$ is multiplication by $\det(T)$.

Recall that the exterior algebra of V is $\bigwedge(V) = \oplus_{m \geq 0} \bigwedge^m(V)$. It is graded by the degree of alternating tensors.

Since V is finite dimensional, say of dimension d, we actually have

$$\bigwedge(V) = \bigoplus_{m=0}^{d} \bigwedge^m(V)$$

and since each $\bigwedge^m(V)$ has dimension $\binom{d}{m}$, we deduce that

$$\dim(\bigwedge(V)) = \sum_{m=0}^{d} \binom{d}{m} = 2^d = 2^{\dim(V)}.$$

[Lecture 7, 16.12. 2015]

Remark 3.35. There is a linear map $\mu' : \bigwedge^r(V) \otimes \bigwedge^s(V) \to \bigwedge^{r+s}(V)$ induced by the (bilinear) multiplication map $\mu : \bigwedge^r(V) \times \bigwedge^s(V) \to \bigwedge^{r+s}(V)$ (cf. Definition 3.7), defined on pairs of decomposable alternating tensors as follows

$$\mu(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_s) = v_1 \wedge \cdots \wedge v_s \wedge w_1 \wedge \cdots \wedge w_s$$

and extended to all of $\bigwedge^r(V) \times \bigwedge^s(V)$. We write $\mu(x, y)$ (with $x \in \bigwedge^r(V)$, $y \in \bigwedge^s(V)$) as $x \wedge y$. waffle ex. 3.6:
Geometrically, this corresponds to combining an r-dimensional volume with an s-dimensional volume to get an $r+s$-dimensional volume. We write $x \wedge y$ for $\mu(x, y)$.

The multiplication

$$\bigwedge^m(V) \times \bigwedge^n(V) \to \bigwedge^{m+n}(V)$$

is skew-symmetric in the following sense:

Theorem 3.36. *For all $\alpha \in \bigwedge^m(V)$ and for all $\beta \in \bigwedge^n(V)$, we have*

$$\beta \wedge \alpha = (-1)^{mn} \alpha \wedge \beta$$

Proof. Since $v \wedge u = -u \wedge v$ for all $u, v \in V$, the claim follows by induction. □

Remark 3.37. 6 lines of text at the end of page 776 Note that $\alpha \wedge \alpha = 0$ for every simple tensor $\alpha = u_1 \wedge \cdots \wedge u_n$. What about $\alpha \wedge \alpha = 0$ for an arbitrary element $\alpha \in \bigwedge(V)$?. If $\dim(V) \leq 3$, one can show that $\alpha \wedge \alpha = 0$ for all $\alpha \in \bigwedge(V)$. In dimension 4, this is not true anymore, e.g. if $\{u_1, u_2, u_3, u_4\}$ is a basis for V, and if $\alpha = u_1 \wedge u_2 + u_3 \wedge u_4$, we have

$$\begin{aligned} \alpha \wedge \alpha =& (u_1 \wedge u_2 + u_3 \wedge u_4) \wedge (u_1 \wedge u_2 + u_3 \wedge u_4) \\ =& \cdots = u_1 \wedge u_2 \wedge u_3 \wedge u_4 + u_3 \wedge u_4 \wedge u_1 \wedge u_2 \\ =& 2\, u_1 \wedge u_2 \wedge u_3 \wedge u_4 \end{aligned}$$

Pairings

Definition 3.38. Let V and W be vector spaces. A *pairing* is a bilinear map $\langle\, ,\rangle : V \times W \to K$. The pairing is *non-degenerate* if

$$\begin{aligned} \langle v_0, w\rangle = 0 \;\forall w \;\in W \implies v_0 = 0 \\ \langle v, w_0\rangle = 0 \;\forall v \;\in V \implies w_0 = 0 \end{aligned}$$

An example for a non-degenerate pairing is the so-called *evaluation map*:

$$V^* \times V \to K, \quad (f, v) \mapsto f(v)$$

Theorem 3.39. *If $b : V \times W \to K$, $(v, w) \mapsto b(v, w)$, is a non-degenerate pairing, then $V \cong W^*$ and $W \cong V^*$*

Proof. Define the map $f : V \to W^*$ as follows

$$(f(v))(w) = b(v, w)$$

This map is linear and

$$\ker f = \{v \in V : f(v) = 0\} = \{v \in V : b(v, w) = 0 \;\forall\; w \;\in W\} = \{0\}.$$

(Note that $f(v) = 0$ means that $f(v)$ is the zero map.) The last equality follows from the non-degeneracy of the pairing. So $\dim V \leq \dim W^* = \dim W$. Similarly, define a linear map $g : W \to V^*$ by $g(w)(v) = b(v, w)$. Its kernel is also 0. So $\dim W \leq \dim V^*$, therefore $\dim V^* = \dim V = \dim W^* = \dim W$ and we get that f and g are isomorphisms. ☐

Corollary 3.40. *Let $1 \leq k \leq n$. The map $\langle \cdot, \cdot \rangle$: $\bigwedge^k(V^*) \times \bigwedge^k(V) \to K$ given by*

$$\langle v_1^* \wedge \cdots \wedge v_k^*, v_1 \wedge \cdots \wedge v_k \rangle = \det\left(v_i^*(v_j)\right)$$

gives a non-degenerate pairing, hence

$$\bigwedge^k(V^*) \cong \bigwedge^k(V)$$

(The pairing is defined in simple alternating tensors - and then extended to all of $\bigwedge^k(V^*) \times \bigwedge^k(V)$)

Proof. Let $b : (V^*)^k \times V^k \to K$ be (the $2k$-linear map) defined (on simple elements) by

$$b(v_1^*, \ldots, v_k^*, v_1, \ldots, v_k) = \det(v_i^*(v_j)_{ij})$$

For fixed $(v_1^*, \ldots, v_k^*)$ in $(V^*)^k$, b is alternating in the v_j's. So there is a map $\overline{b} : (V^*)^k \times \bigwedge^k(V) \to K$ with

$$(v_1^*, \ldots, v_k^*, v_1 \wedge \cdots \wedge v_k) \mapsto \det(l_i(v_j))$$

(Theorem 3.29). Similarly, for fixed $v_1 \wedge \cdots \wedge v_k$ in $\bigwedge^k(V)$, $\overline{b}$ is alternating in the v_i^*'s (and k-linear), so there exists a pairing $b' : \bigwedge^k(V^*) \times \bigwedge^k(V) \to K$.
To check non-degeneracy, evaluate the pairing on the respective bases. The claim then follows from Theorem 3.39. ☐

Remark 3.41. Explicitely, every element $v_1^* \wedge \cdots \wedge v_k^* \in \bigwedge^k(V^*)$ defines a k-linear alternating map by setting

$$v_1^* \wedge \cdots \wedge v_k^*\,(v_1, \ldots, v_k) := \det(l_i(v_j)_{ij})$$

for all $(v_1, \ldots, v_k) \in V^k$. In particular,

$$v_1^* \wedge v_2^*\,(v_1, v_2) = v_1^*(v_1)v_2^*(v_2) - v_1^*(v_2)v_2^*(v_1).$$

Exercises for Section 3.6

1. Identify $\bigwedge^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $e_1 \wedge e_2$ with e_3, $e_2 \wedge e_3$ with e_1 and $e_3 \wedge e_1$ with e_2. Show that under this identification, the exterior product $v \wedge w \in \bigwedge^2(\mathbb{R}^3) = \mathbb{R}^3$ is the same as the cross product $u \times w \in \mathbb{R}^3$.

2. exercise 3.2 of Waffle
   Let V have basis $\{e_1, e_2\}$ and let $T : V \to V$ be given by $T(e_1) = ae_1 + ce_2$ and $T(e_2) = be_1 + de_2$. Compute $\bigwedge^2 T : \bigwedge^2(V) \to \bigwedge^2(V)$ in terms of this basis. What is $\det(T)$?

3. Exercise 3.4 of Waffle
   Let $\mathrm{Sym}^2(V)$ be the quotient of $V \otimes V$ by the subspace spanned by elements of the form $v \otimes w - w \otimes v$. We write vw for the image of $v \otimes w$ under the quotient map $V \otimes V \to \mathrm{Sym}^2(V)$. If $\{e_i\}$ is a basis for V, show that $\{e_i e_j\}_{i \le j}$ is a basis for $\mathrm{Sym}^2(V)$.
   Hint: imitate the proof of theorem 3.26

4. Exercise 3.7 of Waffle
   Let $x \in \bigwedge^r(V)$, $y \in \bigwedge^s(V)$ and $z \in \bigwedge^t(V)$. Show that $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and that $x \wedge y = (-1)^{rs} y \wedge x$ where $x \wedge y$ is defined as in Lemma 3.35.

5. Proposition 29.20 of Gallier
   For any vector space V, the vectors $u_1, \dots, u_n \in V$ are linearly independent iff $u_1 \wedge \cdots \wedge u_n \neq 0$.

3.7 Reminder from linear algebra

3.7.1 Reminder on inner products

Let V be a vector space over $\mathbb{C}$. An inner product on V is a function $(\cdot, \cdot)\colon V \times V \to \mathbb{C}$ such that
1. $(u, v) = \overline{(v, u)}$ for all $u, v \in V$.
2. $(\alpha_1 v_1 + \alpha_2 v_2, u) = \alpha_1 (v_1, u) + \alpha_2 (v_2, u)$ for all $v_i, u \in V$, $\alpha_i \in \mathbb{C}$.
3. $(v, v) \geq 0$ for all $v \in V$ and $(v, v) = 0$ if and only if $v = 0$.

V is then called an inner product space. The *norm* induced by the inner product is defined as

$$\|v\| = \sqrt{(v, v)}, \quad v \in V$$

Vectors v with $\|v\| = 1$ are *unit vectors*. Two vectors $u, v \in V$ are *orthogonal* if $(u, v) = 0$, denoted by $u \perp v$. A basis $E = \{e_1, \dots, e_n\}$ is an *orthogonal basis*, if the vectors are pairwise orthogonal. It is *orthonormal*, if $(e_i, e_j) = \delta_{ij}$.

