DISCRETE AND ALGEBRAIC STRUCTURES

Master Study Mathematics

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Parts I & II

Contents

I Combinatorics and Graph Theory **1**

1 Combinatorics **2**

1.1 Basics 2
- 1.1.1 Binomial Coefficient and Binomial Theorem 2
- 1.1.2 Landau- and Asymptotic Notations 5
- 1.1.3 Useful Inequalities and Asymptotic Estimates 6

1.2 Recursions 7
- 1.2.1 Recursions 7
- 1.2.2 Linear recurrences 7
- 1.2.3 Non-linear recurrences 12

1.3 Generating Functions 13
- 1.3.1 Formal Power Series 13
- 1.3.2 Ordinary and Exponential Generating Functions 17

1.4 Symbolic method 20
- 1.4.1 Unlabelled combinatorial objects 20
- 1.4.2 Labelled combinatorial objects 25

1.5 Analytic Methods 28
- 1.5.1 Analytic functions 28
- 1.5.2 Cauchy's coefficient formula 29
- 1.5.3 Lagrange Inversion Theorem 29
- 1.5.4 Singularities 33
- 1.5.5 Meromorphic functions 33
- 1.5.6 Newton's generalised binomial theorem 34
- 1.5.7 Transfer theorem 35
- 1.5.8 Multiple singularities 36

2 Graph Theory **39**

2.1 Matchings, Eulerian Tour and Hamiltorian Cycles 40
- 2.1.1 Basic Terminologies 40
- 2.1.2 Matchings 42
- 2.1.3 Eulerian Tours and Hamiltorian Cycles 44

2.2 Graph Decompositions 46
- 2.2.1 Connected Graphs 46
- 2.2.2 2-Connected Graphs 46

2.2.3 3-Connected Graphs 48
2.3 Planar Graphs and Trees 49
2.3.1 Planar and Plane Graphs 49
2.3.2 Trees 51

3 Stochastic Aspects 55
3.1 Basics 56
3.2 Ramsey Number 59
3.3 Independence Number 63
3.4 Subgraphs and Second Moment Method 64

Part I

Combinatorics and Graph Theory

Chapter 1

Standard Methods of Enumerative Combinatorics

Lecture 1

1.1 Basics

1.1.1 Binomial Coefficient and Binomial Theorem

Definition 1 (Set of integers)**.**

$$[n] := \{1, 2, \ldots, n\}$$

Definition 2 (Factorial)**.** For any $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} n! &:= n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \prod_{i=0}^{n-1} (n-i) \\ 0! &:= 1 \end{aligned}$$

Definition 3 (Falling factorial)**.** For any $n, k \in \mathbb{N} \cup \{0\}$ satisfying $k \leq n$,

$$\begin{aligned} (n)_k &:= n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \prod_{i=0}^{k-1} (n-i) \\ (n)_0 &:= 1 \end{aligned}$$

Definition 4 (Binomial coefficient)**.** For any $n, k \in \mathbb{N} \cup \{0\}$,

$$\binom{n}{k} := \begin{cases} \frac{n!}{(n-k)!k!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Definition 5 (Multinomial coefficient)**.** For any $n, k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\}$ satisfying $n = k_1 + k_2 + \ldots + k_m$,

$$\binom{n}{k_1, k_2, \ldots, k_m} := \frac{n!}{k_1! k_2! \ldots k_m!}$$

Properties 1. *For any $n, k \in \mathbb{N} \cup \{0\}$ satisfying $k \leq n$,*

$$\binom{n}{k} = \binom{n}{n-k}$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

For any $k, m, n \in \mathbb{N}$,

$$\binom{n+m}{k} = \sum_{\ell=0}^{k} \binom{n}{\ell} \cdot \binom{m}{k-\ell}$$

Theorem 1 (Binomial theorem)**.** *For any $n \in \mathbb{N}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where x is a formal variable or $x \in \mathbb{C}$.

Theorem 2 (Multinomial theorem)**.** *For any $n \in \mathbb{N}$,*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_2}$$

where the sum is over all $k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\}$ satisfying $n = k_1 + k_2 + \ldots + k_m$.

Applying the binomial theorem with $x = 1$ and $x = -1$ we have

$$2^n = \sum_{i=0}^{n} \binom{n}{k}$$
$$0 = \sum_{i=0}^{n} (-1)^k \binom{n}{k}$$

Example 1 (Subsets)**.** *The number of all subsets of $[n]$ is equal to*

$$\sum_{i=0}^{n} \binom{n}{k} = 2^n$$

Example 2 (Subsets of odd sizes)**.**

To calculate the number of subsets of $[n]$ of odd sizes, we observe that

$$\begin{aligned} 2^n &= \sum_{i=0}^{n} \binom{n}{k} + \sum_{i=0}^{n} (-1)^k \binom{n}{k} \\ &= 2\left[\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2 \cdot \lfloor n/2 \rfloor}\right] \end{aligned}$$

This implies that the number of subsets of $[n]$ of even sizes is 2^{n-1}, and therefore the number of subsets of $[n]$ of odd sizes is equal to

$$2^n - 2^{n-1} = 2^{n-1}$$

Definition 6 (Generalized binomial coefficient)**.** For any $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$,

$$\binom{\alpha}{k} := \frac{\alpha \cdot (\alpha - 1) \cdot \ldots \cdot (\alpha - k + 1)}{k!}$$

(1) Note that the denominator has k terms.
(2) If α is in $\mathbb{N}$ and $k > \alpha$, $\binom{\alpha}{k} = 0$ as in Definition 4.

Example 3.

$$\begin{aligned} \binom{\frac{1}{2}}{k} &:= \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdot \ldots \cdot \left(\frac{1}{2} - k + 1\right)}{k!} \\ &= \frac{1 \cdot (1-2) \cdot \ldots \cdot (1 - 2(k-1))}{2^k\, k!} \\ &= \frac{(-1)^{k-1}}{4^k\, (2k-1)} \binom{2k}{k} \end{aligned}$$

Theorem 3 (Generalized binomial theorem)**.** *For any $\alpha \in \mathbb{C}$ and $x \in \mathbb{C}$ with $|x| < 1$,*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Example 4.

$$\begin{aligned} (1-x)^{\frac{1}{2}} &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-x)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{4^k\, (2k-1)} \binom{2k}{k} (-x)^k \\ &= -\sum_{k=0}^{\infty} \frac{1}{4^k\, (2k-1)} \binom{2k}{k} x^k \end{aligned}$$

1.1.2 Landau- and Asymptotic Notations

(1) $O(g(n)) := \{f(n): \quad \exists c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \leq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty\right\}$

$f(n) = O(g(n))$
"$f(n)$ is Big Oh of $g(n)$"
"$f(n)$ grows at most as fast as $g(n)$"'

(2) $\Omega(g(n)) := \{f(n): \exists c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \geq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \liminf_{n\to\infty} \frac{|f(n)|}{|g(n)|} > 0\right\}$

$f(n) = \Omega(g(n))$
"$f(n)$ is Big Omega of $g(n)$"
"$f(n)$ grows at least as fast as $g(n)$"'

(3) $\Theta(g(n)) := O(g(n)) \cap \Omega(g(n))$

$f(n) = \Theta(g(n))$
"$f(n)$ is Big Theta of $g(n)$"
"$f(n)$ grows as fast as $g(n)$"'

(4) $o(g(n)) := \{f(n): \forall c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \leq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0\right\}$

$f(n) = o(g(n)) \quad or \quad f(n) \ll g(n)$
"$f(n)$ is Small Oh of $g(n)$"
"$f(n)$ grows slower than $g(n)$"'

(5) $\omega(g(n)) := \{f(n): \forall c>0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \ |f(n)| \geq c \cdot |g(n)|\}$
$= \left\{f(n): \quad \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = \infty\right\}$

$f(n) = \omega(g(n)) \quad or \quad f(n) \gg g(n)$
"$f(n)$ is omega of $g(n)$"
"$f(n)$ grows faster than $g(n)$"'

(6)

$$f(n) \sim g(n) \quad \Leftrightarrow \quad \lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$$

"$f(n)$ is of order of $g(n)$"
"$f(n)$ is asymptotically equal to $g(n)$"'

Example 5.

$$1+\tfrac{1}{n} = 1+o(1) \quad \text{vs} \quad 1+\tfrac{1}{n^2} = 1+o(1) \quad \text{vs} \quad 1+\tfrac{1}{n^2+1000} = 1+o(1)$$

1.1.3 Useful Inequalities and Asymptotic Estimates

Theorem 4 (GM-AM-RMS inequality). *For any $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$,*

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \quad \leq \quad \frac{1}{n}\sum_{i=1}^{n} x_i \quad \leq \quad \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

In words, "the geometric mean is smaller than the arithmetic mean, which is smaller than the root mean square".

Theorem 5 (Stirlings formula). *For any $n \in \mathbb{N}$,*

$$\begin{aligned} n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right) \\ &= (1+o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \end{aligned}$$

Lemma 6 (Binomial coefficient approximation). *For any $n, k \in \mathbb{N}$ satisfying $k \leq n$,*

$$\begin{aligned} \left(\frac{n}{k}\right)^k &\leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k \\ \binom{n}{k} &= 2^{H\left(\frac{n}{k}\right)n+O(\log k)} \end{aligned}$$

where H is the Entropy function

$$H(x) := -x\log_2 x - (1-x)\log_2(1-x)$$

Lecture 2

1.2 Recursions

1.2.1 Recursions

Example 6. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0 \end{cases}$$

The sequence begins with $0, 1, 3, 7, 15, 31, \ldots$

We add 1 in both sides of the latter equation to obtain

$$a_{n+1} + 1 = 2(a_n + 1), \quad n \geq 0.$$

Then we have $a_n + 1 = 2^n(a_0 + 1) = 2^n$ *and therefore the sequence is explicitly given by*

$$a_n = 2^n - 1, \quad n \geq 0.$$

Example 7. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_1 &= 1 \\ a_{n+2} &= a_{n+1} + a_n, \quad n \geq 0. \end{cases}$$

This sequence is called Fibonacci sequence and begins with $0, 1, 1, 2, 3, 5, 8, 13, \ldots$.

How can we derive the explicit form of the sequence?

1.2.2 Linear recurrences

Definition 7. We say that a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfies a k-th order recurrence (or recursion) if a_n can be written as

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}), \quad n \geq k$$

for a function $f : \mathbb{C}^k \to \mathbb{C}$.

First-order linear recurrences

Theorem 7. *A first-order linear recurrence with constant coefficients of the form*

$$\begin{cases} a_0 = 0 \\ a_n = c_n \cdot a_{n-1} + d_n, \quad n \geq 1, \end{cases}$$

with $c_i \neq 0$ has an explicit solution,

$$a_n = d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n.$$

Proof. Divide both sides by $c_n c_{n-1} \cdots c_1$:

$$\frac{a_n}{c_n c_{n-1} \cdots c_1} = \frac{a_{n-1}}{c_{n-1} c_{n-2} \cdots c_1} + \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Change of variables: let

$$b_n = \frac{a_n}{c_n c_{n-1} \cdots c_1}.$$

We get the difference relation

$$b_n = b_{n-1} + \frac{d_n}{c_n c_{n-1} \cdots c_1},$$

that is,

$$b_n - b_{n-1} = \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Summing up, we get

$$b_n = \sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1}.$$

Thus, for a_n we have

$$\begin{aligned} a_n &= c_n c_{n-1} \cdots c_1 \cdot \left(\sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1} \right) \\ &= d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n. \end{aligned}$$ □

Example 8. *Returning to Example 6 we have*

$$\begin{cases} a_0 & = \ 0 \\ a_{n+1} & = \ 2a_n + 1, \quad n \geq 0, \end{cases}$$

so taking $c_i = 2, d_i = 1$ for all i in Theorem 7 we have

$$a_n = 1 + \sum_{i=1}^{n-1} 2^{n-i} = 1 + 2^n \left(1 - \left(\frac{1}{2} \right)^{n-1} \right) = 2^n - 1,$$

because $\sum_{i=1}^{n} r^i = \frac{r(1-r^n)}{1-r}$ for any $r \neq 1$.

Higher order linear recurrences with constant coefficients

Theorem 8. *All solutions to the k-th order linear recurrence with constant coefficients* $c_1, c_2, \dots, c_k$

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \dots + c_k \cdot a_{n-k}, \quad n \geq k,$$

can be expressed as linear combinations of terms of the form

$$\alpha^n, n\alpha^n, n^2\alpha^n, \dots, n^{m-1}\alpha^n$$

where α *is a root of order m of the characteristic polynomial*

$$q(z) = z^k - c_1 \cdot z^{k-1} - c_2 \cdot z^{k-2} - \dots - c_k.$$

Remark 1. If $q(z) = (z-\alpha)^m \cdot p(z)$ for some polynomial $p(z)$ where $(z-\alpha) \nmid p(z)$, i.e. α is a root of multiplicity m, then $q(\alpha) = q'(\alpha) = \dots = q^{(m-1)}(\alpha) = 0$.

Proof of Theorem 8. Let α be a simple root of $q(z)$ (i.e. the multiplicity is 1) and $a_n = \alpha^n$. We want to check that a_n is a solution to the recurrence, i.e.

$$\begin{aligned}
0 &\overset{!}{=} \alpha^n - c_1\alpha^{n-1} - c_2\alpha^{n-2} - \dots - c_k\alpha^{n-k} \\
&= \alpha^{n-k} \cdot \left(\alpha^k - c_1\alpha^{k-1} - \dots - c_k\right) \\
&= \alpha^{n-k} \cdot q(\alpha) = 0.
\end{aligned}$$

Let α now be a root of multiplicity of m of $q(z)$. Then, for $0 \leq i \leq m-1$, $a_n = n^i\alpha^n$ is a solution to the recurrence because

$$\begin{aligned}
0 &\overset{!}{=} a_n - c_1 \cdot a_{n-1} - c_2 \cdot a_{n-2} - \dots - c_k a_{n-k} \\
&= n^i\alpha^n - c_1(n-1)^i\alpha^{n-1} - c_2(n-2)^i\alpha^{n-2} - \dots c_k(n-k)^i\alpha^{n-k} \\
&= \alpha^{n-k} \cdot \left(n^i\alpha^k - c_1(n-1)^i\alpha^{k-1} - \dots - c_k(n-k)^i\right) \\
&= \alpha^{n-k} \cdot \big((n-k)^i q(\alpha) + \alpha\left((n-k)^i - (n-k-1)^i\right) q'(\alpha) + \\
&\qquad \alpha^2\left(b_0(n-k)^i + b_1(n-k-1)^i + b_2(n-k-2)^i\right) q''(\alpha) + \dots\big) \\
&= \alpha^{n-k} \cdot \left(\sum_{0 \leq j \leq i} \cdot \left(\sum_{0 \leq \ell \leq j} b_{j,l}(n-k-\ell)^i\right) q^{(j)}(\alpha)\right) = 0,
\end{aligned}$$

because of the previous remark; the $b_{j,\ell}$ are constants.

Furthermore, a linear combination of $\alpha^n, n \cdot \alpha^n, \dots, n^{m-1}\alpha^n$ is also a solution to the recurrence.

If $q(z)$ has distinct roots $\alpha_1, \dots, \alpha_s$ with multiplicities $m_1, \dots, m_s$ where $m_1 + \dots + m_s = k$, then a linear combination of all these $n^j \cdot \alpha_i^n$ for $0 \leq j \leq m_i - 1$, $1 \leq i \leq s$, i.e.

$$a_n = \sum_{i=1}^{s} \left(b_{i,0} + b_{i,1}n + \dots + b_{i,m_i-1}n^{m_i-1}\right) \cdot \alpha_i^n$$

is also a solution to the recurrence.

We claim the opposite is also true. Let s be the set of sequences $\{a = (a_n \in \mathbb{R})_{n\geq 0}\}$ and R be the set of solutions to the recurrence. Then R is closed under addition and scalar multiplication, and $R \neq \emptyset$. Therefore, R is a vector space.

We claim that R has dimension k. Consider a map f from R to $\mathbb{R}^k$:

$$f : R \to \mathbb{R}^k, \quad a \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}.$$

This is a linear map and also an isomorphism, because any solution to the recurrence is uniquely determined by the k initial values. So, R has dimension k.

Therefore, R is given (generated) by linear combinations of any k linearly independent solutions to the recurrence. Now it suffices to show that the set of k solutions $\{n^j \alpha_i^n \mid 1 \leq j \leq m_i - 1, 1 \leq i \leq s\}$ to the recurrence is linearly independent. But this is true because these solutions have different orders of growth (in particular at ∞). $\square$

Example 9. *Returning to Example 7 we consider the Fibonacci sequence*

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, \quad n \geq 2 \\ a_0 = 0, a_1 = 1. \end{cases} \tag{1.1}$$

Its characteristic polynomial is

$$q(z) = z^2 - z - 1 = \left(z - \frac{1+\sqrt{5}}{2}\right) \cdot \left(z - \frac{1-\sqrt{5}}{2}\right),$$

and the solution to the recurrence for a_n is

$$\begin{cases} a_n = r_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + r_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \\ a_0 = 0, a_1 = 1. \end{cases}$$

From the initial conditions, we get $r_1 = \frac{1}{\sqrt{5}}$, $r_2 = -\frac{1}{\sqrt{5}}$. The explicit solution to (1.1) *is*

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example 10. *Consider the second order linear recurrence*

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \geq 2$$

with the initial conditions $a_0 = 0, a_1 = 1$.

The characteristic polynomial is

$$q(z) = z^2 - 5z + 6 = (z-2)(z-3).$$

The solution will be of the form

$$a_n = r_1 2^n + r_2 3^n.$$

Due to the initial conditions, $r_1 = -1$ *and* $r_2 = 1$ *and the solution is*

$$a_n = -2^n + 3^n.$$

This can be solved in Maple as follows:

```
rsolve({a(n) = 5 * a(n-1) - 6 * a(n-2), a(0) = 0, a(1) = 1}, a(n));
```

In Mathematica:

```
RSolve[{a[n] == 5 * a[n-1] - 6 * a[n-2], a[0] == 0, a[1] == 1}, a[n], n]
```

Exercise 1. Find initial conditions a_0, a_1, a_2 for which the growth rate of the solution to the recurrence

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n \geq 3$$

is (a) constant, (b) exponential, and (c) fluctuating in sign.

Higher order linear recurrences with non-constant coefficients

Example 11. *Consider the recurrence*

$$\begin{cases} a_n = na_{n-1} + n(n-1)a_{n-2}, & n \geq 2 \\ a_0 = 0, a_1 = 1. \end{cases}$$

Divide by $n!$ *to receive*

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + \frac{a_{n-2}}{(n-2)!}.$$

Change variables: let $b_n = \frac{a_n}{n!}$. *Then we get the recurrence*

$$\begin{cases} b_n = b_{n-1} + b_{n-2}, & n \geq 2 \\ b_0 = 0, b_1 = 1. \end{cases}$$

for the Fibonacci sequence. From Example 9 we have

$$b_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

This yields

$$a_n = \frac{n!}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Exercise 2. Solve the recurrence

$$\begin{cases} n(n-1)a_n = (n-1)a_{n-1} + a_{n-2}, & n \geq 2 \\ a_0 = 0, \ a_1 = 1. \end{cases}$$

1.2.3 Non-linear recurrences

Example 12. *Consider the second order non-linear recurrence*

$$\begin{cases} a_n = \sqrt{a_{n-1} \cdot a_{n-2}}, & n \geq 2 \\ a_0 = 1,\ a_1 = 2. \end{cases}$$

We use the logarithm function, since

$$\log_a xy = \log_a x + \log_a y.$$

Let $b_n = \log a_n$. *We get*

$$\begin{cases} b_n = \frac{1}{2}\left(b_{n-1} + b_{n-2}\right), & n \geq 2 \\ b_0 = 0,\ b_1 = 1. \end{cases}$$

This can be solved similar to the previous linear recurrences.

Lecture 3

1.3 Generating Functions

1.3.1 Formal Power Series

Now let us return to Example 6.

Example 13. *Let a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0 \end{cases} \tag{1.2}$$

and let z be a formal indeterminate variable.

Multiplying both sides of (1.2) by z^{n+1} and summing over $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\sum_{n=0}^{\infty} a_{n+1} z^{n+1} = \sum_{n=0}^{\infty} (2a_n + 1)\, z^{n+1}.$$

If it were true that $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} = 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n$, we would obtain

$$\sum_{n=1}^{\infty} a_n z^n = 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n$$

Defining $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ we have

$$A(z) = 2zA(z) + z\sum_{n=0}^{\infty} z^n \quad \textit{equiv.} \quad (1-2z)A(z) = z\sum_{n=0}^{\infty} z^n$$

because $a_0 = 0$. From this we we would further obtain

$$\begin{aligned} A(z) &\stackrel{?}{=} \frac{z}{1-2z} \sum_{n=0}^{\infty} z^n \stackrel{?}{=} \frac{z}{1-2z} \cdot \frac{1}{1-z} = 2z\,\frac{1}{1-2z} - z\,\frac{1}{1-z} \\ &\stackrel{?}{=} 2z \sum_{n=0}^{\infty} (2z)^n - z \sum_{n=0}^{\infty} z^n \\ &\stackrel{?}{=} \sum_{n=1}^{\infty} (2z)^n - \sum_{n=1}^{\infty} z^n \\ &\stackrel{?}{=} \sum_{n=1}^{\infty} (2^n - 1)\, z^n \\ &= \sum_{n=0}^{\infty} (2^n - 1)\, z^n, \end{aligned}$$

if the following were true:

$$(1-2z)A(z) = z\sum_{n=0}^{\infty} z^n \quad \overset{?}{\iff} \quad A(z) = \frac{z}{1-2z}\sum_{n=0}^{\infty} z^n \tag{1.3}$$

$$\sum_{n=0}^{\infty} (az)^n \overset{?}{=} \frac{1}{1-az}, \quad a \in \mathbb{K} \tag{1.4}$$

$$\sum_{n=0}^{\infty} (2a_n+1)\, z^{n+1} \overset{?}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n. \tag{1.5}$$

Summing up, we would have $\sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty}(2^n-1)\, z^n$ *and therefore* $a_n = 2^n - 1$ *for every* $n \in \mathbb{N}\cup\{0\}$*, by comparing the coefficients, if* (1.3)–(1.5) *were true. When are they true?*

Definition 8 (Formal Power Series)**.** Let $\mathbb{K}$ denote a commutative ring, usually we take $\mathbb{K} = \mathbb{C}$. Given a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ with $a_n \in \mathbb{K}$ and a formal variable z, we call an infinite sum of the form

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{n\geq 0} a_n\, z^n$$

a *formal power series*. We call a_n the coefficient of z^n in $A(z)$ and use the notation

$$[z^n]\, A(z) := a_n.$$

The ring of formal power series is denoted by $\mathbb{K}[[z]]$ and endowed with the operations of addition and product:

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) + \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty} (a_n+b_n)\, z^n \tag{1.6}$$

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \tag{1.7}$$

We often skip $\cdot$ for the product of two formal power series. As a special case of (1.7) we have, for any $a \in \mathbb{K}$,

$$a\left(\sum_{n=0}^{\infty} b_n\, z^n\right) = \sum_{n=0}^{\infty} a\, b_n\, z^n. \tag{1.8}$$

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ we also define the following algebraic operations:

- Differentiation: $A'(z) := \sum_{n\geq 1} n a_n\, z^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1}\, z^n$.
- Integration: $\int_0^z A(t)\,\mathrm{d}t := \sum_{n\geq 0} \frac{a_n}{n+1}\, z^{n+1} = \sum_{n\geq 1} \frac{a_{n-1}}{n}\, z^n$.

A formal power series $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ is called a *reciprocal* of a formal power series $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ (and vice versa) if $A(z)B(z) = B(z)A(z) = 1$.