Fact:
Let $F := \{f_1, \dots, f_n\}$ be a basis of the $\mathbb{C}$-vector space V. Then there exists a unique inner product $(\cdot, \cdot)$ on V such that F is an orthonormal basis. (cf. Problem 7.3 (a))

Proof of this fact: Let $(\cdot, \cdot)$ be an inner product with ONB $E = \{e_1, \dots, e_n\}$. Define $S \in \mathrm{End}V$ by $Sf_i = e_i$. The endomorphism S is invertible. (Why? please give an argument - e.g. by definition, S has rank $n = \dim V$). Set $T := S^*S > 0$. (positive semi-definiteness, i.e. $S^*S \geq 0$, holds for any homomorphism between inner produc spaces. Positive definiteness, i.e. $S^*S > 0$, holds for invertible homomorphisms. View S as a square matrix. Then the adjoint S^* of S is the complex conjugate transpose of S.) So $\langle u, v\rangle := (Tu, v)$ is an inner product. (Fact: For $T \in \operatorname{End} V$, $\langle u, v\rangle := (Tu, v)$ defines an inner product if and only if T is pos. definite w.r.t. $(\cdot,\cdot)$). The elements $f_1, \ldots, f_n$ form an ONB w.r.t. $\langle\cdot,\cdot\rangle$. Uniqueness: straightforward.

Exercise:
Let $E = \{e_1, \ldots, e_n\}$ be a basis of V. For any $u = \sum_{i=1}^n a_i e_i$ and $v = \sum_{i=1}^n b_i e_i$, show that $(u, v) := \sum_{i=1}^n a_i \overline{b_i}$ is the unique inner product on V so that E is an orthonormal basis (cf. Problem 7.3 (b)).

3.7.2 Reminder on Adjoints

Let V, W be inner product spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$ respectively. For each $T \in \operatorname{Hom}(V, W)$, the *adjoint* of T is $S \in \operatorname{Hom}(W, V)$ such that $(Tv, w)_W = (v, Sw)_V$ for all $v \in V$, $w \in W$ and is denoted by T^*. Clearly, $(T^*)^* = T$.
Fact:
Let W, V be inner product spaces. Each $T \in \operatorname{Hom}(V, W)$ has a unique adjoint.

This is Theorem 1.4.1 in [5]. To prove it, use thm 1.3.4 of [5]: we can find an ONB w.r.t. $(\cdot,\cdot)_V$. Let $E = \{e_1, \ldots, e_n\}$ be an ONB of V. Then for $w \in W$, one defines $S \in \operatorname{Hom}(W, V)$ by

$$Sw := \sum_{i=1}^n (w, Te_i)_W e_i.$$

etc. And show uniqueness.

And then another fact:
If E and $F = \{f_1, \ldots, f_m\}$ are ONB's of the inner product spaces V and W and $T \in \operatorname{Hom}(V, W)$ with matrix A representing T w.r.t. the bases E and F, then T^* is represented by A^* (the complex conjugate transpose of the matrix A), a matrix w.r.t. the bases F and E.

Chapter 4

Rings and modules

Modules over rings generalize the concept of vector spaces over a field. Every vector space has a basis, but for modules, this is not always the case.

4.1 Basic definitions

Recall that a ring R (with unit) is a set with two operations, denoted $+$ and $\cdot$, $R \times R \to R$, such that

R1 $(R, +)$ is a commutative group.

R2 The multiplication $\cdot$ is associative and has a unit element.

R3 for all $x, y, z \in R$, we have $(x + y)z = xz + yz$ and $z(x + y) = zx + zy$.

The last condition is called the *distributivity.*

The unit element for the addition is denoted by 0 or 0_R, the unit element for multiplication by 1 or 1_R.

Definition 4.1. Let R be a ring. A *left module* over R or a *left R-module* M is an abelian group, usually written additively, together with an operation of R on M, such that for all $r, s \in R$ and $x, y \in M$, we have

$$(r + s)x = rx + sx, \quad r(x + y) = rx + ry, \quad r(sx) = (rs)x \quad \text{and} \quad 1_R x = x$$

As an exercise: check that $a(-x) = -(ax)$ and $0_R x = 0_M$, $a0_M = 0_M$ (for arbitrary $a \in R$, $x \in M$).

Right R-modules are defined analogously. We will work with left R-modules unless mentioned otherwise. There are cases where it is important to distinguish between the action of R from left and the one from right. In the following examples, switching from left to right modules can mostly be done by a simple notational change. This is not the case in Example 4.2 (f).

Example 4.2. (a) If M is a vector space over a field K, then M is a K-module.

(b) Any ring R is a module over itself. Informally: elements of a ring can be added and subtracted (abelian group part of definition), we can multiply every $r \in R$ by $x \in R$ and the distributivity yields the conditions for R-modules.

(c) Let $(R, +, \cdot) = (\mathbb{Z}, +, \cdot)$ where $+$ and $\cdot$ are addition and multiplication of integers, R is commutative. Every abelian group A is a $\mathbb{Z}$-module. Addition and subtraction are carried out according to the group structure of A; the key point is that we can multiply $x \in A$ by the integer n. If $n > 0$, $nx = x + x + \cdots + x$ (n times); if $n < 0$, then $nx = -x - x - \cdots - x$ ($|n|$ times). Concrete examples:
Let $M_1 = \mathbb{Z}$, $M_2 = \langle a \rangle = a\,\mathbb{Z} = \{\dots, -2a, -a, 0, a, 2a, \dots\}$ and $M_3 = \mathbb{Z}/a\,\mathbb{Z}$ for $a \geq 0$. Then the M_i are abelian groups and left and right $\mathbb{Z}$-modules.

(d) Let R be any ring and R^n the set of all n-tuples with components in R. Then R^n is an R-module with the usual definitions of addition and scalar multiplication (e.g. $r(x_1, \dots, x_n) = (rx_1, \dots, rx_n)$).

(e) Let $M = R_{m,n}$ be the set of all $m \times n$ matrices with entries in R. Then M is an R-module, where addition is ordinary matrix addition and multiplication of the scalar $c \in R$ by the matrix A means multiplication of each entry of A by c.

(f) Let I be a left ideal of the ring R; then I is a left R-module (if $x \in I$ and $r \in R$, then rx belongs to I, but xr is not necessarily in I! Similarly, a right ideal is a right R-module, and a two-sided ideal is both a left and a right R-module.

(g) Let J be a two-sided ideal of R. Then the factor ring R/J is a module over A: if $r \in R$ and $r + J$ is a coset of J in R, then one defines the operation to be $r(x + J) = rx + J$. This defines an R-module structure on R/J. We will do this more generally below: if M is a module and N a submodule (cf. Definition 4.7), we can define the quotient module M/N. (Section 4.4, see Definition 4.37 and discussion afterwards).

(h) An additive group consisting of 0 alone is a module over any ring.

[Lecture 8, 11.1.2016]

Definition 4.3. Let R be a commutative ring (with unit). An R-*algebra* (an *algebra over* R) is an R-module A together with composition $A \times A \to A$, $(x, y) \mapsto xy$ called *multiplication* which is distributive over addition and is linked to the action of R on A by the identities

$$a(xy) = (ax)y = x(ay) \quad \forall\, a \in R, \forall\, x, y \in A$$

If the multiplication above satisfies additional conditions, we obtain different types of algebras: If the multiplication is associative, then A is called an *associative algebra*. (In this case, A is a ring under the addition and multiplication). If the multiplication is commutative, A is a *commutative algebra*. If A contains a multiplicative identity element, A is *unitary*. A unitary associative algebra in which every non-zero element has an inverse is called a *division algebra*.

Example 4.4. $\mathbb{C}$ is a division algebra over $\mathbb{R}$.

Example 4.5. Let R be a commutative ring (with unit). Let $R^{\mathbb{N}}$ be the set of all maps $f : \mathbb{N} \to R$ (i.e. the set of all sequences of elements of R), with 0 included in $\mathbb{N}$. Endow $R^{\mathbb{N}}$ with the obvious addition, namely for $f, g \in R^{\mathbb{N}}$ let $f + g$ be defined by

$$(f+g)(n) = f(n) + g(n)$$

$R^{\mathbb{N}}$ with this addition is an abelian group. We define $R \times R^{\mathbb{N}} \to R^{\mathbb{N}}$ by $(r, f) \mapsto rf$, where $rf \in R^{\mathbb{N}}$ is the element given by

$$(rf)(n) = rf(n)$$

This makes $R^{\mathbb{N}}$ into an R-module. We now make it into an R-algebra, we need to define a multiplication for this:
Given $f, g \in R^{\mathbb{N}}$, define the product map $fg : \mathbb{N} \to R$ by

$$(fg)(n) = \sum_{i=0}^{n} f(i)g(n-i)$$

One checks that this makes $R^{\mathbb{N}}$ a R-algebra. It is called the *algebra of formal power series with coefficients in R*. mehr text dazu: seiten 5,6 von Blyth

Example 4.6. If R is a ring (with 1), then the set $R_{n,n}$ of $n \times n$-matrices over R is a unitary associative R-algebra.

Exercises for Section 4.1

1. Let M be an abelian group and let $\operatorname{End} M$ be the set of all endomorphisms on M, i.e. the set of all group homomorphisms $f : M \to M$. Show that $\operatorname{End} M$ is an abelian group under the operation $(f, g) \mapsto f + g$ where $(f + g)(x) := f(x) + g(x)$.

   Show also that
   1) $(\operatorname{End} M, +, \circ)$ is a ring with unit;
   2) M is an $\operatorname{End} M$-module under the action $\operatorname{End} M \times M \to M$ given by $(f, m) \mapsto f \cdot m = f(m)$;
   3) if R is a ring (with unit) and $\mu : R \to \operatorname{End} M$ a ring morphism with $\mu(1_R) = \mathrm{id}_M$, then M is an R-module under the action $R \times M \to M$ given by $(\lambda, m) \mapsto \lambda m = (\mu(\lambda))(m)$.

2. Let G be a finite abelian group with $|G| = m$. Show that if $n, t \in \mathbb{Z}$ then

   $$n \equiv t \mod m \Longrightarrow ng = tg \quad \forall\, g \in G$$

   Deduce that G is a $\mathbb{Z}/m\mathbb{Z}$-module under the action $\mathbb{Z}/m\mathbb{Z} \times G \to G$ given by $(n + m\mathbb{Z}, g) \mapsto ng$. Conclude that every finite abelian group whose order is a prime p can be regarded as a vector space over a field of p elements.

3. what exactly is $P_n(R)$??

   If R is a commutative ring (with unit) show that the set $P_n(R)$ of all polynomials over R of degree less than or equal to n is an R-module. Show also that the set $P(R)$ of all polynomials over R is a unitary associative R-algebra.

4. If A is a ring (with unit) define its *centre* to be

$$Z(A) := \{x \in A : xy = yx \ \forall\ y \in A\}$$

   Show that $Z(A)$ is a ring (with unit). If R is a commutative ring (with unit) prove that A is a unitary associative R-algebra if and only if there is a 1-preserving (sending 1_R to $1_{Z(A)}$) ring morphism $\varphi : R \to Z(A)$.

   (Hint: $\Longrightarrow$: denoting the action of R on A by $(r, a) \mapsto r \cdot a$, define φ by $\varphi(r) = r \cdot 1_A$. $\Longleftarrow$: Define an action by $(r, a) \mapsto r \cdot a = \varphi(r)a$.)

4.2 Submodules; intersections and sums

Definition 4.7. By a *submodule* of an R-module M we mean a subgroup N of M that is stable under the action of R on M, in the sense that if $x \in N$ and $r \in R$, then $rx \in N$.

A *non-empty* subset N of an R-module M is a submodule of M if and only if

$$x - y \in N \quad \text{and} \quad rx \in N \ \forall\ x, y \in N,\ r \in R \qquad \text{(“submodule criterion”)}$$

These two conditions are equivalent to the condition

$$rx + sy \in N \quad \forall\ r, s \in R,\ x, y \in N$$

(check the equivalence yourself)
if the first two conditions hold, then $rx \in N$ and $sy = (-s)y \in N$, hence $rx - sy \in N$. If the 3rd condition holds, then for $r = 1_R$ and $s = -1_R$, we obtain $x - y \in N$, and for $s = 0_R$, we obtain $rx \in N$.

Definition 4.8. A non-empty subset B of an R-algebra A is a *subalgebra of* A if

$$x - y \in B, xy \in B, rx \in B \quad \forall\ x, y \in B,\ \forall\ r \in R$$

Example 4.9. Let R be a ring (with unit) and consider R as an R-module (as in Example 4.2 (b)). The submodules of R are precisely the left ideals of R. Similarly, if we consider R as a right R-module, then the submodules of R are precisely the right ideals of R.

Example 4.10. If G is an abelian group, then the submodules of the $\mathbb{Z}$-module G are the subgroups of G.

Theorem 4.11. *If $(M_i)_{i \in I}$ is a family of submodules of an R-module M, then $\bigcap_{i \in I} M_i$ is a submodule of M.*

proof skipped in class

Proof. First observe that $\bigcap_{i\in I} M_i \neq \emptyset$, since every M_i is a subgroup of M and hence contains the identity element 0_M. Since each M_i is a submodule, we have

$$\begin{aligned} x, y \in \bigcap_{i\in I} M_i &\Longrightarrow x, y \in M_i \ \forall\ i \in I \\ &\Longrightarrow x - y \in M_i \ \forall\ i \in I \\ &\Longrightarrow x - y \in \bigcap_{i\in I} M_i \end{aligned}$$

and

$$x \in \bigcap_{i\in I} M_i, r \in R \Longrightarrow rx \in M_i \quad \forall\ i \in I \Longrightarrow rx \in \bigcap_{i\in I} M_i$$

Consequently, $\bigcap_{i\in I} M_i$ is a submodule of M. □

Now let S be a subset of an R-module M, possibly $S = \emptyset$ and consider the collection of all submodules of M that contain S. By Theorem 4.11, the intersection of this collection is again a submodule of M and it contains S. It is the smallest submodule of M containing S. We call it the *submodule generated by* S, denoted $\langle S\rangle$. We can give an explicit description of this submodule:

$$\langle S\rangle = \begin{cases} \{0_M\} \text{ (the zero submodule)} & \text{if } S = \emptyset \\ \text{all linear combinations (over } R\text{) of elements of } S & \text{else} \end{cases} \tag{4.1}$$

Definition 4.12. The R-module M is *generated by* the subset S of M (S is a *set of generators for* M) if $\langle S\rangle = M$. By a *finitely generated R-module* we mean an R-module which has a finite set of generators.

Let $(M_i)_{i\in I}$ be a family of submodules of an R-module M. We consider the submodule $\langle \cup_{i\in I} M_i\rangle$ generated by $\cup_{i\in I} M_i$. It is the smallest submodule of M containing every M_i. It is often called (by abuse of notation) the *submodule generated by the family* $(M_i)_{i\in I}$. It can be characterised as in the theorem below. For this, let $\mathcal{P}^*(I)$ be the set of all non-empty finite subsets of the set I.

Theorem 4.13. *Let $(M_i)_{i\in I}$ be a family of submodules of an R-module M. Then the submodule generated by $\cup_{i\in I} M_i$ consists of all finite sums of the form $\sum_{j\in J} m_j$ where $J \in \mathcal{P}^*(I)$ and $m_j \in M_j$.*

Proof. A linear combination of elements of $\cup_{i\in I} M_i$ is precisely a sum of the form $\sum_{j\in J} m_j$ for some $J \in \mathcal{P}^*(I)$. The result is thus an immediate consequence of the description of $\langle S\rangle$ in (4.1). □

Because of Theorem 4.13, we call the submodule generated by the family $(M_i)_{i\in I}$ the *sum of the family* and denote it by $\sum_{i\in I} M_i$. If I is finite, say $I = \{1, 2, \ldots, n\}$, we often write $\sum_{i=1}^{n} M_i$ or $M_1 + M_2 + \cdots + M_n$.

We have two immediate consequences of this:

Corollary 4.14. *(1) (Commutativity of $\sum$) If $\sigma : I \to I$ is a bijection then*

$$\sum_{i\in I} M_i = \sum_{i\in I} M_{\sigma(i)}$$

(2) (Associativity of $\sum$) If $(I_k)_{k\in A}$ is a family of non-empty subsets of I with $I = \cup_{k\in A} I_k$ then

$$\sum_{i\in I} M_i = \sum_{k\in A}(\sum_{i\in I_k} M_i).$$

proof skipped in class

Proof. (1) is clear.
(2) A typical element of the right-hand side is $\sum_{k\in J}(\sum_{i\in J_k} m_i)$ with $J_k \in \mathcal{P}^*(I_k)$ and $J \in \mathcal{P}^*(A)$. By associativity of addition in M this can be written as $\sum_{i\in K} m_i$ where $K = \cup_{k\in J} J_k \in \mathcal{P}^*(I)$. Thus the right-hand side is contained in the left-hand side.

As for the converse inclusion, a typical element of the left-hand side is $\sum_{i\in I} m_i$ where $J \in \mathcal{P}^*(I)$. Now $J = J \cap I = \cup_{k\in A}(J \cap I_k)$ so that if we define $J_k = J \cap I_K$ we have $J_k \in \mathcal{P}^*(I_k)$ and, by the associativity of addition in M, $\sum_{i\in J} m_i = \sum_{k\in B}(\sum_{i\in J_k} m_i)$ where $B \in \mathcal{P}^*(A)$. Thus the left-hand side is contained in the right-hand side. □

If we take $A = \{1, 2\}$ in (2) of Corollary 4.14 and set $I_1 = \{i\}$, $I_2 = I \setminus I_1$, we obtain as a further corollary, for all $i \in I$, the following:

$$\sum_{i\in I} M_i = M_i + \sum_{j\neq i} M_j$$

Note that $\cup_{i\in I} M_i$ need not be a submodule of M, so in general, $\sum_{i\in I} M_i \neq \cup_{i\in I} M_i$: Take $I = \{1, 2\}$, let M_1 and M_2 be the subspaces of the vector space $\mathbb{R}^2$ given by $M_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $M_2 = \{(0, y) : y \in \mathbb{R}\}$. We have $M_1 + M_2 = \mathbb{R}^2$ whereas $M_1 \cup M_2 \subsetneq \mathbb{R}^2$.

Let M be an R-module, let A and B be submodules of M. Then $A + B$ is the smallest submodule of M containing both A and B. $A \cap B$ is the largest submodule of M contained in A and in B. The set of submodules of M, ordered by set inclusion, is therefore such that every two-element subset $\{A, B\}$ has a supremum (namely $A + B$) and an infimum (namely $A \cap B$). We say that the set of submodules of M forms a *lattice*. An important property of this lattice is that it is *modular*, in the following sense:

Theorem 4.15 (Modular law)**.** *If M is an R-module and if A, B, C are submodules of M with $C \subseteq A$ then*

$$A \cap (B + C) = (A \cap B) + C$$

Proof. Since $C \subseteq A$ we have $A+C = A$. Now $(A\cap B)+C \subseteq A+C$ and $(A\cap B)+C \subseteq B+C$ and so we have

$$(A \cap B) + C \subseteq (A + C) \cap (B + C) = A \cap (B + C).$$

To obtain the reverse inclusion, let $a \in A\cap(B+C)$. Then $a \in A$ and there exists $b \in B$, $c \in C$ such that $a = b + c$. Since $C \subseteq A$ we have $c \in A$ and therefore $b = a - c \in A$. Consequently, $b \in A \cap B$ and so $a = b + c \in (A \cap B) + C$. □

Exercises for Section 4.2

1. Let M be an R-module. If S is a non-empty subset of M, define the *annihilator of S in R* to be

$$\operatorname{Ann}_R S = \{r \in R : rx = 0_M \ \forall\, x \in S\}.$$

   Show that $\operatorname{Ann}_R S$ is a left ideal of R and that it is a two-sided ideal whenever S is a submodule of M.

2. Describe the kernel of the ring morphism μ from Exercise 1 (3) to Section 4.1.

3. Prove that the ring of endomorphisms of the abelian group $\mathbb{Z}$ is isomorphic to the ring $\mathbb{Z}$, and that the ring of endomorphisms of the abelian group $\mathbb{Q}$ is isomorphic to the field $\mathbb{Q}$.