Propostion 1. *A formal power series $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case, the reciprocal is unique and so we denote the reciprocal of $A(z)$ by $\frac{1}{A(z)}$.*

Proof. Exercise! □

In Example 13 we wished to have

$$\begin{aligned}
(1-2z)A(z) = 1 \quad &\overset{?}{\Longleftrightarrow} \quad A(z) = \frac{1}{1-2z} \\
\sum_{n=0}^{\infty} z^n \;&\overset{?}{=}\; \frac{1}{1-z}, \\
\sum_{n=0}^{\infty} (2a_n+1)\, z^{n+1} \;&\overset{?}{=}\; 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n.
\end{aligned}$$

From definition, equality $(1-2z)A(z) = 1$ and Proposition 1, $A(z)$ is the (unique) reciprocal of $1-2z$ and vice versa, so

$$(1-2z)A(z) = 1 \quad \Longleftrightarrow \quad A(z) = \frac{1}{1-2z}.$$

For the second equality $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, we shall show that the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} z^n$ is $B(z) := 1-z$ and vice versa. Letting $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ with $a_n = 1$ for $n \geq 0$ and $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ with $b_0 = 1, b_1 = -1$ and $b_n = 0$ for $n \geq 2$, we have

$$A(z)B(z) \overset{(1.7)}{=} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n = a_0 b_0 + \sum_{n=1}^{\infty} (a_{n-1} b_1 + a_n b_0)\, z^n = 1.$$

By Proposition 1, $A(z)$ is the unique reciprocal of $B(z)$ and so $A(z) = \frac{1}{B(z)} = \frac{1}{1-z}$, that is, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

in the ring $\mathbb{K}[[z]]$.

Analogously one can show that for any $a \in \mathbb{K}$, the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} (az)^n$ is $B(z) := 1-az$ and vice versa. In other words, for any $a \in \mathbb{K}$,

$$\sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az} \tag{1.9}$$

in the ring $\mathbb{K}[[z]]$.

For the third equality $\sum_{n=0}^{\infty} (2a_n+1)\, z^{n+1} = 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n$, we use (1.6) and (1.7) to have

$$\sum_{n=0}^{\infty} (2a_n+1)\, z^{n+1} \overset{(1.6)}{=} \sum_{n=0}^{\infty} 2a_n\, z^{n+1} + \sum_{n=0}^{\infty} z^{n+1} \overset{(1.7)}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n.$$

That is, $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} = 2z\, \sum_{n=0}^{\infty} a_n z^n + z\, \sum_{n=0}^{\infty} z^n$ in the ring $\mathbb{K}[[z]]$.

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ and a constant $\beta \in \mathbb{K}$ we have the following properties:

- $A(\beta z) = \sum_{n\geq 0} a_n \beta^n\, z^n$, so $[z^n]\, A(\beta z) = a_n \beta^n = \beta^n a_n = \beta^n\, [z^n]\, A(z)$ (scaling)
- $(A(z)-a_0)/z = \sum_{n\geq 1} a_n\, z^{n-1} = \sum_{n\geq 0} a_{n+1}\, z^n$ (left shift)
- $z\, A(z) = \sum_{n\geq 0} a_n\, z^{n+1} = \sum_{n\geq 1} a_{n-1}\, z^n$ (right shift)
- $A(z)/(1-z) = A(z)\cdot \sum_{n\geq 0}\, z^n = \sum_{n\geq 0}\left(\sum_{0\leq k\leq n} a_k\right) z^n$ (partial sum)

Example 14. *Let us study the recursion for the Fibonacci sequence*

$$\begin{cases} a_0 & = 0 \\ a_1 & = 1 \\ a_{n+2} & = a_{n+1} + a_n, \quad n \geq 0, \end{cases}$$

using the generating function $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$. Multiplying both sides of the recursion by z^{n+2} and summing over $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}\, z^{n+2} \;&=\; \sum_{n=0}^{\infty} (a_{n+1}+a_n)\, z^{n+2} \\
\iff \quad \sum_{n=2}^{\infty} a_n z^n \;&=\; z \sum_{n=1}^{\infty} a_n\, z^n \;+\; z^2 \sum_{n=0}^{\infty} a_n\, z^n \\
\iff \quad A(z) - z \;&=\; z\, A(z) \;+\; z^2\, A(z) \\
\iff \quad (1-z-z^2)\, A(z) \;&=\; z \\
\iff \quad A(z) \;=\; \frac{z}{1-z-z^2} \;&=\; \frac{1}{\beta_1-\beta_2}\left(\frac{1}{1-\beta_1 z} - \frac{1}{1-\beta_2 z}\right) \\
&\overset{(1.9)}{=} \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}(\beta_1 z)^n - \sum_{n=0}^{\infty}(\beta_2 z)^n\right) \\
&\overset{(1.7)}{=} \left(\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\beta_1^n\, z^n - \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\beta_2^n\, z^n\right) \\
&\overset{(1.6)}{=} \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\,(\beta_1^n - \beta_2^n)\; z^n,
\end{aligned}$$

where $\beta_1 = \frac{1+\sqrt{5}}{2}$ and $\beta_2 = \frac{1-\sqrt{5}}{2}$. Therefore we have

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \quad n \geq 0.$$

1.3.2 Ordinary and Exponential Generating Functions

Throughout the lecture we take $\mathbb{K} = \mathbb{C}$.

Definition 9 (Generating Functions)**.** The *ordinary generating function*, OGF, of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n.$$

The *exponential generating function*, EGF, of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!}.$$

Definition 10 (Combinatorial Class, Counting Sequence and Generating Function)**.**

- A *combinatorial class* $\mathscr{A}$ is a finite or denumerable set on which a size function is defined such that the size $|\alpha|$ of an element α is a non-negative integer and the number of elements of any given size is finite.
- An element of size 1 in a combinatorial class is an *atom*.
- Given a combinatorial class $\mathscr{A}$ we denote by $\mathscr{A}_n$ the set of elements in $\mathscr{A}$ of size n for any $n \in \mathbb{N}\cup\{0\}$.
- The *counting sequence of a combinatorial class* $\mathscr{A}$ is the sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ where a_n is the number of elements in $\mathscr{A}_n$, which is often denoted by $|\mathscr{A}_n|$ ($\mathscr{A}_n$ is also a combinatorial class).
- The *ordinary generating function* of a combinatorial class $\mathscr{A}$ is the ordinary generating function of the sequence of numbers $a_n = |\mathscr{A}_n|$. Therefore we have

  $$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}.$$

  Notation: $[z^n]A(z) := a_n$.
- The *exponential generating function of a combinatorial class* $\mathscr{A}$ is the exponential generating function of the sequence of numbers $a_n = |\mathscr{A}_n|$. Therefore we have

  $$A(z) := \sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!} = \sum_{\alpha\in\mathscr{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

  Notation: $[z^n]A(z) := \frac{a_n}{n!}$.

We say the variable *z marks the size* in the generating function $A(z)$.

Two combinatorial classes are said to be *combinatorially equivalent* if their counting sequences are identical.

Propostion 2. *For the product of two exponential generating functions we have*

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{z^n}{n!}.$$

Proof. Exercise! □

Example 15 (Triangulations of Convex Polygon)**.** *Let $\mathscr{T}$ denote the class of all triangulation of convex polygons with one distinguished edge, in which the size of a triangulation in $\mathscr{T}$ is defined as the number of triangles it is composed of. Then $\mathscr{T}$ is a combinatorial class. For $n \geq 1$ we let $\mathscr{T}_n$ denote the class of all triangulation of convex polygons of size n and let $t_n := |\mathscr{T}_n|$. Set $t_0 := 1$. That is, $\mathscr{T}_n$ is the set of all triangulations of convex $(n+2)$-gons (i.e. with n triangles) with one distinguished edge.*

The sequence $(t_n)_{n\in\mathbb{N}\cup\{0\}}$ begins with $1,1,2,5,14,42,\ldots$

By deleting the triangle incident to the distinguished edge we obtain the recursion

$$t_n = \sum_{k=0}^{n-1} t_k t_{n-1-k}, \quad n \geq 1.$$

Let $T(z) := \sum_{n=0}^{\infty} t_n z^n$ be the ordinary generating function of $\mathscr{T}$. Multiplying both sides of $t_n = \sum_{k=0}^{n-1} t_k\, t_{n-1-k}$ by z^n and summing over $n \geq 1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} t_n\, z^n &= \sum_{n=1}^{\infty} (\sum_{k=0}^{n-1} t_k\, t_{n-1-k})\, z^n \\
\iff \quad T(z) - t_0 &= z \sum_{n=1}^{\infty} (\sum_{k=0}^{n-1} t_k\, t_{n-1-k})\, z^{n-1} \\
\iff \quad T(z) - 1 &= z \sum_{n=0}^{\infty} (\sum_{k=0}^{n} t_k\, t_{n-k})\, z^n \\
T(z) - 1 &\overset{(1.8)}{=} z\, T(z)^2.
\end{aligned}$$

Therefore $T(z)$ satisfies the quadratic equation

$$z\, T(z)^2 - T(z) + 1 = 0,$$

among whose two solutions we choose

$$T(z) = \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative.

From this, we get for $n \geq 1$:

$$\begin{aligned}
t_n = [z^n]T(z) &= -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} \\
&\overset{rescaling}{=} -\frac{1}{2}(-4)^{n+1}\, [z^{n+1}](1+z)^{1/2}.
\end{aligned}$$

Using the generalised binomial theorem

$$[z^{n+1}](1+z)^{1/2} = \binom{\frac{1}{2}}{n+1} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!} = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}(n+1)!},$$

we obtain

$$\begin{aligned} t_n &= -\frac{1}{2}(-4)^{n+1}\ \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}(n+1)!} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2^n}{(n+1)!} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2 \cdot 4 \cdot 6 \cdots (2n)}{(n+1)!\, n!} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Using Stirling's formula

$$n! = \left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of triangulations of size n (i.e. of a convex $n+2$-gon)

$$t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\ \frac{(2n)!}{n!\, n!} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}.$$

Lecture 4

1.4 Symbolic method

1.4.1 Unlabelled combinatorial objects

For a given combinatorial class $\mathscr{A}$, denote by $\mathscr{A}_n$ the set of elements of size n in $\mathscr{A}$, and let $a_n = |\mathscr{A}_n|$. We say the OGF

$$A(z) = \sum_{n\geq 0} a_n z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}$$

enumerates $\mathscr{A}$.

Basic constructions and OGF's

(1) $\mathscr{E}$ is the *neutral class* that consists of a single element of size 0.
The OGF of $\mathscr{E}$ is 1.

(2) $\mathscr{Z}$ is the *atomic class* that consists of a single element of size 1.
The OGF of $\mathscr{Z}$ is z.

(3) The *combinatorial sum (disjoint union)* $\mathscr{A}+\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A}\cap\mathscr{B}=\emptyset$ is the set of objects consisting of two disjoint copies of $\mathscr{A}$ and $\mathscr{B}$, in which the size of an element $\alpha\in\mathscr{A}+\mathscr{B}$ is defined as $|\alpha|_{\mathscr{A}}$ if $\alpha\in\mathscr{A}$ and $|\alpha|_{\mathscr{B}}$ if $\alpha\in\mathscr{B}$ (i.e. the size of an element in $\alpha\in\mathscr{A}+\mathscr{B}$ is inherited from its size in its class of origin). In order to formalise $\mathscr{A}+\mathscr{B}$ we introduce red marker to $\mathscr{A}$ and blue marker to $\mathscr{B}$. The combinatorial sum $\mathscr{A}+\mathscr{B}$ is a well-defined combinatorial class. Its OGF satisfies

$$\sum_{\alpha\in\mathscr{A}+\mathscr{B}} z^{|\alpha|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|} + \sum_{\alpha\in\mathscr{B}} z^{|\alpha|}.$$

(4) *The Cartesian product* $\mathscr{A}\times\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A}\times\mathscr{B} = \{(\alpha,\beta) \mid \alpha\in\mathscr{A}, \beta\in\mathscr{B}\},$$

in which the size of a pair (α,β) is defined as $|\alpha|+|\beta|$. The Cartesian product $\mathscr{A}\times\mathscr{B}$ is a well-defined combinatorial class. Its OGF satisfies

$$\sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|\alpha|+|\beta|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|} + \sum_{\beta\in\mathscr{B}} z^{|\beta|} = A(z)\cdot B(z).$$

For any $n\geq 1$, we define $\mathscr{A}^n = \mathscr{A}^{n-1}\times\mathscr{A} = \mathscr{A}\times\cdots\times\mathscr{A}$ recursively.

(5) The *sequence* $\mathrm{SEQ}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0=\emptyset$ that consists of sequences of elements from $\mathscr{A}$ is the infinite sum

$$\mathscr{E}+\mathscr{A}+\mathscr{A}^2++\mathscr{A}^3+\ldots.$$

Note that the condition $\mathscr{A}_0 = \emptyset$ (equiv. $a_0 = 0$) guarantees the finiteness condition for sizes, and therefore the sequence construction $\mathrm{SEQ}(\mathscr{A})$ is a well-defined combinatorial class. Its OGF satisfies

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)},$$

where the latter equality is because $[z^0]A(z) = 0$.

(6) The *multiset construction* $\mathrm{MS{\scriptsize ET}}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite multisets (i.e. repetition allowed) of elements from $\mathscr{A}$, more precisely, we define

$$\mathrm{MS{\scriptsize ET}}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is the equivalence class of sequences defined by

$$(\alpha_1, \ldots, \alpha_n) \sim_{\mathrm{R}} (\beta_1, \ldots, \beta_n)$$

iff there is a permutation σ of $[1 \ldots n]$ such that for all $1 \le i \le n$, $\beta_i = \alpha_{\sigma(i)}$.

We determine its generating sequence.

<u>Case $\mathscr{A}$ finite</u>

For a finite set $\mathscr{A}$, we let $\alpha_1, \alpha_2, \ldots$ be a canonical listing of the elements of $\mathscr{A}$. Then any multiset can be sorted in such a way that it can be viewed as formed by a sequence of repeated elements of α_1, followed by a sequence of repeated elements of α_2, and so on. It follows that

$$\mathrm{MS{\scriptsize ET}}(\mathscr{A}) = \prod_{\alpha \in A} \mathrm{S{\scriptsize EQ}}(\alpha).$$

(a finite product). Therefore, the OGF of $\mathscr{C} = \mathrm{MS{\scriptsize ET}}(\mathscr{A})$ satisfies

$$C(z) = \prod_{\alpha \in \mathscr{A}} \frac{1}{1 - z^{|\alpha|}} = \prod_{n \ge 1} \left(\frac{1}{1 - z^n} \right)^{a_n},$$

where the latter equality holds because $a_0 = 0$.

Consider the formal power series

$$\exp(z) := \sum_{n \ge 0} \frac{z^n}{n!}, \quad \ln(z) := \sum_{n \ge 1} \frac{(-1)^{n-1}}{n} (z-1)^n.$$

Applying the exp-ln transformation we have

$$\begin{aligned} C(z) &= \exp\left(\ln\left(\prod_{n \ge 1} (1 - z^n)^{-a_n} \right) \right) = \exp\left(\sum_{n \ge 1} a_n \ln\left(\frac{1}{1 - z^n} \right) \right) \\ &= \exp\left(\sum_{n \ge 1} a_n \sum_{k \ge 1} \frac{(z^n)^k}{k} \right) = \exp\left(\sum_{k \ge 1} \frac{1}{k} \sum_{n \ge 1} a_n (z^k)^n \right) = \exp\left(\sum_{k \ge 1} \frac{1}{k} A(z^k) \right) \\ &= \exp\left(A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \ldots \right). \end{aligned}$$

<u>Infinite class $\mathscr{A}$</u>: The case of infinite class $\mathscr{A}$ follows by a limit argument.

(7) The *power set construction* $\mathrm{PSET}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite subsets of $\mathscr{A}$ (without repetition).

For a finite set $\mathscr{A}$, we have

$$\mathrm{PSET}(\mathscr{A}) = \prod_{\alpha \in \mathscr{A}} (\mathscr{E} + \{\alpha\}),$$

because the distributing the products in all possible ways forms all possible combinations of elements of $\mathscr{A}$ (i.e. the sets of elements from $\mathscr{A}$ without repetition). So, the OGF of $\mathscr{C} = \mathrm{PSET}(\mathscr{A})$ satisfies

$$\begin{aligned} C(z) &= \prod_{\alpha \in \mathscr{A}} (1 + z^{|\alpha|}) = \prod_{n \geq 1} (1 + z^n)^{a_n} = \exp\Big(\sum_{n \geq 1} a_n \ln(1 + z^n)\Big) \\ &= \exp\Big(\sum_{n \geq 1} a_n \sum_{k \geq 1} (-1)^{k-1} \frac{z^{nk}}{k}\Big) = \exp\Big(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} A(z^k)\Big) \\ &= \exp\left(A(z) - \frac{A(z^2)}{2} + \frac{A(z^3)}{3} \pm \ldots\right). \end{aligned}$$

(8) A further construction is the *substitution* (see Chapter 2, Example 1 for an example with substitution). Let $\mathscr{B}, C$ be combinatorial classes. Then the substitution of $\mathscr{C}$ into $\mathscr{B}$ (or the *composition of $\mathscr{B}$ and $\mathscr{C}$*), note d $\mathscr{B} \circ \mathscr{C}$ or $\mathscr{B}[\mathscr{C}]$ is defined as

$$\mathscr{B} \circ \mathscr{C} = \mathscr{B}[\mathbb{C}] = \sum_{k \geq 0} \mathscr{B}_k \times \mathrm{SEQ}_k(\mathscr{C})$$

$\mathscr{B}[\mathscr{C}]$ means "substitute elements of $\mathscr{C}$ for atoms of $\mathscr{B}$".

Applications

Example 16. *A* binary tree *is a combinatorial structure that is recursively defined such that*

- *It is either a single external node $\circ$, or*
- *it consists of an internal node (the root $\bullet$) and two binary trees attached to the root (left tree $\triangle_\ell$ and right tree $\triangle_r$),*

$$\circ \qquad \text{or} \qquad \begin{array}{c} \bullet \\ / \quad \backslash \\ \triangle_\ell \quad \triangle_r \end{array} .$$

Let $\mathscr{B}$ denote the class of all binary trees, in which the size of a binary tree is defined as the number of internal nodes. Then $\mathscr{B}$ is a combinatorial class. For $n \geq 0$ we let $\mathscr{B}_n$ denote the class of all binary trees of size n and let $b_n := |\mathscr{B}_n|$. (Note that if a binary tree has n internal nodes, then it has $n+1$ external nodes. Thus b_n counts the number of binary trees with $n+1$ external nodes.)

The sequence $(b_n)_{n\in\mathbb{N}\cup\{0\}}$ *begins with* $1,1,2,5,14,42,\ldots$ *Let* $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ *be the ordinary generating function of the combinatorial class* $\mathscr{B}$*. We have* $b_0 = 1$*, since the only (binary) tree is* $\circ$ *and for* $n \geq 1$,

$$\begin{array}{c} \bullet \\ / \quad \backslash \\ \triangle_\ell \quad \triangle_r \end{array}.$$

Thus its OGF satisfies

$$B(z) = 1 + zB(z)^2.$$

The solution for the quadratic equation

$$zB(z)^2 - B(z) + 1 = 0$$

is

$$B(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Since $b_n \geq 0$*,* $B(z)$ *increases along the real axis and therefore it increases. Thus the right solution of* $B(z)$ *is*

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

As in Example 15 we get for $n \geq 1$*:*

$$b_n = [z^n]B(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 9 (Number of binary trees)**.** *The number* b_n *of binary trees with n internal nodes (equiv.* $n+1$ *external nodes) is given by the so-called Catalan number*

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

Remark 2. Using Stirling's formula

$$n! = \left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of binary trees

$$b_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}.$$

Example 17. *Let us return to triangulations of convex polygons we saw in Example 15.*

Let $\mathscr{T}_n$ *be the set of all triangulations of convex* $(n+2)$*-gons (n triangles) with one edge distinguished. Using the basic constructions described above we have*

$$\mathscr{T} = \biguplus_{n\geq 0} \mathscr{T}_n = \sum_{n\geq 0} \mathscr{T}_n, \qquad \mathscr{T}_0 = \mathscr{E}, \quad \mathscr{T}_1 = \mathscr{Z}.$$

$\mathscr{T}_0$ contains only one edge, $\mathscr{T}_1$ one triangle. By removing one edge of the convex $(n+2)$-gon, we end up with two separate convex triangulations (sharing one node), so

$$\mathscr{T} \;=\; \mathscr{E} + \mathscr{T} \times \mathscr{Z} \times \mathscr{T}$$

$$T(z) \;=\; 1 + zT^2(z),$$

among whose two solutions we choose

$$T(z) \;=\; \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative.

As in Example 15,

$$t_n = [z^n]T(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \ldots = \frac{1}{n+1}\binom{2n}{n}.$$

This suggests a bijection between binary trees and triangulations, where each node corresponds to one triangle (or alternatively, the root node of a binary tree B corresponds to the distinguished edge of a triangulation T, and each internal node of B to a diagonal edge of T, and each external node to the external edges of T except the distinguished edge of T.

Example 18. *Let $\mathscr{S}$ be the set of binary strings with no two consecutive 0 bits; for example, $\emptyset, 0, 1, 01, 10, 11, 010, 011, \ldots$.*

$$\mathscr{S} = \mathscr{E} + \{0\} + \{1\} \times \mathscr{S} + \{01\} \times \mathscr{S}.$$

From (3) and (4) in Section 1.4.1 we get

$$S(z) = 1 + z + zS(z) + z^2 S(z)$$

and thus

$$S(z) = \frac{1+z}{1-z-z^2}.$$

Exercise 3. Let $S(z)$ be as above.

- Find the closed solution form of $s_n = |\mathscr{S}_n|$.
- Find a recurrence of s_n that leads to $S(z)$.

1.4.2 Labelled combinatorial objects

A labelled combinatorial class $\mathscr{A}$ is a combinatorial class, where each atom carries an integer label in such a way that the labels of atoms occurring in an object are distinct, and the collection of the labels of atoms occurring in an object of size n is the complete integer interval $[n]$.

Given a labelled combinatorial class $\mathscr{A}$, we consider the EGF

$$A(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!} = \sum_{\alpha \in \mathscr{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

Basic constructions and EGFs

(1) Neutral class: $\mathscr{E}$. The EGF of $\mathscr{E}$ is 1.

(2) Atomic class: $\mathscr{Z}$. The EGF of $\mathscr{Z}$ is z.

(3) Disjoint union: $\mathscr{A} + \mathscr{B}$, for $\mathscr{A}$ and $\mathscr{B}$ labelled combinatorial classes. The EGF of $\mathscr{A} + \mathscr{B}$ is $A(z) + B(z)$.

(4) Labelled product $\mathscr{A} * \mathscr{B}$ of two labelled combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A} * \mathscr{B} := \sum_{\alpha \in \mathscr{A}, \beta \in \mathscr{B}} \alpha * \beta.$$

Given $\alpha \in \mathscr{A}, \beta \in \mathscr{B}$, let $\alpha * \beta$ denote the set of all pairs (α', β') where the atoms of (α', β') get distinct labels from $[n]$ where $n = |\alpha| + |\beta|$, such that the labelling preserves the relative order of labels of α, β. There are $\binom{|\alpha|+|\beta|}{|\alpha|}$ possibilities for such labelling (choose which labels go to the first substructure, then the previous relative order determines which label belongs to which element). In other words, if $\mathscr{C} = \mathscr{A} * \mathscr{B}$, $C(z) = A(z) \cdot B(z)$.

(5) Labelled sequence $\mathrm{SEQ}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\mathrm{SEQ}(\mathscr{A}) = \mathscr{E} + \mathscr{A} + \mathscr{A} * \mathscr{A} + \ldots = \sum_{k \geq 0} \textsc{Seq}_k(\mathscr{A}),$$

where $\textsc{Seq}_k(\mathscr{A}) = \mathscr{A} * \cdots * \mathscr{A}$ is the labelled product of k copies of $\mathscr{A}$. The EGF of $\mathrm{SEQ}(\mathscr{A})$ satisfies

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)}.$$

(6) Set $\textsc{Set}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\textsc{Set}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is an equivalence relation such that $(\alpha_1,\ldots,\alpha_n) \sim_{\mathscr{R}} (\beta_1,\ldots,\beta_n)$ if there exists a permutation σ of $[n]$ such that for each $1 \le i \le n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\textsc{Set}(\mathscr{A})$ satisfies

$$1 + A(z) + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \ldots = \exp(A(z)).$$

(7) Cylces $\textsc{Cyc}(\mathscr{A})$ of a (labelled) combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

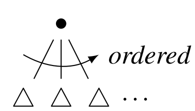
$$\textsc{Cyc}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{S},$$

where $\mathscr{S}$ is the equivalence relation such that $(\alpha_1,\ldots,\alpha_n) \sim_{\mathscr{R}} (\beta_1,\ldots,\beta_n)$ if there exists a cyclic permutation σ of $[n]$ such that for each $1 \le i \le n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\textsc{Cyc}(\mathscr{A})$ satisfies

$$1 + A(z) + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \ldots = \log\frac{1}{1 - A(z)}.$$

Recall that a tree is a connected graph without cycles.