   (Hint: Use Exercise 1 to Section 4.1 and Exercise 2 above. Note that if $f \in \operatorname{End}\mathbb{Z}$ then $f = \mu(f(1))$.)

4. Let M be an R-module. If $r, s \in R$ show that

$$r - s \in \operatorname{Ann}_R M \Longrightarrow rx = sx \ \forall\, x \in M$$

   Deduce that M can be considered as an $R/\operatorname{Ann}_R M$-module. Show that the annihilator of M in $R/\operatorname{Ann}_R M$ is zero.

4.3 Morphisms; exact sequences

[Lecture 9, 12.1.2016]

In the theory of groups, group homomorphisms preserve the group structure. In K-vector spaces, K-linear maps preserve the vector space structure. In ring theory, ring morphisms preserve the ring structure. We now define structure-preserving maps between modules over a ring.

Definition 4.16. If M and N are R-modules, then $f : M \to N$ is a *R-morphism* if
(1) $f(x + y) = f(x) + f(y) \quad \forall\, x, y \in M$;
(2) $f(rx) = rf(x) \quad \forall\, x \in M, \forall\, r \in R$.

An R-morphism is a *R-monomorphism* if it is injective and it is an *R-epimorphism* if it is surjective. It is an *R-isomorphism* if it is bijective. An R-morphism $f : M \to M$ is often called an *R-endomorphism*.

If f is an R-morphism $f : M \to N$ we have $f(0_M) = 0_N$ and $f(-x) = -f(x)$ for all $x \in M$.

Example 4.17. If M and N are abelian groups regarded as $\mathbb{Z}$-modules then a $\mathbb{Z}$-morphism $f : M \to N$ is simply a group homomorphism. more in blyth, ex. 3.1

Example 4.18. Let M be an R-module and $n > 0$. Then M^n is also an R-module (n copies of M). For $1 \leq i \leq n$, the map $pr_i : M^n \to M$ described by

$$pr_i(x_1, \ldots, x_n) = x_i$$

is an R-epimorphism, the *i-th projection of M^n onto M.*

Next we describe how R-morphisms $f : M \to N$ induce maps between the lattices of submodules (cf. discussion above Theorem 4.15).

Theorem 4.19. *Let $f : M \to N$ be a morphism between R-modules. Then for every submodule X of M, the set $f(X)$ is a submodule of N and for every submodule Y of N, the set $f^{-1}(Y)$ is a submodule of M.*

Proof. Note first that $f(X)$ and $f^{-1}(Y)$ are not empty, the first, since X contains 0_M and so $f(0_M) = 0_N \in f(X)$. Analoguously, $0_M \in f^{-1}(Y)$.

Since f is in particular a group homomorphism from M to N, and since as submodules, X and Y are subgroups of M and N respectively, we get that the image $f(X)$ is a subgroup of N and that the preimage $f^{-1}(Y)$ is a subgroup of M (facts about group homomorphisms, introductory algebra course, e.g. [2, Satz 2.13]). The only thing that remains to be checked is that $f(X)$ and $f^{-1}(Y)$ are stable under the action of R (in M or in N respectively).

So let $y \in f(X)$. There exists $x \in M$ such that $f(x) = y$. For $r \in R$ arbitrary, we then have

$$ry = rf(x) \overset{f\ R\text{-morph.}}{=} f(rx) \in f(X)$$

Now if $x \in f^{-1}(Y)$, then $f(x) \in Y$. For $r \in R$ arbitrary, we have

$$f(rx) \overset{f\ R\text{-morph.}}{=} rf(x) \in Y \quad (Y \text{ is a submodule of } N), \quad \text{hence } rx \in f^{-1}(Y)$$

□

In particular, $\operatorname{im} f = f(M)$ is a submodule of N and $\ker f = f^{-1}(\{0_N\})$ is a submodule of M.

We write $\mathcal{L}(M)$ for the lattice of submodules of M. By Theorem 4.19, we can define maps $f^{\rightarrow} : \mathcal{L}(M) \to \mathcal{L}(N)$ and $f^{\leftarrow} : \mathcal{L}(N) \to \mathcal{L}(M)$ through $f^{\rightarrow} : X \mapsto f(X)$, $f^{\leftarrow} : Y \mapsto f^{-1}(Y)$. We will often simply write f for $f^{\rightarrow} : \mathcal{L}(M) \to \mathcal{L}(N)$ and f^{-1} for the map $f^{\leftarrow} : \mathcal{L}(N) \to \mathcal{L}(M)$.

These maps are inclusion-preserving: if X_1, X_2 are submodules of M such that $X_1 \subseteq X_2$, then $f^{\rightarrow}(X_1) \subseteq f^{\rightarrow}(X_2)$; if Y_1, Y_2 are submodules of N with $Y_1 \subseteq Y_2$, then $f^{\leftarrow}(Y_1) \subseteq f^{\leftarrow}(Y_2)$.

Lemma 4.20. *Let $f : M \to N$ be an R-morphism. If A is a submodule of M and B a submodule of N then*
(a) $f(f^{-1}(B)) = B \cap \operatorname{im} f$;
(b) $f^{-1}(f(A)) = A + \ker f$.

Proof. One can show more generally that the following two properties hold:
(a') $f\left(A \cap f^{-1}(B)\right) = f(A) \cap B$
and that
(b') $f^{-1}(B + f(A)) = f^{-1}(B) + A$.

(a) then follows by setting $A = M$ in (a'), (b) follows from (b') by taking $B = \{0_N\}$.
For *(a')*: First note that $f(f^{-1}(B)) \subseteq B$ since for $y \in f^{-1}(B)$, $f(y) \in B$. Since $f = f^{\rightarrow}$ is inclusion-preserving, $\subseteq$ follows.
To obtain $\supseteq$, let $y \in f(A) \cap B$. Then $y = f(a)$ for some $a \in A$ and $y \in B$. Since $f(a) \in B$ we have $a \in f^{-1}(B)$, so $a \in f^{-1}(B) \cap A$, hence $y \in f\left(A \cap f^{-1}(B)\right)$.
For *(b')*: We have $A \subseteq f^{-1}(f(A))$ since for $a \in A$, we have $f(a) \in f(A)$. Since $f^{-1} = f^{\leftarrow}$ is inclusion-preserving, we get $\supseteq$.
To obtain $\subseteq$, let $x \in f^{-1}(B + f(A))$. Then $f(x) \in B + f(A)$ and so $f(x) \in B$ and $f(x) = f(A)$ for some $a \in A$. This gives $x - a \in \ker f = f^{-1}(\{0_N\}) \subseteq f^{-1}(B)$ and therefore $x \in f^{-1}(B) + A$. □

Just as with group homorphisms, we can compose R-morphisms to obtain new R-morphisms. The basic facts concerning this are the following:

(a) If $f : M \to N$ and $g : N \to P$ are R-morphisms, then the composition $g \circ f : M \to P$ is also an R-morphism.
(To see this: we know this is true for group homomorphisms. So one only need to check that for all $r \in R$, for all $x \in M$, we have $(g \circ f)(rx) = \cdots = r(g \circ f)(x)$.)

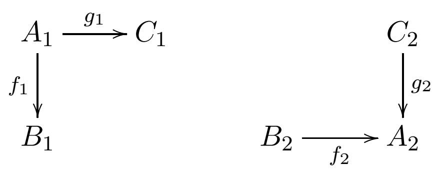
(b) If $f : M \to N$ and $g : N \to P$ are R-epimorphisms, then so is $g \circ f$.

(c) If $f : M \to N$ and $g : N \to P$ are R-monomorphisms, then so is $g \circ f$.

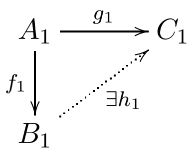
(d) If $g \circ f$ is an epimorphism, then so is g.

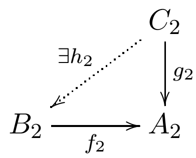
(e) If $g \circ f$ is a monomorphism, then so is f.

Concerning compositions of morphisms, there are two 'diagram-completion' problems that we will consider now:



If we are given a diagram with R-modules and R-morphisms A_1, B_1, C_1 and f_1, g_1, under which conditions does there exist a morphism $h_1 : B_1 \to C_1$ such that $h_1 \circ f_1 = g_1$? The dual problem is a diagram as on the right, with modules A_2, B_2, C_2 and morphisms f_2, g_2: what are the conditions ensuring the existence of a morphism $h_2 : C_2 \to B_2$ such that $f_2 \circ h_2 = g_2$?





$$h_1 \circ f_1 = g_1 \qquad f_2 \circ h_2 = g_2$$

We first consider these questions on the level of sets and maps between them.

Theorem 4.21. *(a) Let A, B, C be non-empty sets, $f : A \to B$, $g : A \to C$ maps. Then the following conditions are equivalent:*

(1) There exists a map $h : B \to C$ with $h \circ f = g$;

(2) $f(x) = f(y) \implies g(x) = g(y)$ for all $x, y \in A$

(b) Let A, B, C be non-empty sets and $f : B \to A$, $g : C \to A$ maps. Then the following conditions are equivalent:

(3) There exists a map $h : C \to B$ such that $f \circ h = g$;

(4) $\operatorname{im} g \subseteq \operatorname{im} f$.

(do the proof?)

Proof. (a) The implication (1) $\implies$ (2) is clear.
For (1) $\impliedby$ (2) consider the subset G of $\operatorname{im} f \times C$ given by

$$G := \{(x, z) : y = f(x), z = g(x) \text{ for some } x \in A\}$$

The subset G is non empty, as for any $x \in A$, $(f(x), g(x)) \in G$. Now given any $y \in \operatorname{im} f$ there is a unique $z \in C$ such that $(y, z) \in G$. In fact, if $y = f(x)$, choose $z = g(x)$ to see that such an element z exists. To see that it is unique, suppose that $(y, z) \in G$ and that $(y, z') \in G$. By the definition of G, we have $y = f(x) = f(x')$ and $z = g(x), z' = g(x')$ for some x, $x' \in A$. From this, by (2), $g(x) = g(x')$, and so $z = z'$.

We can therefore define a map $t : \operatorname{im} f \to C$ as follows:

$$t(f(x)) := g(x) \quad \forall\, x \in A$$

We need to extend this to a map on all of B: With this, we can construct $h : B \to C$:

$$h(y) = \begin{cases} t(y) & \text{if } y \in \operatorname{im} f; \\ \text{some } c \in C & \text{otherwise.} \end{cases}$$

Then for every $x \in A$ we have $h(f(x)) = t(f(x)) = g(x)$, and so $h \circ f = g$.

Similarly, the implication (3) $\implies$ (4) is clear.
For (4) $\impliedby$ (3): If (4) holds, then for every $x \in C$ there exists $y \in B$ such that $g(x) = f(y)$. Given any $x \in C$, choose one of these $y \in B$ and label it y_x (by the axiom of choice), i.e. y_x is an element of B with $g(x) = f(y_x)$. This gives us a map $h : C \to B$ by setting $h(x) = y_x$. This satisfies $f(h(x)) = f(y_x) = g(x)$, hence $f \circ h = g$. ☐

Corollary 4.22. *(a) If A, B are non-empty sets and $f : A \to B$ is a map, then the following statements are equivalent:*

(α) f is injective;

(β) there exists $g : B \to A$ such that $g \circ f = \mathrm{id}_A$;

(γ) f is left cancellable, in the sense that for every non-empty set C and all maps $h, k : C \to A$,

$$f \circ h = f \circ k \implies h = k.$$

(b) If A, B are non-empty sets and $f : A \to B$ is a map, then the following statements are equivalent:

(α') f is surjective;

(β') there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$;

(γ') f is right cancellable, in the sense that for every non-empty set C and all maps $h, k : B \to C$,

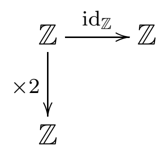
$$h \circ f = k \circ f \implies h = k.$$

Proof. $(\alpha) \iff (\beta)$ is immediate from $(1) \iff (2)$.
The implication $(\beta) \implies (\gamma)$ is straightforward (try it out!), $(\gamma) \implies (\alpha)$ requires more work. $(\alpha') \iff (\beta')$ is immediate from $(3) \iff (4)$.
$(\beta') \implies (\gamma')$ is straightforward, $(\gamma') \implies (\alpha')$ requires more work.
Most work: $(\gamma) \implies (\alpha)$ and $(\gamma') \implies (\alpha')$

For details: This is the Corollary on page 20 in [3]. ☐

The statements from Theorem 4.21 and Corollary 4.22 cannot be made into module-theoretic results by simply replacing "non-empty set" by R-module and "map" by R-morphisms. The following examples illustrates this.

Example 4.23. Consider the diagram of $\mathbb{Z}$-modules and $\mathbb{Z}$-morphisms



in which $\mathrm{id}_\mathbb{Z}$ is the identity morphism and $\times 2$ the $\mathbb{Z}$-morphism acting as $n \mapsto 2n$. By Theorem 4.21 (a) there is a *map* $h : \mathbb{Z} \to \mathbb{Z}$ such that $h \circ (\times 2) = \mathrm{id}_\mathbb{Z}$. But no such $\mathbb{Z}$-morphism can exist: Suppose h were such a $\mathbb{Z}$-morphism. Then for every $n \in \mathbb{Z}$, we would have $2h(n) = h(2n) = n$. In particular, $2h(1) = 1$; and this is impossible, since the equation $2x = 1$ has no solution in $\mathbb{Z}$.

Example 4.24. Let p be a prime, consider the subgroup $\mathbb{Q}_p$ of $\mathbb{Q}$ defined as follows

$$\mathbb{Q}_p := \{x \in \mathbb{Q} : x = \frac{k}{p^n} \text{ for some } k \in \mathbb{Z} \text{ and some } n \in \mathbb{N}\}.$$

Note that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}_p$. We can thus form the quotient group $\mathbb{Q}_p / \mathbb{Z}$. As abelian groups, they are $\mathbb{Z}$-modules. Consider the diagram

$$\begin{array}{ccc} & & \mathbb{Q}_p / \mathbb{Z} \\ & & \downarrow \text{id} \\ \mathbb{Q}_p / \mathbb{Z} & \xrightarrow[f]{} & \mathbb{Q}_p / \mathbb{Z} \end{array}$$

of $\mathbb{Z}$-modules and $\mathbb{Z}$-morphisms, where f is defined as $x \mapsto px$. Since for all $k \in \mathbb{Z}$ and for all $n \in \mathbb{Z}$ we have

$$\frac{k}{p^n} + \mathbb{Z} = p(\frac{k}{p^{n+1}} + \mathbb{Z})$$

we see that $\operatorname{im} f = \mathbb{Q}_p / \mathbb{Z} = \operatorname{im} \text{id}$. By Theorem 4.21 (b) there is therefore a *map* $h : \mathbb{Q}_p / \mathbb{Z} \to \mathbb{Q}_p / \mathbb{Z}$ such that $f \circ h = \text{id}$. However, no such $\mathbb{Z}$-morphism can exist: If h were such a $\mathbb{Z}$-morphism, we would have

$$\begin{aligned} \frac{1}{p} + \mathbb{Z} = f\left(h(\frac{1}{p} + \mathbb{Z})\right) &= p\left(h(\frac{1}{p} + \mathbb{Z})\right) \\ &= h\left(p(\frac{1}{p} + \mathbb{Z})\right) = h(1 + \mathbb{Z}) = 0 + \mathbb{Z} \end{aligned}$$

which is nonsense since $x + \mathbb{Z} = 0 + \mathbb{Z}$ if and only if $x \in \mathbb{Z}$.

[Lecture 10, 18.1.2016]

In certain situation, there are module-theoretic analogues of Theorem 4.21. The following two are examples of such. [3] will give more later.

Theorem 4.25. *Consider the diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \\ B & & \end{array}$$

of R-modules and R-morphisms in which f is an R-epimorphism. Then the following are equivalent:

(1) there is a unique R-morphism $h : B \to C$ such that $h \circ f = g$;

(2) $\ker f \subseteq \ker g$.

Moreover, such an R-morphism h is a monomorphism if and only if $\ker f = \ker g$.

Proof. (1) $\implies$ (2): Suppose that (1) holds, take $x \in \ker f$. Then $g(x) = h(f(x)) = h(0) = 0$.

(2) $\implies$ (1): Suppose that $\ker f \subseteq \ker g$. Given $x, y \in A$ we have

$$\begin{aligned} f(x) = f(y) &\implies f(x - y) = f(x) - f(y) = 0_B \\ &\implies x - y \in \ker f \subseteq \ker g \\ &\implies g(x) - g(y) = g(x - y) = 0_C \\ &\implies g(x) = g(y) \end{aligned}$$

By Theorem 4.21 (a) we can therefore define a map $h : B \to C$ such that $h \circ f = g$. Since f is surjective, it follows by Corollary 4.22 that f is right cancellable and so h is unique. It remains to show that h is an R-morphism. Since f is surjective[1], this follows from the equalities

$$h(f(x) + f(y)) = h(f(x + y)) = g(x + y) = g(x) + g(y) = h(f(x)) + h(f(y))$$
$$h(rf(x)) = h(f(rx)) = g(rx) = rg(x) = rh(f(x))$$

(for all $x, y \in A$, $r \in R$).

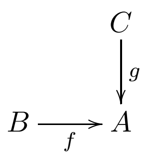
To the last statement: if h is injective, then since $g(x) = h(f(x))$ we have

$$x \in \ker g \Longrightarrow f(x) \in \ker h = \{0_B\} \Longrightarrow x \in \ker f,$$

and so we have equality in (2). Conversely, suppose that $\ker g = \ker f$ and let $x \in \ker h$. Since f is surjective we have $x = f(y)$ for some $y \in A$ and so $0_B = h(x) = h(f(y)) = g(y)$ and so $y \in \ker g = \ker f$. From this, $x = f(y) = 0_B$ and h is injective. □

Only the proof of Theorem 4.25 was done in class, the proof of 4.26 was omitted.

Theorem 4.26. *Consider the diagram*



of R-modules and R-morphisms in which f is an R-monomorphism. Then the following are equivalent:

(1) there is a unique R-morphism $h : C \to B$ such that $f \circ h = g$;

(2) $\operatorname{im} g \subseteq \operatorname{im} f$.

Moreover, such an R-morphism h is an epimorphism if and only if $\operatorname{im} g = \operatorname{im} f$.

Proof. (1) $\Longrightarrow$ (2): If (1) holds then $g(c) = f(h(c)) \in \operatorname{im} f$ for every $c \in C$.

(2) $\Longrightarrow$ (1): If (2) holds then by Theorem 4.21 (b) there is a map $h : C \to B$ such that $f \circ h = g$. Since f is injective, it follows by Corollary 4.22 that f is left cancellable and so h is unique. Now for all $c, d \in C$ and $r \in R$ we have the equalities

$$f(h(c + d)) = g(c + d) = g(c) + g(d) = f(h(c)) + f(h(d)) = f(h(c) + h(d)),$$
$$f(h(rc)) = g(rc) = rg(c) = rf(h(c)) = f(rh(c))$$

Since f is injective[2], we deduce that $h(c + d) = h(c) + h(d)$ and $h(rc) = rh(c)$, so that h is indeed an R-morphism.

[1] compare with Example 4.23!

[2] compare with Example 4.24!

For the last claim: if h is surjective, then for every $b \in B$ there exists $b \in B$ with $b = h(c)$. So $f(b) = f(h(c)) = g(c)$ and we have $\operatorname{im} f \subseteq \operatorname{im} g$.

Conversely, if $\operatorname{im} f = \operatorname{im} g$, then for every $b \in B$ there exists $c \in C$ such that $f(b) = g(c) = f(h(c))$ whence $b = h(c)$, since f is injective. Consequently, h is surjective. $\square$

It is costumary to say, in the situations of Theorems 4.25 and 4.26, that the diagrams commute. This notion is often used for diagrams of sets and maps (groups and homomorphisms, R-modules and R-morphisms, etc.) between them, it is a convenient abbreviation.

Definition 4.27. Given a diagram of sets and maps we say that the diagram is *commutative* if all compositions of maps between a starting set to an ending set are equal.