Example 19 (Plane trees, ordered trees)**.** Plane trees *are trees embedded in the plane, so that subtrees attached to each vertex are ordered, say from left to right. There is a root vertex, implicitly defined; to a root vertex, subtrees are attached in a specified order.*



So,

$$\mathscr{P} = \mathscr{Z} * \textsc{Seq}(\mathscr{P}).$$

The generating function is then

$$P(z) = z \cdot \frac{1}{1 - P(z)}.$$

This is a quadratic equation. As previously, since $P(z)$ increases along the real axis, the negative solution is the correct one,

$$P(z) = \frac{1 - \sqrt{1 - 4z}}{2} = z \cdot B(z),$$

where $B(z)$ is the OGF of binary trees.

Let p_n be the number of plane trees on n vertices, and b_n the number of binary trees on $n+1$ external nodes, so $p_{n+1} = b_n$. $P(z) = zB(z)$ suggests that there is a combinatorial bijection between plane trees on $n+1$ vertices and binary trees on $n+1$ external nodes.

Exercise 4. Find such a bijection between plane trees and binary trees.

Example 20 (2-regular graphs)**.** *Let $\mathscr{R}$ be the set of all 2-regular labelled graphs, i.e. each vertex has exactly 2 neighbours. Note that connected 2-regular graphs are undirected cycles of length ≥ 3.*

$$\mathscr{R} = \textsc{Set}(\textit{connected 2-regular graphs}) = \textsc{Set}(\mathrm{U}\textsc{Cyc}_{\geq 3}(\mathscr{Z})),$$

where $\mathrm{U}\textsc{Cyc}$ *is the undirected cyclic construction. Then, the generating function is*

$$R(z) = \exp\left(\mathrm{U}\textsc{Cyc}_{\geq 3}(z)\right) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}},$$

since the generating function for $\mathrm{U}\textsc{Cyc}_{\geq 3}(\mathscr{Z})$ *is* $\frac{1}{2}\left(\log(\frac{1}{1-z}) - z - \frac{z^2}{2}\right)$.

How can we derive $[z^n]R(z)$?

Lecture 5

1.5 Analytic Methods

1.5.1 Analytic functions

Definition 11. Let $\Omega \subset \mathbb{C}$ be a region (i.e. open and connected). A function $f : \mathbb{C} \to \mathbb{C}$ is called *complex differentiable* or *holomorphic at* $z_0 \in \Omega$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. It is called *holomorphic in* Ω if f is holomorphic for every point in Ω.

Definition 12. Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \mathbb{C} \to \mathbb{C}$ is called *analytic at* $z_0 \in \Omega$ if $\exists\, \varepsilon > 0$ such that $\forall z \in B_\varepsilon(z_0)$, $f(z)$ is representable by a convergent power series expansion around z_0

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n, \quad \text{for some } c_n \in \mathbb{C}.$$

It is called *analytic in* Ω if f is analytic for every point in Ω.

Theorem 10. *Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in Ω iff it is analytic in Ω.*

Definition 13. Let $\Omega \subset \mathbb{C}$ be a region and $f : \mathbb{C} \to \mathbb{C}$ be holomorphic at $z_0 \in \Omega$. Set $f^{(0)}(z_0) := f(z_0)$. Then the power series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is called the *Taylor series expansion of $f(z)$ around z_0*.

Exercise 5. Find the maximum regions, in which the functions $z \mapsto e^z$, $z \mapsto \frac{1}{1-z}$, $z \mapsto (1+z)^m$ (for fixed $m \in \mathbb{N}$) are analytic. Show that their Taylor series expansions around the origin are given by

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}, \qquad \frac{1}{1-z} = \sum_{n \geq 0} z^n, \qquad (1+z)^m = \sum_{n \geq 0} \binom{m}{n} z^n.$$

Exercise 6. Find the Taylor series expansions of

$$\frac{1}{\sqrt{1-4z}}, \quad \sin z, \quad \cos z, \quad z \cdot e^z, \quad \ln z, \quad \frac{1}{1-z} \ln \frac{1}{1-z}, \quad \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}$$

around the origin and find the maximum regions where these power series converge.

Exercise 7. Prove

$$\sum_{n\geq m}\binom{n}{m}z^n = \frac{z^m}{(1-z)^{m+1}}.$$

Exercise 8. Prove

$$\sum_{n\geq 0} H_n z^n = \frac{1}{1-z}\ln\frac{1}{1-z}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the n-th *harmonic number*.

Exercise 9. Find OGF for each of the following sequences:

$$\{2^{n+1}\}_{n\geq 0}, \qquad \{n2^{n+1}\}_{n\geq 0}, \qquad \{nH_n\}_{n\geq 1}, \qquad \{n^3\}_{n\geq 2}.$$

1.5.2 Cauchy's coefficient formula

Theorem 11 (Cauchy's coefficient formula, 1st version)**.** *Let $\Omega \subset \mathbb{C}$ be a region, f be analytic in Ω and γ be a simple loop in Ω. Then*

$$\int_\gamma f = 0.$$

Theorem 12 (Cauchy's coefficient formula, 2nd version)**.** *Let $\Omega \subset \mathbb{C}$ be a region and f be analytic in Ω. Let $z_0 \in \Omega$ and γ be a simple loop encircling z_0 such that the image of γ is contained in Ω. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

In general, for $n \geq 0$

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

1.5.3 Lagrange Inversion Theorem

Theorem 13 (Langrange Inversion, 1st version)**.** *Let $\phi(u) = \sum_{k\geq 0}\phi_k u^k$ be a power series in $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then the equation*

$$A = z\phi(A)$$

admits a unique solution in $\mathbb{C}[[z]]$ and the coefficients of

$$A(z) = \sum_{n\geq 0} a_n z^n$$

are given by

$$a_n = \frac{1}{n}[u^{n-1}]\phi(u)^n.$$

Furthermore, for any holomorphic function H,

$$[z^n]H(A(z)) = \frac{1}{n}[u^{n-1}]\left(H'(u)\phi(u)^n\right).$$

In particular, for $k \geq 1$,

$$[z^n]A(z)^k = \frac{k}{n}[u^{n-k}]\phi(u)^n.$$

Theorem 14 (Langrange Inversion, 2nd version)**.** *Let the generating function*

$$A(z) = \sum_{n \geq 0} a_n z^n$$

satisfy the functional equation

$$z = \psi(A(z)),$$

where $\psi(0) = 0$ but $\psi'(0) \neq 0$ (i.e. ψ is the compositional inverse of A).

If

$$\psi(A) = \frac{A}{\phi(A)},$$

i.e. $A = \psi(A) \cdot \phi(A) = z\phi(A)$, then

$$a_n = \frac{1}{n}[z^{n-1}]\phi(z)^n = \frac{1}{n}[z^{n-1}]\left(\frac{z}{\psi(z)}\right)^n.$$

Application of Lagrange Inversion to Binary Trees

Let $\tilde{\mathscr{B}}$ denote the class of all binary trees, in which the size of a binary tree is defined as the total number of all vertices (internal nodes and leaves alike). For $n \geq 1$ we let $\tilde{\mathscr{B}}_n$ denote the class of all binary trees of size n and let $\tilde{b}_n := |\tilde{\mathscr{B}}_n|$ and set $\tilde{b}_0 = 0$.

Let $\tilde{B}(z)$ be the ordinary generating function of the combinatorial class $\tilde{\mathscr{B}}$. Then

$$\tilde{B}(z) \;=\; z \,+\, z\,\tilde{B}(z)^2 \tag{1.10}$$

Let $\tilde{B} = \tilde{B}(z)$ and $\phi(u) := 1 + u^2$. Then (1.10) can be rewritten as

$$\tilde{B} \;=\; z\,\phi(\tilde{B}) \tag{1.11}$$

Note that $\phi(u) := 1 + u^2 = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.11) admits a unique solution $\tilde{B}(z) := \sum_{n=0}^{\infty} \tilde{b}_n\, z^n$ in the ring $\mathbb{C}[[z]]$ where the coefficients are given by

$$\begin{aligned}
\tilde{b}_n \;&=\; \frac{1}{n}\,[u^{n-1}]\;\phi(u)^n \\
&=\; \frac{1}{n}\,[u^{n-1}]\;(1+u^2)^n \\
&=\; \frac{1}{n}\,[u^{n-1}]\;\left(\sum_{k=0}^{n}\binom{n}{k}\,u^{2k}\right) \\
&=\; \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n}\,\binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

In other words, we have, for any $n \in \mathbb{N}$,

$$\tilde{b}_{2n+1} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

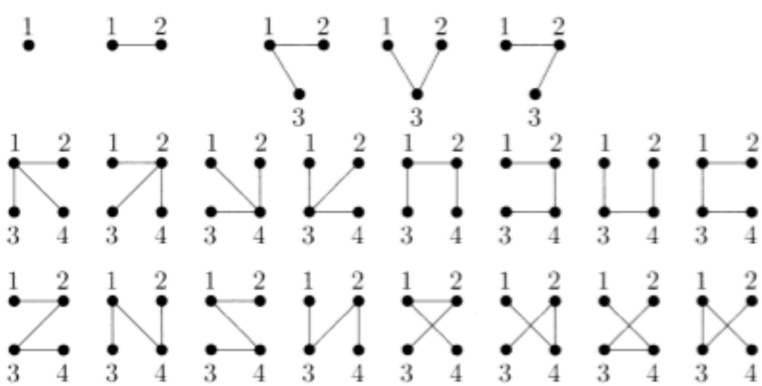
Remark 3. Recall that b_n was defined in Theorem 9 as the number of binary trees of size n, in which the size of a binary tree is defined as the number of internal nodes. If a binary tree has n internal nodes, then it has $n+1$ external nodes and therefore $2n+1$ vertices in total. Thus we have $\tilde{b}_{2n+1} = b_n$.

Application of Lagrange Inversion to Labelled Trees

Definition 14.

- A *labelled tree* (also called *Cayley tree*) is a tree whose vertices are distinctly labelled by numbers in $\mathbb{N}$. The vertices of a Cayley tree on n vertices are labelled by distinct numbers from $[n]$.
- A *rooted labelled tree* on n vertices is a labelled tree on n vertices, in which one vertex is distinguished by a mark from the other vertices.

Let $\mathscr{C}$ denote the class of all labelled trees, in which the size of a tree is defined as the number of vertices, and let $\mathscr{C}_n$ denote the set set of all Cayley tree on n vertices. Then $\mathscr{C}$ is a (labelled) combinatorial class. ($\mathscr{C}_n$ is also a (labelled) class, it is a *subclass* of $\mathscr{C}$). For $n \geq 1$ we let $c_n := |\mathscr{C}_n|$ and set $c_0 = 0$. Let $C(z) := \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial class $\mathscr{C}$. The sequence $(c_n)_{n\geq 0}$ begins with $0, 1, 1, 3, 16, 125\ldots$

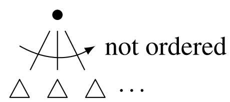


Theorem 15 (Cayley's formula (Cayley 1889))**.**

$$c_n = n^{n-2}, \quad n \geq 2.$$

Proof. Let $\mathscr{T}$ denote the (labelled combinatorial) class of all rooted labelled trees (i.e. the set of all Cayley trees in $\mathscr{C}$, in which one vertex is distinguished by a mark from the other vertices) and let $\mathscr{T}_n$ denote the class of all rooted labelled tree on n vertices. For $n \geq 1$ we let $t_n := |\mathscr{T}_n|$ and set $t_0 = 0$. Because there are n ways to choose a root vertex of a labelled tree on n vertices, the number of Cayley trees on n vertices is equal to $t_n = n\, c_n$.

Let $T(z) := \sum_{n=0}^{\infty} t_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial labelled class $\mathscr{T}$.



$$\mathscr{T} = \mathscr{Z} * \textsc{Set}(\mathscr{T}).$$

Its EGF

$$T(z) \;=\; z\, \exp(T(z)). \tag{1.12}$$

So $T(z)$ is implicitly defined.