Example. The triangle

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & \nearrow_{h} & \\ B & & \end{array}$$

is commutative if and only if $h \circ f = g$. The diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

is commutative if and only if $f' \circ \alpha = \beta \circ f$ and $g' \circ \beta = \gamma \circ g$, i.e. if and only if each of the squares commutes.

Another important concept is the following.

Definition 4.28. A *sequence of modules and morphisms* is a diagram of the form

$$\dots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \dots$$

The sequence is said to be *exact at M_i* if $\operatorname{im} f_{i-1} = \ker f_i$. The sequence is *exact* if it is exact at each M_i.

The zero module is usually written as 0 in a sequence of modules and morphisms. From the definition, we have:

Theorem 4.29. *Let $f : M \to N$ be an R-morphism and let $0 \to M$, $N \to 0$ denote the inclusion map and the zero map respectively. Then f is*

(1) a monomorphism iff $0 \longrightarrow M \xrightarrow{f} N$ *is exact;*

(2) an epimorphism iff $M \xrightarrow{f} N \longrightarrow 0$ *is exact;*

(3) an isomorphism iff $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact.*

Example 4.30. If $f : A \to B$ is a group homomorphism of abelian groups, then we have the exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\iota} A \xrightarrow{\pi} A/\ker f \longrightarrow 0$$

where ι is the inclusion map and π is the natural epimorphism. Likewise, we have the exact sequence

$$0 \longrightarrow \operatorname{im} f \longrightarrow B \longrightarrow B/\operatorname{im} f \longrightarrow 0$$

An exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is called a *short exact sequence.*

In an exact sequence, the composition of two successive morphisms is the zero morphism. The converse is not true in general, as $f_i \circ f_{i-1} = 0$ is equivalent to $\operatorname{im} f_{i-1} \subseteq \ker f_i$. Sequences (of modules and morphisms) with $f_i \circ f_{i-1} = 0$ for all i are called *semi-exact*[3].

The following result is a useful property of the kernel of an R-morphism. It illustrates the notions we have just introduced.

Theorem 4.31. *Given the diagram of R-modules and R-morphisms*

$$\begin{array}{ccccccc} & & & & A & & \\ & & & & \big\downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

with exact row and with $g \circ \alpha = 0$ there is a unique R-morphism $h : A \to X$ such that the completed diagram is commutative.

Proof. Since $g \circ \alpha = 0$ and since the row is exact we have that $\operatorname{im} \alpha \subseteq \ker g = \operatorname{im} f$. The morphism f is injective (Theorem 4.29) and so the result follows from Theorem 4.26. ☐

Theorem 4.32. *Let $f : M \to N$ be an R-morphism. If $\iota : \ker f \to M$ is the inclusion map then*

(1) $f \circ \iota = 0$;

(2) if P is an R-module and if $g : P \to M$ is an R-morphism such that $f \circ g = 0$ then there is a unique R-morphism $\beta : P \to \ker f$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \exists!\beta}\swarrow & \big\downarrow{\scriptstyle g} & & \\ \ker f & \xrightarrow[\iota]{} & M & \xrightarrow[f]{} & N \end{array}$$

[3]compare with the notion of a chain complex.

Proof. (1) is clear and (2) is covered by Theorem 4.31 □

It can be shown that the pair $(\ker f, \iota)$ is characterised by the properties of Theorem 4.32. exercise 3.5 of [3].

We now present an example of a technique known as *diagram chasing.*

Theorem 4.33 (The four lemma)**.** *Suppose that the diagram of modules and morphisms*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

is commutative and has exact rows. Then the following holds:

(1) If α, γ are epimorphisms and δ is a monomorphism then β is an epimorphism;

(2) If α is an epimorphism and β, δ are monomorphisms then γ is a monomorphism.

Proof. (1) Let $b' \in B'$. Since γ is surjective, there exists $c \in C$ such that $g'(b') = \gamma(c)$. By the commutativity of the right-hand square we have

$$\delta(h(c)) = h'(\gamma(c)) = h'(g'(b')) = 0$$

since $h' \circ g' = 0$. Thus $h(c) \in \ker \delta = 0$ and so $h(c) = 0$ giving $c \in \ker h = \operatorname{im} g$ so that $c = g(b)$ for some $b \in B$. Then, by the commutativity of the middle square,

$$g'(b') = \gamma(c) = \gamma(g(b)) = g'(\beta(b)).$$

Consequently, $b' - \beta(b) \in \ker g' = \operatorname{im} f'$, so that $b' - \beta(b) = f'(a')$ for some $a' \in A'$. Since α is surjective, there exists $a \in A$ with $a' = \alpha(a)$. Now the commutativity of the left-hand square yields $b' - \beta(b) = f'(\alpha(a)) = \beta(f(a))$. We thus have

$$b' = \beta(b) + \beta(f(a)) = \beta(b + f(a)) \in \operatorname{im} \beta$$

and β is surjective.

part (2) is analoguous and thus was skipped in class

(2) Let $c \in \ker \gamma$. Then $\delta(h(c)) = h'(\gamma(c)) = h'(0) = 0$ and so $h(c) \in \ker \delta = 0$. Thus $c \in \ker h = \operatorname{im} g$ so that $c = g(b)$ for some $b \in B$. Now $0 = \gamma(c) = \gamma(g(b)) = g'(\beta(b))$ so $\beta(b) \in \ker g' = \operatorname{im} f'$ whence $\beta(b) = f'(a')$ for some $a' \in A'$. Now $a' = \alpha(a)$ for some $a \in A$, so $\beta(b) = f'(\alpha(a)) = \beta(f(a))$. Since β is a monomorphism, we deduce that $b = f(a)$ whence $c = g(b) = g(f(a)) = 0$ since $f \circ g = 0$. □

Theorem 4.34 (The five lemma)**.** *Suppose that the diagram of modules and morphisms*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_3} & & \downarrow{\scriptstyle\alpha_4} & & \downarrow{\scriptstyle\alpha_5} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

is commutative and has exact rows. If α_1, α_2, α_4 and α_5 are isomorphisms, then so is α_3.

Proof. Applying Theorem 4.33 (1) to the right-hand three squares shows that α_3 is an epimorphism. Applying Theorem 4.33 (2) to the left-hand three squares shows that α_3 is a monomorphism. □

Corollary 4.35. *Suppose that the diagram of modules and morphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is commutative and has exact rows. If α and γ are isomorphisms, then so is β.

Proof. Take $A = A' = E = E' = 0$ in Theorem 4.34 □

[Lecture 11, 19.1.2016]

Exercises for Section 4.3

1. Let R be a commutative ring (with unit). Prove that a map $f : R \times R \to R$ is an R-morphism if and only if there exist $\alpha, \beta \in R$ such that

$$f(x, y) = \alpha x + \beta y \quad \text{for all } x, y \in R$$

2. If A and B are submodules of an R-module M, establish a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{\gamma} A \times B \xrightarrow{\pi} A + B \longrightarrow 0$$

(Hint: Observe that the ‘obvious’ definitions of γ and π, namely $\gamma(x) = (x, x)$ and $\pi(x, y) = x + y$ do not work. Try $\pi(x, y) = x - y$)

3. A short exact sequence of the form

$$(f, E, g) : \quad 0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} B \longrightarrow 0$$

is called an *extension of A by B*.

(a) Given any R-modules A and B show that at least one extension of A by B exists.

(b) Two extensions (f_1, E_1, g_1) and (f_2, E_2, g_2) of A and B are said to be *equivalent* if there exists an R-morphism $h : E_1 \to E_2$ such that $h \circ f_1 = f_2$ and $g_2 \circ h = g_1$. Prove that such an R-morphism is an isomorphism.

(c) Show that there the following two are non-equivalent short exact sequences

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$

i.e. that these are extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_4$ that are not equivalent.

4.4 Quotient modules; basic isomorphism theorem

In this section, we will see an important way of constructing new modules from old ones. This arises from the following problem: Suppose that M is an R-module and that E is an equivalence relation on M. Precisely when can we define operations on the set M/E of equivalence classes so that M/E becomes an R-module with the natural surjection $\pi : M \to M/E$? This important question is settled in the following result , in which we denote the class of x modulo E by $[x]_E$.

Theorem 4.36. *Let M be an R-module and E an equivalence relation on M. The following statements are equivalent:*

(1) there is a unique addition $([x]_E, [y[_E) \mapsto [x]_E + [y]_E$ and a unique R-action $(r, [x]_E) \mapsto r[x]_E$ such that M/E is an R-module and the natural surjection is an R-epimorphism, i.e. the following identities hold:

$$[x]_E + [y]_E = [x + y]_E, \quad r[r]_E = [rx]_E \quad \forall x, y \in M \quad \forall\, r \in R$$

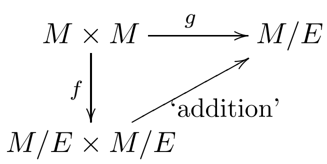
(2) E is compatible with the structure of M, in the sense that

$$\begin{aligned} x \equiv a \mod E,\ y \equiv b \mod E \implies x + y \equiv a + b \mod E \\ x \equiv a \mod E, r \in R \implies rx \equiv ra \mod E \end{aligned}$$

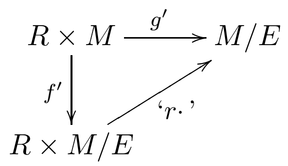
(3) There is a submodule M_E of M such that

$$x \equiv y \mod E \iff x - y \in M_E$$

Proof. (1) $\iff$ (2) is immediate on applying Theorem 4.21 to the diagram



(for $h =$"addition") where f is the map $(x, y) \mapsto ([x]_E, [y]_E)$ and g is given by $(x, y) \mapsto [x + y]_E$ and to the diagram



with f' given by $(r, x) \mapsto (r, [x]_E)$ and g' by $(r, x) \mapsto [rx]_E$. The uniqueness of these operations follows from the fact that both vertical maps are surjective and so are right cancellable.