Let $T = T(z)$ and $\phi(u) := \exp(u) = \sum_{k\geq 0} \frac{u^k}{k!}$. Then (1.12) can be rewritten as

$$T \;=\; z\, \phi(T) \tag{1.13}$$

Note that $\phi(u) := \exp(u) = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.13) admits a unique solution in the ring $\mathbb{C}[[u]]$ whose coefficients are given by $T(z) = \sum_{n=0}^{\infty} t_n\, z^n$, where

$$\begin{aligned} t_n \;&=\; \frac{1}{n}\,[u^{n-1}]\;\phi(u)^n \;=\; \frac{1}{n}\,[u^{n-1}]\;(\exp(u))^n \;=\; \frac{1}{n}\,[u^{n-1}]\;\exp(un) \\ &=\; \frac{1}{n}\,[u^{n-1}]\left(\sum_{k=0}^{\infty}\frac{(un)^k}{k!}\right) \;=\; \frac{1}{n}\;\frac{n^{n-1}}{(n-1)!} \;=\; \frac{n^{n-1}}{n!}. \end{aligned}$$

In other words, for any $n \in \mathbb{N}$,

$$t_n = n^{n-1}.$$

Therefore, we have

$$c_n = \frac{t_n}{n} = n^{n-2}.$$

□

Remark 4. There are several interesting proofs for this in The Book by M. Aigner and G. Ziegler. We shall see one proof in Section 2.3.2 (Chapter 2).

Lecture 6

1.5.4 Singularities

Definition 15. Let f be an analytic function in a region Ω and z_0 be a point on the boundary of Ω.

We say f is *analytically continuable at* z_0 if there is an analytic function g defined in a region Ω' containing z_0 such that $g(z) = f(z)$ in $\Omega \cap \Omega'$.

Example 21. $f(z) = \frac{1}{1-z}$ *is analytic for* $|z| < 1$, *and is analytically continuable except for* $z_0 = 1$.

Definition 16. A function f is said to be *singular at* $z_0 \in \overline{\Omega}$ or z_0 is called *a singularity of* f if f is not analytically continuable at z_0.

Theorem 16 (Boundary singularities)**.** *Let f be analytic at the origin and let R be the finite radius of convergence of a power series expansion of f at the origin. Then f has necessarily a singularity on the boundary of the disc of convergence.*

Definition 17. If f is analytic at the origin with radius R of convergence then a *dominant singularity* is a singularity on the circle around 0 with radius $R > 0$.

Theorem 17 (Pringsheim's Theorem)**.** *Let f be analytic at the origin and $f(z) = \sum_{n\geq 0} f_n z^n$ be its convergence power series expansion at the origin with the radius of convergence $R > 0$. If $f_n \geq 0, \forall n \geq 0$, then the point $z = R$ is a singularity of f; in this case, $z = R$ is a dominant singularity of f.*

The singularity $z = R$ of Theorem 17 is often called *the* dominant singularity of f.

Remark 5. The radius R of convergence of $\sum_{n\geq 0} f_n z^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} |f_n|^{\frac{1}{n}}},$$

so we have

$$f_n = R^{-n}\theta(n),$$

where

$$\limsup_{n\to\infty} |\theta(n)|^{\frac{1}{n}} = 1.$$

1.5.5 Meromorphic functions

Definition 18. A function $f(z)$ is *meromorphic at* z_0 if $\forall z$ in a neighbourhood $B_\varepsilon(z_0)$ of z_0 with $z \neq z_0$, it can be represented as

$$f(z) = \frac{h(z)}{g(z)},$$

where $h(z)$ and $g(z)$ are analytic at z_0 and $g(z) \not\equiv 0$. In this case $f(z)$ admits an expansion of the form

$$f(z) = \sum_{n \geq -M} f_n (z - z_0)^n$$

for $z \in B_\varepsilon(z_0)$, for some $M \in \mathbb{Z}$. If $f_{-M} \neq 0$, $M \geq 1$, we say $f(z)$ has a *pole of order M at z_0*. In this case, we have

$$\begin{aligned} f(z) &= f_{-M}(z-z_0)^{-M} + O((z-z_0)^{-M+1}) \\ &= f_{-M}(-z_0)^{-M}\left(1-\frac{z}{z_0}\right)^{-M} + O\left(\left(1-\frac{z}{z_0}\right)^{-M+1}\right). \end{aligned}$$

Remark 6. Scaling rule: If $f(z)$ analytic around 0 and singular at z_0, then $g(z) = f(z_0 z)$ is singular at 1. If $z_0 > 0$ is a dominant singularity of $f(z)$, then $f(z)$ admits a convergent series expansion (around 0):

$$f(z) = \sum_{n \geq 0} f_n z^n, \quad |z| < z_0.$$

Therefore, $g(z)$ admits a convergent series expansion

$$g(z) = \sum_{n \geq 0} g_n z^n, \quad |z| < 1, \quad g_n = f_n z_0^n.$$

1.5.6 Newton's generalised binomial theorem

For any $\alpha \in \mathbb{C}$,

$$[z^n](1-z)^{-\alpha} = \binom{-\alpha}{n} \overset{?}{=} \binom{n+\alpha-1}{\alpha-1} \overset{?}{=} \Theta(n^{\alpha-1}).$$

Theorem 18. *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.*

$$[z^n](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right),$$

where the Γ-function is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, \mathrm{d}t.$$

Properties of the Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \, \mathrm{d}t = \sqrt{\pi}.$$

$$\begin{cases} \Gamma(t+1) = t\,\Gamma(t) \\ \Gamma(1) = 1. \end{cases}$$

For $n \in \mathbb{N}$,

$$\Gamma(n+1) = n\Gamma(n) = \ldots = n!\,.$$

1.5.7 Transfer theorem

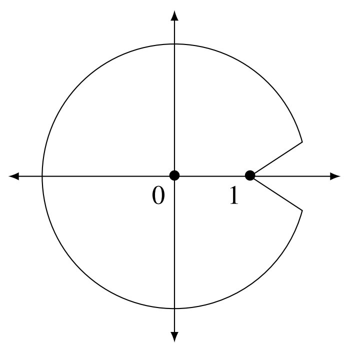
Transfer theorems allow to extract the asymptotit order of coefficients of error terms in singular expansions. For details see chapter VI in [2].

Definition 19. Given two numbers R, ϕ with $R > 1$ and $0 < \phi < \frac{\pi}{2}$, then the set

$$\Delta = \Delta(R, \phi) = \{z \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi\}$$

is called Δ-domain at 1.

A function f is called Δ-analytic at 1 if it is analytic at some Δ-domain at 1.



Theorem 19 (Transfer theorem)**.** *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and assume that $f(z)$ is Δ-analytic at 1.*

If $f(z) = O\left((1-z)^{-\alpha}\right)$, then

$$[z^n]f(z) = O(n^{\alpha-1}).$$

If $f(z) = o\left((1-z)^{-\alpha}\right)$, then

$$[z^n]f(z) = o\left(n^{\alpha-1}\right).$$

Therefore, if $f(z) = (1-z)^{-\alpha} + o\left((1-z)^{-\alpha}\right)$, then

$$[z^n]f(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right) + o(n^{\alpha-1}).$$

Example 22 (2-regular graphs)**.** *Consider the EGF of 2-regular graphs,*

$$\mathscr{G} = \textsc{Set}(\mathrm{U}\textsc{Cyc}_{\geq 3}\mathscr{Z}).$$

Then

$$G(z) = \exp\left(\frac{1}{2}\left(\log\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2}\right)\right) = \frac{1}{\sqrt{1-z}}e^{-\frac{z}{2}-\frac{z^2}{4}}.$$

$\frac{1}{\sqrt{1-z}}$ *is analytic in $\mathbb{C} \setminus [1, \infty)$. $G(z)$ is in particular Δ-analytic at 1.*

To derive the singular expansion of G (an expansion of the singularity of G) near 1, we use the Taylor expansion of $e^{-\frac{z}{2}-\frac{z^2}{4}}$ *at 1.*

$$\begin{aligned} h(z) = e^{-\frac{z}{2}-\frac{z^2}{4}} &= h(1) + h'(1)(z-1) + \frac{h''(1)}{2}(z-1)^2 + \dots \\ &= e^{-\frac{3}{4}} + e^{-\frac{3}{4}}(1-z) + e^{-\frac{3}{4}}(z-1)^2 + \dots \end{aligned}$$

The singular expansion of $G(z)$ *near 1 is*

$$\begin{aligned} G(z) &= \frac{e^{-\frac{3}{4}}}{\sqrt{1-z}} + e^{-\frac{3}{4}}(1-z)^{\frac{1}{2}} + e^{-\frac{3}{4}}(1-z)^{\frac{3}{2}} + \dots \\ &= e^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}} + o\left((1-z)^{-\frac{1}{2}}\right). \qquad (z \to 1) \end{aligned}$$

Using the transfer theorem, we get

$$[z^n]G(z) = e^{-\frac{3}{4}} \frac{n^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{n}\right)\right) + o\left(n^{-\frac{1}{2}}\right) = \frac{e^{-\frac{3}{4}}}{\sqrt{n\pi}}\left(1 + o(1)\right).$$

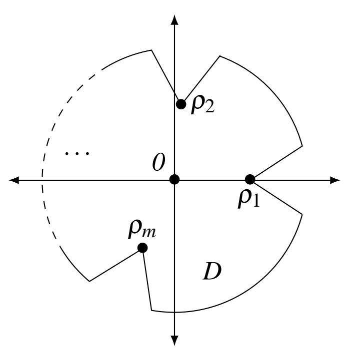
The first term corresponds to $h(1)$*, the second is from the theorem.*

1.5.8 Multiple singularities

Theorem 20. *Let* $f(z)$ *be analytic in* $|z| < \rho$*. Suppose that* f *has a finite number of singularities on the circle* $|z| = \rho$*, at points* $\rho_j = \rho \cdot e^{i\theta_j}$ *for* $j = 1, \dots, m$*. Assume there exists a* Δ*-domain* Δ_0 *at 1 such that* $f(z)$ *is analytic in the region*

$$D = \bigcap_{j=1}^{m} \rho_j \Delta_0,$$

where $\rho_j\Delta_0$ *is the dilation (rotation) of the* Δ*-domain* Δ_0 *for each* ρ_j*.*



Let

$$S = \{(1-z)^{-\alpha} \mid \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}\}.$$

Assume there exist m functions $g_1(z),\dots,g_m(z)$, each a linear combination of the functions in S, and there exists a function $h(z)=(1-z)^{-\alpha_0}$ such that

$$f(z)=g_j\left(\frac{z}{\rho_j}\right)+O\left(h\left(\frac{z}{\rho_j}\right)\right)\qquad \text{as } z\to\rho_j \text{ in } D.$$

Then

$$\begin{aligned}[z^n]f(z)&=\sum_{j=1}^{m}\left([z^n]g_j\left(\frac{z}{\rho_j}\right)+O\left([z^n]h\left(\frac{z}{\rho_j}\right)\right)\right)\\&=\sum_{j=1}^{m}\rho_j^{-n}[z^n]g_j(z)+O\left(\rho_j^{-n}n^{\alpha_0-1}\right).\end{aligned}$$

Remark 7.

$$[z^n]g_j(z)=[z^n]\sum_{k=1}^{\ell}c_{jk}(1-z)^{-\alpha_{j,k}}=\sum_{k=1}^{\ell}c_{jk}\frac{n^{\alpha_{j,k}-1}}{\Gamma(\alpha_{j,k})}.$$

Remark 8 (Recipe for singularity analysis)**.**

- Find the grammar that determines the combinatorial class and the corresponding generating function
- Locate the singularities
- Check the Δ-analyticity of the generating function
- Do the singular expansion at each singularity separately
- Extract and sum the asymptotics using the Transfer Theorem and basic scaling.

Example 23. *Let $\mathscr{P}$ be the class of permutations with cycles of odd length. Then*

$$\begin{aligned}\mathscr{P}&=\textsc{Set}(\mathscr{A}),\\ \mathscr{A}(\mathscr{Z})&=\textsc{Cyc}_{odd}(\mathscr{Z}).\end{aligned}$$

For the generating functions, this means

$$\begin{aligned}A(z)&=\sum_{k=odd}\frac{z^k}{k}=\frac{1}{2}\log\frac{1+z}{1-z},\\ P(z)&=e^{A(z)}=\exp\left(\frac{1}{2}\log\frac{1+z}{1-z}\right)=\sqrt{\frac{1+z}{1-z}}.\end{aligned}$$

The dominating singularities of this function are $\{\pm1\}$. Let Δ_0 be a Δ-domain at 1.