(2)$\implies$ (3): Suppose that E is compatible with the structure of M. Then $[0]_E$ (the class of 0 modulo E) is a submodule of M. In fact, if $x \equiv 0 \mod E$ and $y \equiv 0 \mod E$

then, by compatibility, $x - y \equiv 0 - 0 = 0 \mod E$ and if $x \equiv 0 \mod E$ and $r \in R$ then $rx \equiv r0 = 0 \mod E$. Moreover, we have

$$\Longrightarrow: \quad x \equiv y \mod E \Longrightarrow x - y \equiv y - y = 0 \mod E$$
$$\Longleftarrow: \quad x - y \equiv 0 \mod E \Longrightarrow x = (x - y) + y \equiv 0 + y = y \mod E$$

so that $x \equiv y \mod E \Longleftrightarrow x - y \in [0]_E$.

(3)$\Longrightarrow$ (2): Suppose that M_E is a submodule of M such that $x \equiv y \mod E$ is equivalent to $x - y \in M_E$. Then from $x \equiv a \mod E$ and $y \equiv b \mod E$ we have $x-a \in M_E$ and $y-b \in M_E$ so that, M_E being a submodule, $x+y-(a+b) =\in M_E$ whence $x + y \equiv a + b \mod E$. Similarly, from $x \equiv a \mod E$ we have $rx - ra = r(x - a) \in M_E$ for all $r \in R$ so that $rx \equiv ra \mod E$. Thus E is compatible with the structure of M. □

Definition 4.37. In the situation of Theorem 4.36 we say that M/E is the *quotient module of M by the compatible equivalence relation E.*

If we identify equivalence relations on M that yield the same quotient set, we get a *bijection from the set of compatible equivalences on M to the set of submodules of M*: For every compatible equivalence relation E on M define $\varphi(E)$ to be the submodule $[0]_E$.

(a) That φ is surjective follows from the fact that if N is a submodule of M then the relation F given by

$$x \equiv y \mod F \Longleftrightarrow x - y \in N$$

is (as can be checked) a compatible equivalence relation on M with

$$x \equiv 0 \mod F \Longleftrightarrow x \in N$$

so that $\varphi(F) = [0]_F = N$.

(b) That φ is also injective results from the fact that if E and F are compatible equivalence relations on M such that $\varphi(E) = \varphi(F)$ then $[0]_E = [0]_F$ and so, by Theorem 4.36 (3), $x \equiv y \mod E$ is equivalent to $x \equiv y \mod F$, whence $E = F$ by the agreed identification.

Because of this bijection, it is standard practice to write M/N for the quotient module M/E where N is the submodule corresponding to E (namely $N = [0]_E$). This abuse of notation yields a corresponding abuse of language: M/N is called the *quotient module of M by the submodule N*. In this case the equivalence class of x will be written $[x]_N$. Note that, as in the case of quotient groups, $[x]_N$ coincides with the coset $x + N = \{x + n : n \in N\}$: we have

$$\begin{aligned} y \in [x]_N &\Longleftrightarrow [y]_N = [x]_N \\ &\Longleftrightarrow x - y \in N \\ &\Longleftrightarrow y = x + n \text{ for some } n \in N \end{aligned}$$

We now consider the question of how to identify the submodules of a quotient module.

Theorem 4.38 (Correspondence theorem)**.** *If N is a submodule of an R-module M then there is an inclusion-preserving bijection from the set of submodules of M/N to the set of the submodules of M that contain N.*

Proof. Suppose that A is a submodule of M that contains N. Then the set

$$A/N := \{[a]_N : a \in A\}$$

is a submodule of M/N. Consider the map ψ from the set of all such submodules A to the set of submodules of M/N described by $\psi(A) = A/N$. Since ψ so defined is the restriction (to the set of submodules that contain N) of the natural surjection π_N, it is clear that ψ is inclusion-preserving.

We observe from Corollary 4.22 (applied to $f = \psi$) that if $N \subseteq A$ then

$$\pi_N^{-1}(\psi(A)) = \pi_N^{-1}(\pi_N(A)) \stackrel{Lm\ 4.20(b)}{=} A + \ker \pi_N = A + N = A.$$

So part (β) from Corollary 4.22 holds. Consequently, ψ is injective.

We now observe that if P is any submodule of M/N then, again by Corollary 4.22,

$$\psi(\pi_N^{\leftarrow}(P)) = \pi_N^{\rightarrow}(\pi_N^{\leftarrow}(P)) \stackrel{Lm\ 4.20(a)}{=} P \cap \operatorname{im} \pi_N = P,$$

so part (β') of Corollary 4.22 holds. Consequently, ψ is also surjective. □

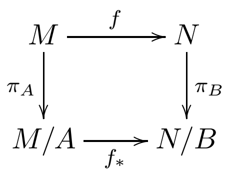
Corollary 4.39. *Every submodule of M/N is of the form A/N where A is a submodule of M that contains N*

Our aim now is to consider certain induced morphisms from one quotient module to another, and to establish some fundamental isomorphisms.

Theorem 4.40. *Let $f : M \to N$ be an R-morphism. If A and B are submodules of M and N respectively then the following statements are equivalent:*

(1) $f(A) \subseteq B$:

(2) There is a unique R-morphism $f_* : M/A \to N/B$ such that the diagram



is commutative.

Moreover, when such an R-morphism f_* exists, it is

(a) a monomorphism if and only if $A = f^{-1}(B)$;

(b) an epimorphism if and only if $B + \operatorname{im} f = N$.

Proof. Applying Theorem 4.25 to the diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi_B\circ f} & N/B \\ {\scriptstyle \pi_A}\big\downarrow & & \\ M/A & & \end{array}$$

we see that (2) holds if and only if

$$ker\pi_A \subseteq \ker(\pi_B \circ f).$$

Now

$$x \in \ker \pi_A \iff [x]_A = [0]_A \iff x \in A,$$

and similarly

$$x \in \ker(\pi_B \circ f) \iff [f(x)]_B = [0]_B \iff f(x) \in B.$$

Thus we see that (2) holds if and only if $x \in A$ implies $f(x) \in B$, which is (1).

Last part skipped in class.

As for the last statements, we observe that $f(A) \subseteq B$ is equivalent to $A \subseteq f^{-1}(B)$ and that therefore

$$\begin{aligned} \ker f_* =& \{[x]_A : f(x) \in B\} \\ =& \{[x]_A : x \in f^{-1}(B)\} \\ =& f^{-1}(B)/A, \end{aligned}$$

so that f_* is injective if and only if $A = f^{-1}(B)$.

Finally,

$$\operatorname{im} f_* = \{[f(x)]_B : x \in M\}$$

and so f_* is surjective if and only if for every $n \in N$ there exists $x \in M$ with

$$[n]_B = [f(x)]_B,$$

which is equivalent to the condition that for every $n \in N$ there exists $m \in M$ with

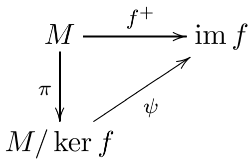
$$n - f(x) \in B,$$

which is equivalent to $N = B + \operatorname{im} f$. □

[Lecture 12, 25.1.2016]

If $f : M \to N$ is an R-morphism then we shall denote by $f^+ : M \to \operatorname{im} f$ which is given by the same prescription as f, namely $f(x) = f^+(x)$. Note that although f and f^+ have the same effect on elements of M, we distinguish between them since they have different arrival sets. f^+ is by definition surjective, f need not be.

Theorem 4.41 (First isomorphism theorem)**.** *If $f : M \to N$ is an R-isomorphism then there is a unique R-morphism $\psi : M/\ker f \to \operatorname{im} f$ such that the diagram*



is commutative

Proof. Applying Theorem 4.40 in the case where $N = \operatorname{im} f$, $B = \{0_N\}$ and $A = \ker f$ we obtain the existence of a unique R-morphism $\psi : M/\ker f \to \operatorname{im} f$ such that $\psi \circ \pi = f^+$. Since f^+ is surjective, so is ψ. Moreover,

$$\ker f = f^{-1}\{0\} = f^{-1}(B)$$

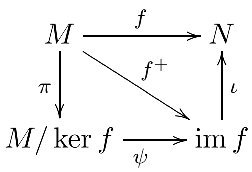
and so ψ is also injective. Thus ψ is an isomorphism. □

Corollary 4.42. *If $M \to N$ is an R-morphism then there is an inclusion-preserving bijection from the set of submodules of* $\operatorname{im} f$ *to the set of submodules of M that contain* $\ker f$

Proof. Immediate by Theorem 4.38 □

Corollary 4.43 (Canonical decomposition of morphisms)**.** *Every morphism can be expressed as the composite of an epimorphism, an isomorphism, and a monomorphism.*

Proof. With the above notation, the diagram



is commutative, ι being the natural inclusion. It follows from this that $f = \iota \circ \psi \circ \pi$. □

Although the above decomposition is called *canonical* (or *natural*) it is by no means unique, but if

$$M \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} N$$

is another such decomposition of f then necessarily $A \cong M/\ker f$ and $B \cong \operatorname{im} f$. see exercise 4.6

Theorem 4.44 (Second isomorphism theorem)**.** *If M is an R-module and if N, P are submodules of M such that $P \subseteq N$ then N/P is a submodule of M/P and there is a unique R-isomorphism*

$$h : M/N \longrightarrow (M/P)/(N/P)$$

such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\pi_P} & M/P \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi} \\ M/N & \xrightarrow[h]{} & (M/P)/(N/P) \end{array}$$

Proof. We know by Corollary 4.39 that N/P is a submodule of M/P. Since $\pi_P(N) = \{[n]_P : n \in N\} = N/P$, we can apply Theorem 4.40 to the above diagram to obtain the existence of a unique R–morphism $h : M/N \to (M/P)/(N/P)$ making the diagram commutative. Now since, by the commutativity, $h \circ \pi_N$ is an epimorphism, so is h. To show that h is also a monomorphism, it suffices to note that $\pi_P^{-1}(N/P) = N$ and appeal to Theorem 4.40 again. □

The third isomorphism theorem is a consequence of the following.

Given an R-module M and a submodule A of M, it is clear that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\iota_A} M \xrightarrow{\pi_A} M/A \longrightarrow 0$$

in which ι is the natural inclusion and π_A the natural surjection. This generalises to arbitrary R-modules the situation of Example 4.30 in which the abelian groups are considered as $\mathbb{Z}$-modules.

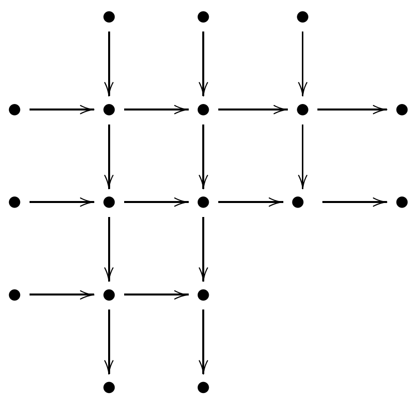
Theorem 4.45. *If A and B are submodules of an R-module M then there is a commutative diagram (with exact rows and columns) of the form*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A \cap B & \longrightarrow & B & \longrightarrow & B/(A \cap B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & M/A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A/(A \cap B) & \longrightarrow & M/B & \longrightarrow & M/(A+B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Proof. Let ι_A be the natural inclusion. Then we have $\iota_A(A \cap B) \subseteq B$ and so we can apply Theorem 4.40 to obtain the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & M \\ {\scriptstyle \pi_{A\cap B}}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ A/(A \cap B) & \xrightarrow[(\iota_A)_*]{} & M/B \end{array}$$

Considering likewise the inclusion $\iota_B : B \to M$ we obtain a similar commutative diagram. These diagrams can be joined together and extended to form all but the bottom right-hand corner of the big diagram, namely



We can complete the bottom right-hand corner by defining maps $\psi_B : M/A \to M/(A+B)$ and $\psi_A : M/B \to M/(A+B)$ by

$$\psi_B([x]_A) := [x]_{A+B}, \quad \psi_A([x]_B) := [x]_{A+B}.$$

It is clear that ψ_B and ψ_A are R-morphisms which make the completed diagram commutative.

We now show that the bottom row

$$0 \longrightarrow A/(A \cap B) \xrightarrow{(\iota_A)_*} M/B \xrightarrow{\psi_A} M/(A+B) \longrightarrow 0$$

is exact. By symmetry, the right-hand column will then also be exact. Since ψ_A is surjective and since $(\iota_A)_*$ is injective (Theorem 4.40) it only remains to show that $\mathrm{im}(\iota_A)_* = \ker \psi_A$. For this purpose, we note that $\mathrm{im}(\iota_A)_* = \{[x]_B : x \in A\}$ and $\ker \psi_A : \{[x]_B : x \in A + B\}$. Observing that

$$x \in A + B \Longrightarrow x = a + b \text{ for some } a \in A, \text{ some } b \in B \Longrightarrow [x]_B = [a+b]_B = [a]_B$$

we obtain $\ker \psi_A \subseteq \mathrm{im}(\iota_A)_*$ and by observing that

$$x \in A \Longrightarrow x = a \quad \text{for some } a \in A \Longrightarrow [x]_B = [a]_B = [a+b]_B$$

we obtain the reverse inclusion. □

Corollary 4.46 (Third isomorphism theorem)**.** *If A and B are submodules of an R-module M then*

$$A/(A \cap B) \cong (A+B)/B$$

Proof. Since A and B are submodules of $A+B$ we can apply the above in the case where $M = A+B$. The bottom row of the diagram becomes

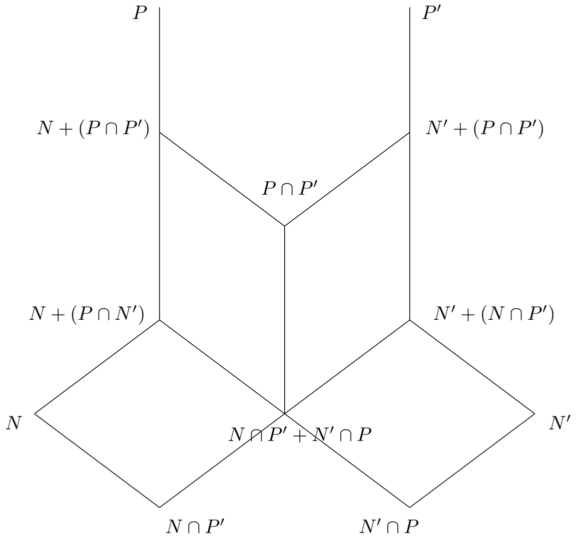
$$0 \longrightarrow A/(A \cap B) \longrightarrow (A+B)/B \longrightarrow (A+B)/(A+B) \longrightarrow 0$$

Since $(A+B)/(A+B)$ is a zero module, the exactness of this row together with Theorem 4.29 (3) gives the required isomorphism. □

The lectures went only up to the third isomorphism theorem and then stopped

The last of the isomorphism theorems that are often used is the *Butterfly Lemma* (or *Lemma of Zassenhaus*). It uses the *Hasse diagram*, a diagram with non-horizontal edges where an ascending line segment from A to B means that A is a submodule of B.

Theorem 4.47 (Butterfly Lemma). *Let M be an R-module with submodules N, P, N', P' such that $N \subseteq P$ and $N' \subseteq P'$. Then relative to the Hasse diagram*



the following quotient modules are isomorphic

$$\frac{N + (P \cap P')}{N + (P \cap N')} \cong \frac{P \cap P'}{(N \cap P') + (N' \cap P)} \cong \frac{N' + (P \cap P')}{N' + (P \cap P')}$$

Proof. The proof uses the modular law (Theorem 4.15), the third isomorphism theorem (Corollary 4.46) for $A = P \cap P'$ and $B = N + (P \cap N')$ and the second isomorphism theorem (Theorem 4.44). For details: [3, §4] □

Exercises for Section 4.4

1. An R-module is *cyclic* if it is generated by a singleton subset. Let $M = Rx$ be a cyclic R-module. Recall that the annihilator of x is the submodule $\operatorname{Ann}_R(\{x\}) = \{r \in R : rx = 0\}$. Prove that $M \cong R/\operatorname{Ann}_R(\{x\})$.

   Deduce that if R is a principal ideal domain (a commutative integral domain in which every ideal is generated by a singleton subset) and if $x \in R$ is such that $\operatorname{Ann}_R(x) = p^k R$ for some $p \in R$ (see exercise 2.2) then the only submodules of M are those in the chain

   $$0 = p^k M \subset p^{k-1} M \subset \cdots \subset pM \subset p^0 = M$$

   (Hint: use the correspondence theorem)

   check whether its really $x \in R$ might be - as $M \cong R/\operatorname{Ann}_R x$

2. Let $f : M \to N$ be an R-morphism and suppose that f can be expressed as the composite map
$$M \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} N$$
where α is an epimorphism, β an isomorphism and γ a monomorphism. Prove that $A \cong M/\ker f$ and $B \cong \operatorname{im} f$.

3. Let R be a commutative unitary ring and let $R_n[X]$ be the R-module of all polynomials of dgree at most n with coefficients in R. Show that, for $n \geq 1$,
$$R_{n-1}[X] \cong R_n[X]/R$$
(Hint: Consider the differentiation map)

Notation for Chapters 3 and 4

- Let V and W be vector spaces over the field K. We write $\mathrm{Hom}(V, W)$ for the space of K-linear maps between V and W (or *Homomorphisms between V and W*). We write $\mathrm{End}(V)$ for $\mathrm{Hom}(V, V)$, the space of *endomorphisms of V*.

- $\mathbb{C}_{m,n}$ the space of $m \times n$-matrices over $\mathbb{C}$. If R is a ring (with unit), $R_{m,n}$ is the space of $m \times n$-matrices over R.

- $x \in V$, V vector space: x^T the transpose of x (similar for matrices).

- $[x_1, \ldots, x_n]$ for the matrix formed by the column vectors $x_i \in \mathbb{C}^m$.

- $V^m = \times^m(V) = V \times \cdots \times V$ for the product of m factors of the vector space V.

- $M(V_1, \ldots, V_m, W)$ the set of multilinear maps from $V_1 \times \cdots \times V_m$ to W.

- The space of r-linear alternating maps, $\mathrm{Alt}^r(V; W) := \{f : V^r \to W : f \text{ is alternating}\}$.

- Let $V_1, \ldots, V_m$ be K-vector spaces of dimensions $n_1, \ldots, n_m$. Then we define $\Gamma(n_1, \ldots, n_m) := \{\gamma : \gamma = (\gamma(1), \ldots, \gamma(m)) \mid 1 \leq \gamma(i) \leq n_i, i = 1, \ldots, m\}$, sometimes abbreviated as Γ. This notation helps enumerating basis vectors of the tensor product of m vector spaces of dimensions $n_1, \ldots, n_m$.
  If $V_1 = \cdots = V_m =: V$ we have $n_1 = \cdots = n_m = n$ for $n := \dim V$; in this case we just write $\Gamma(n : m)$:
  $\Gamma(n : m) = \{\gamma : \gamma = (\gamma(1), \ldots, \gamma(m)), 1 \leq \gamma(i) \leq n, i = 1, \ldots, m\}$.

- Consider $V_1 \otimes \cdots \otimes V_m$ for $\mathbb{C}$-vector spaces V_i of dimension n_i, and where $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ is a basis of V_i for $i = 1, \ldots, m$. Let $\gamma \in \Gamma = \Gamma(n_1, \ldots, n_m)$. We write
  $e_\gamma^\otimes$ for the basis vector $e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)}$.

  For $(u_1, \ldots, u_m) \in V_1 \times \cdots \times V_m$ we write
  $u^\otimes := u_1 \otimes \cdots \otimes u_m$ (which is, by definition, $\otimes(u_1, \ldots, u_m)$).

- $V^{\otimes m} = \otimes^m(V) = V \otimes \cdots \otimes V$ (m copies of V in the tensor product) and $(V^*)^{\otimes m} = \otimes^m(V^*) = V^* \otimes \cdots \otimes V^*$ (m copies of V^*).

- Let I be a set. We write $\mathcal{P}^*(I)$ for the set of all non-empty finite subsets of I.

- Let R be a ring (with unit), let M be a (left) R-module. We write $\mathcal{L}(M)$ for the lattice of submodules of M.
- $\mathbb{N} = \{0, 1, 2, 3, \dots\} = \mathbb{Z}_{\geq 0}$.

Bibliography

The bibliography is not meant to be exhaustive, a lot of the material can be found in textbooks. This is only a very short list, some of the texts are used in proofs.

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