$P(z)$ is analytic in the region $D=\Delta_0\cap(-1)\Delta_0$ since $\sqrt{1-z}$ is analytic in $\mathbb{C}\setminus[1,\infty)$ and $\sqrt{1+z}$ is analytic in $\mathbb{C}\setminus(-\infty,-1]$.

Singular expansions, with $g(z)=\sqrt{1+z}$ *and writing* $P^{+}(z)$ *for* $P(z)$ *as* $z\to+1$ *in* D*,* $P^{-}(z)$ *for* $P(z)$ *as* $z\to-1$ *in* D*:*

$$\begin{aligned}
P^{+}(z) &= \frac{g(1)}{\sqrt{1-z}}+\frac{g'(1)(z-1)}{\sqrt{1-z}}+\dots && \text{(as } z\to+1 \text{ in } D)\\
&= \frac{\sqrt{2}}{\sqrt{1-z}}-2^{-\frac{3}{2}}\sqrt{1-z}+O\left((1-z)^{\frac{3}{2}}\right)\\
P^{-}(z) &= \frac{1}{\sqrt{2}}\sqrt{1+z}+O\left((1+z)^{\frac{3}{2}}\right). && \text{(as } z\to-1 \text{ in } D)
\end{aligned}$$

Extract asymptotics:

$$\begin{aligned}
[z^n]P(z) = \frac{2^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}n^{-\frac{1}{2}}-\frac{2^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)}n^{-\frac{3}{2}}+O\left(n^{-\frac{5}{2}}\right)\\
+\frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}n^{-\frac{3}{2}}+O\left(n^{-\frac{5}{2}}\right)
\end{aligned}$$

where $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(-\frac{1}{2})=-2\sqrt{\pi}$.

Chapter 2

Graph Theory

Lecture 7

Suggested reading for Chapter 2: [1] and [4].

2.1 Matchings, Eulerian Tours, and Hamiltorian Cycles

2.1.1 Basic Terminologies

Notation 1 (Subsets)**.** *Given a set X and $k \in \mathbb{N}$, we denote by $\binom{X}{k}$ the set of all k-element subsets of X.*

Definition 20 (Graphs)**.** A graph is a pair (V,E) where $E \subseteq \binom{V}{2}$.
The elements of V are called *vertices (or nodes, points)* and the elements of E are called *edges*. Two vertices v,w are called *adjacent or neighbours* if $\{v,w\} \in E$. A vertex v is said to be *incident* with an edge $e \in E$ if $v \in e$. We call v an *end* of e.

Note that all graphs considered will be simple (by the above definition) and finite (finitely many vertices, thus finitely many edges).

If G is a graph we will write $V(G)$ to denote the set of vertices of G and $E(G)$ to denote the set of edges of G.

Definition 21 (Walk, Paths, Cycles, Distance)**.** Let $G = (V,E)$ be a graph.

(1) A *walk* in G is an alternating sequence

$$W = v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k$$

of vertices and edges of G such that $e_i = \{v_i, v_{i+1}\}$ for each $0 \leq i \leq k-1$. It is said to be *closed* if $v_0 = v_k$.

(2) A walk $v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k$ in G is called *a path* if $v_0, v_1, \ldots, v_k$ are all distinct. We write $P = v_0 v_1 v_2 \ldots v_{k-1} v_k$ and call it a v_0-v_k path. The *length of P* is k, i.e. the number of its edges.

(3) A cycle in G is an alternating sequence $C = v_0 e_0 v_1 e_1 v_2 \ldots v_{k-1} e_{k-1} v_k e_k v_0$ of vertices and edges of G such that $v_0, v_1, \ldots, v_k$ are distinct, $e_i = \{v_i, v_{i+1}\}$ for $0 \leq i \leq k-1$, and $e_k = \{v_k, v_0\}$.

(4) The *distance of two vertices v,w* in G is the length of a shortest v-w path (a shortest path starting at v and ending in w) in G.

Definition 22 (Subgraphs, induced subgraphs, spanning subgraphs)**.** Given two graphs $G = (V,E)$ and $G' = (V',E')$, we set

$$\begin{aligned} G \cup G' &:= (V \cup V', E \cup E') \\ G \cap G' &:= (V \cap V', E \cap E'). \end{aligned}$$

(1) If $G \cap G = \emptyset$, then we say that G and G' are *disjoint*.

(2) If $V \subset V'$ and $E \subset E'$, then G' is *a subgraph* of G. We write $G' \subseteq G$.

(3) If $G' \subset G$ and $G' \neq G$, then G' is *a proper subgraph* of G. We write $G' \subsetneq G$.

(4) If $G' \subseteq G$ and G' contains all edges $\{v,w\} \in E$ with $v,w \in V'$, then G' is called *an induced subgraph* of G. We write $G' = G[V']$.

(5) If $G' \subseteq G$ and $V = V'$, then G' is called *a spanning subgraph* of G.

Definition 23 (Components, connectedness)**.** Let $G = (V,E)$ be a graph.

(1) G is called *connected* if G contains a subgraph which is isomorphic to a path from v to w for any two vertices $v,w \in V$.

(2) A maximal connected subgraph of G is called *a component* of G.

(3) For $k \in \mathbb{N}$, G is called k-connected if $|V| > k$ and $G[V \setminus X]$ is connected for every set $X \subseteq V$ with $|X| < k$.

Definition 24 (Degrees)**.** Let $G = (V,E)$ be a graph.

(1) For $v \in V$,
$$d(v) \;:=\; |\,\{w \in V \;:\; \{v,w\} \in E\}\,|$$
is called the *degree* of the vertex v. A vertex of degree 0 is called *isolated*.

(2) The number
$$\delta(G) \;:=\; \min|\,\{d(v) \;:\; v \in V\}\,|$$
is called *minimum degree* of G, the number
$$\Delta(G) \;:=\; \max|\,\{d(v) \;:\; v \in V\}\,|$$
is called *maximum degree* of G, and the number
$$d(G) \;:=\; \frac{1}{|V|}\sum_{v \in V} d(v)$$
is called *average degree* of G.

(3) If all the vertices have the same degree d, then G is called d*-regular*.

A 3-regular graph is also called *cubic*.

Remark 9. Clearly we have
$$|V|d(G) = \sum_{v \in V} d(v) = 2|E|$$
$$\delta(G) \quad \leq \quad d(G) \quad \leq \quad \Delta(G)$$

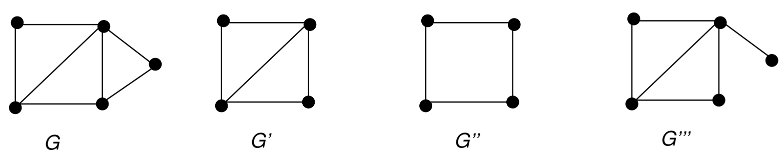
Definition 25 (Graph isomorphism)**.** We say that two graphs $G = (V,E)$ and $G' = (V',E')$ are *isomorphic* and write $G \simeq G'$, if there is a bijection $\phi : V \to V'$ with

$$\{v,w\} \in E \quad \Leftrightarrow \quad \{\phi(v), \phi(w)\} \in E', \forall v,w \in V.$$

Definition 26 (Graph property)**.** A class of graphs that is closed under isomorphism is called a graph property.

An example of a graph property is the class of graphs containing a triangle. A graph property only depends on the abstract structure, not on a representation of a graph (like a drawing).

Example 24.



(1) Determine whether G', G'', G''' are induced subgraphs of G or not

(2) Determine whether G', G'', G''' are spanning subgraphs of G or not

2.1.2 Matchings

Definition 27 (Independent set, matchings, vertex-cover)**.** Let $G = (V,E)$ be a graph.

(1) A set of vertices or a set of edges of G is called *independent* if no two of its elements are adjacent. An independent set is also called a *stable set*.

(2) A set M of independent edges in G is called a *matching* of G. In other words, a matching is a set of pairwise non-adjacent edges.

A vertex is called *matched (or saturated)* if it is an endpoint of an edge in the matching.

(3) A d-regular spanning subgraph of G is called a d-factor of G.

A 1-factor (indeed, its edge set) is called a *perfect matching*.

(4) A set $U \subseteq V$ is called a *vertex cover* of G if every edge of G is incident with a vertex in U.

Definition 28 (Independence number, clique number)**.** Let $G = (V,E)$ be a graph.

(1) The *independence number of* G, denoted by, $\alpha(G)$, is the number of vertices in a maximum independent set (of vertices) in G.

(2) The *clique number of* G, denoted by $\omega(G)$, is the number of vertices in a maximum clique (=complete graph) in G.

Example 25. *In Example 24,* $\alpha(G) =$?? *and* $\omega(G) =$??.

Remark 10 (Perfect matching)**.** Let $H \subseteq G$ be a subgraph of $G = (V, E)$.

$H \subseteq G$ is a 1-factor
$\iff$ $E(H)$ is a perfect matching of G
$\iff$ Every vertex in G is incident to exactly one edge of H

Definition 29 (Bipartite graph)**.** A graph $G = (V, E)$ is called *bipartite* if V admits a partition into two sets, $V = A\dot{\cup}B$, such that every edge $e \in E$ has its ends in different classes.

Lemma 21. *A graph $G = (V, E)$ is bipartite if and only if it contains no odd cycle*

Proof. Exercise! □

Problem 1 (Maximum bipartite matching)**.**

Input: A bipartite graph $G = (A\dot{\cup}B, E)$
Output: A matching in G with as many edges as possible

Definition 30 (Alternating paths, augmenting paths)**.** Let $G = (A\dot{\cup}B, E)$ be a bipartite graph and M be an arbitrary matching in G.

(1) A path which begins with an unmatched vertex and contains alternatingly edges from $E \setminus M$ and from M is called an *alternating path* (with respect to M).

(2) An alternating path that ends also at an unmatched vertex is called an *augmenting path*.

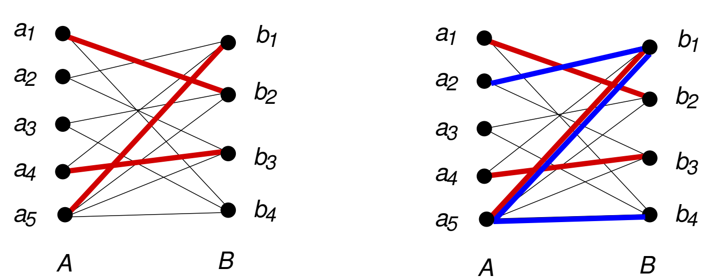
Remark 11. An augmenting path P can be used to turn M into a larger matching

$$M \triangle E(P) := (M \setminus E(P)) \cup (E(P) \setminus M)$$

It has one edge more than M, whose ends are the ends of P.

Example 26.

Let $G = (A\dot{\cup}B, E)$ be a bipartite graph given below.



- $M = \{\{a_1, b_2\}, \{a_4, b_3\}, \{a_5, b_1\}\}$ is a matching
- $P = a_2\ \{a_2, b_1\}\ b_1\ \{b_1, a_5\}\ a_5\ \{a_5, b_4\}\ b_4$ is an augmenting path (with respect to M)
- $M \triangle E(P) = \{\{a_1, b_2\}, \{a_4, b_3\}, \{a_2, b_1\}, \{a_5, b_4\}\}$ is also a matching

Exercise 1. *Let M be a matching in a bipartite graph $G = (A \dot{\cup} B, E)$.*

- *Show that if M contains fewer edges than some other matching in G (i.e. M is suboptimal), then G contains an augmenting path with respect to M.*
- *Describe an algorithm that finds as efficiently as possible a matching of maximum cardinality.*

Theorem 22 (König 1931)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph.*

The number of edges in a maximum matching in G equals the number of vertices in a minimum vertex cover of G.

Proof. Exercise! □

Theorem 23 (Hall 1935)**.** *Let $G = (A \dot{\cup} B, E)$ be a bipartite graph.*

G contains a matching that saturates all vertices in A if and only if

$$|N(S)| \geq |S|, \quad \forall S \subseteq A$$

where $N(S) = \{v \in B : \{u, v\} \in E \text{ for some } u \in S\}$ is the neighborhood of S in B.

Proof. Exercise! □

2.1.3 Eulerian Tours and Hamiltorian Cycles

Definition 31 (Eulerian tour)**.** A closed walk in a graph is called *Eulerian tour* if it traverses (= visits) every edge of the graph exactly once. A graph is called *Eulerian* if it contains an Eulerian tour.

Theorem 1 (Euler 1736)**.** *A connected graph is Eulerian if and only if every vertex has even degree.*

Proof. Exercise! □

Definition 32 (Hamiltonian cycle)**.** A *Hamiltonian cycle* in a graph G is a closed walk that contains every vertex in G exactly once.

If $|V(G)| \geq 3$, any such a walk forms a cycle. Therefore, such a walk is called a Hamiltonian *cycle*.

Problem 2.
When does a graph contain a Hamiltonian cycle?

Theorem 2 (Dirac 1952)**.** *Every graph $G = (V, E)$ with $|V| \geq 3$ and minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamiltonian cycle.*

Proof. Exercise! (cf. Ore's theorem) □

Definition 33 (Graph power)**.**

Given a graph $G = (V, E)$ and $d \in \mathbb{N}$, we denote by G^d (e.g. G^2 is called the square of G) the graph with vertex set V in which two vertices are adjacent if they have distance at most d in G.

Theorem 3 (Fleischner 1974)**.** *If G is a 2-connected graph, then G^2 has a Hamiltonian cycle.*

Remark 12. Problem 2 is still studied in the contemporary research.

Lecture 8

2.2 Graph Decompositions

2.2.1 Connected Graphs

Remark 13. To define connectedness, we can equivalently define a relation $\sim$ on the vertices of $G = (V, E)$ by letting $v \sim w$ if and only if there is a walk from v to w. Then $\sim$ is an equivalence relation, and induces a partition of V into equivalent classes $\{V_1, V_2, \dots, V_l\}$ with $V = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_l$. The induced subgraphs on each equivalence class $V_i, 1 \le i \le l$ are called the *components* of G.

Remark 14 (Decomposition of a graph into components)**.** Let $\mathscr{G}$ denote the set of all graphs in which the size is defined as the number of vertices and $\mathscr{C}$ denote the set of all connected graphs. Both $\mathscr{G}$ and $\mathscr{C}$ are combinatorial classes. Let $G(z)$ and $C(z)$ be the corresponding (exponential) generating functions defined as

$$\begin{aligned} G(z) &:= \sum |\mathscr{G}_n| \frac{z^n}{n!} \\ C(z) &:= \sum |\mathscr{C}_n| \frac{z^n}{n!}. \end{aligned}$$

Note that every graph is the union of its components. This can be formulated as

$$\mathscr{G} = \mathrm{Set}(\mathscr{C}) \Leftrightarrow G(z) = \exp(C(z)).$$

2.2.2 2-Connected Graphs

Definition 34 (Separator, cut vertex, bridge, block)**.** Let $G = (V, E)$ be a graph.

(1) We say a set $X \subseteq V$ *separates* G if $G[V \setminus X]$ is disconnected.

(2) A vertex separating two other vertices in the same component (i.e. increasing the number of connected components) is called a *cut vertex* (or *articulation point*).

(3) An edge separating its ends (i.e. increasing the number of connected components) is called a *bridge*.

(4) A maximal connected subgraph without a cut vertex is called a *block*.

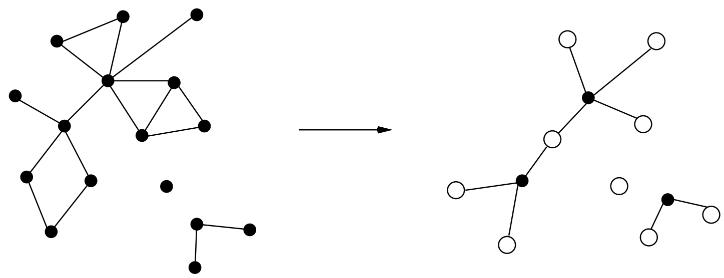
Remark 15.
(1) What does a block look like? Every block is either

- a maximal 2-connected subgraph,
- a bridge (together with its ends), or
- an isolated vertex.

Conversely, every such graph is a block.

(2) By maximality, different blocks of G overlap in at most one vertex, which is a cut vertex of G, and every edge of G lies in a unique block.

Definition 35 (Block graph)**.** Given a graph G, let A be the set of cut vertices in G and B be the set of its blocks. Then there is a natural bipartite graph on the vertex set $A \cup B$, in which there is an edge $\{a,b\}$ for each $b \in B$ and $a \in A \cap V(b)$ (where $V(b)$ is the set of vertices of the block b). This graph is called a *block graph* of G.



Remark 16. The block graph of a connected graph is a bipartite tree.

Example 1 (Combinatorial class of biconnected graphs)**.** *Remark 16 says "every connected graph can be decomposed into 2-connected graphs which are arranged in a tree-like way". This can be made rigorous in terms of compositions of combinatorial classes.*

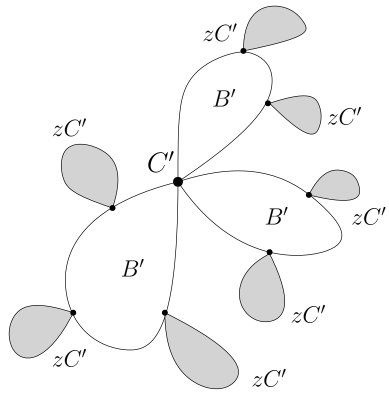
Let $\mathscr{B}_n$ denote the set of all 2-connected graphs on vertex set $[n] := \{1,\dots,n\}$ and $B(z) := \sum b_n \frac{z^n}{n!}$ where $b_n := |\mathscr{B}_n|, n \geq 3$. For an isolated vertex and a bridge we let

$$b_1 = 1$$
$$b_2 = 1.$$

Let $\tilde{\mathscr{B}}$ be the set of all 2-connected graphs with one distinguished vertex and analogously define $\tilde{C}$. Then

$$\tilde{b}_n = n \cdot b_n \quad \Longleftrightarrow \quad \tilde{B}(z) = zB'(z)$$

and the same holds for $\tilde{C}$ as well. Now, given a connected graph, consider a distinguished (cut)vertex $\mathscr{Z}$ and study how the blocks are arranged around this vertex: it is a union of blocks. Note that this particular vertex is contained in every block, so $\mathscr{Z}$ is taken out to avoid multiple counting. Each block consists of vertices and further blocks, i.e. a connected graph.



$$\tilde{\mathscr{C}} = \mathscr{X} + \mathrm{Set}\left(\tilde{\mathscr{B}}\left(\tilde{\mathscr{C}}\right) - \mathscr{X}\right)$$
$$zC'(z) = z \cdot \exp\left(B'\left(zC'(z)\right)\right)$$

Leting $f(z) = zC'(z)$ we have

$$f(z) = z \cdot \exp(B'(f(z)).$$

(Recall that $\tilde{\mathscr{B}}(\tilde{\mathscr{C}})$ is substitution). See the book "Random Trees" (Chapter 9.2) by Michael Drmota for this and related topics.

2.2.3 3-Connected Graphs

Definition 36 (Subdivision, topological minor, minor)**.** Let $G = (V, E)$ and $G' = (V', E')$ be graphs.

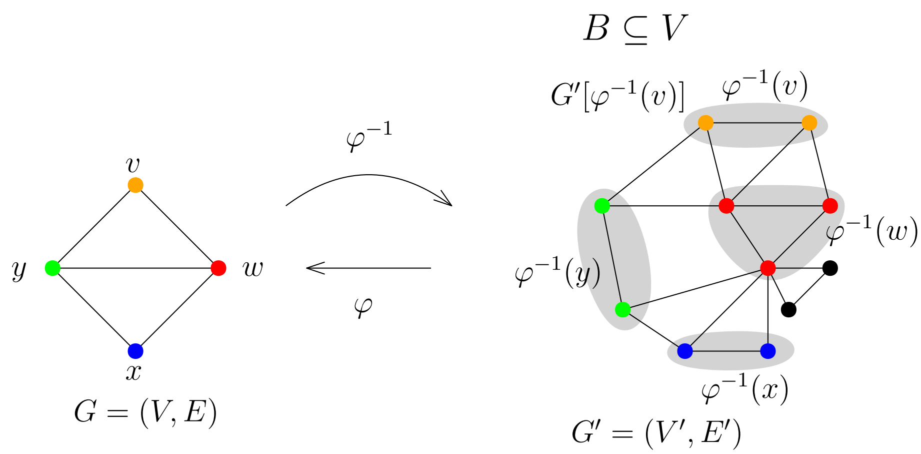
(1) G' is a *subdivision of* G if G' is obtained from G by inserting new vertices on some or all of its edges.

(2) G is a *topological minor of* G' if a subdivision of G is isomorphic to a subgraph of G'.

(3) G is a *minor of* G' if there is a map $\phi : B \subseteq V' \to V$ such that

- for every vertex $v \in V$, the subgraph induced of its pre-image $G'\left[\varphi^{-1}(v)\right]$ is connected in G' and
- for every edge $\{u, w\} \in E$ there is an edge in G' between $\varphi^{-1}(v)$ and $\varphi^{-1}(w)$.

In words: a graph G is a minor of G' if it can be obtained from G' by contracting edges in G'.



In particular, if $G'\left[\varphi^{-1}(v)\right]$ consists of a single edge $e' \in E'$ and every other pre-image $G'\left[\varphi^{-1}(w)\right]$ contains just a single vertex, then we say G arises from G' by contracting the edge e' and denote it by $G = G'/e'$.

Theorem 4 (Tutte 1961)**.** *A graph G is 3-connected if and only if there is a sequence $G_0 = K_4, G_1, \ldots, G_n = G$ of graphs such that G_{i+1} has an edge $e = \{v, w\}$ with $d(v)$, $d(w) \geq 3$ and $G_i = G_{i+1}/e, \quad \forall\, 0 \leq i \leq n-1$.*

2.3 Planar Graphs and Trees

2.3.1 Planar and Plane Graphs

Definition 37 (Planar graph)**.**

(1) A graph is called *planar* if it can be embedded/drawn in the plane without crossing edges.

(2) A planar graph with a given planar embedding is called *plane graph*. A face of a plane graph is an area bounded by vertices and edges (the outside is also a face).

Example 2. *K_5 and $K_{3,3}$ are not planar and neither are topological minors of them.*

Theorem 5 (Kuratowski 1930, Wagner 1937)**.**

- *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a topological minor.*
- *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

Theorem 6 (Euler characteristics)**.** *Let $G = (V, E)$ be a connected plane graph then the number of faces equals $|E| - |V| + 2$.*

Proof. Induction □

Remark 17 (Number of faces)**.** Note that the number of faces is independent of the plane embedding.

Corollary 24.

- *A plane graph with $n \geq 3$ vertices hast at most $3n - 6$ edges.*
- *Every plane triangulation with $n \geq 3$ vertices (i.e. every face is bounded by a triangle) has exactly $3n - 6$ edges.*

Proof. Exercise!
Use double counting argument (each face contains (at most) 3 edges, on the other hand each edge contains exactly 2 faces) and then use Euler characteristics. □

Lecture 9

2.3.2 Trees

Definition 38 (Forest, tree, leaf)**.**

(1) An acyclic graph is called a forest.

(2) A connected forest is called a tree.

(3) The vertices of degree one in a tree are called leaves.

Remark 18. A graph is acyclic if and only if it does not contain a K_3 as a minor.

Lemma 25. *Every tree with at least two vertices (at least one edge) contains at least two leaves.*

Proof. Exercise! □

Lemma 26 (Tree growing lemma)**.** *Let $G = (V,E)$ be a graph and $d(v) = 1$ for some $v \in V$. Then G is a tree if and only if $G[V \setminus \{v\}]$ is a tree.*

Proof. Exercise! □

Theorem 7 (Cayley's formula)**.** *Let $n \geq 2$ be an integer and let $T(K_n)$ denote the number of all spanning trees of the complete graph K_n. Then $T(K_n) = n^{n-2}$.*

For an illustration see the pictures for Theorem 15 in Section 1.5.

Proof (Prüfer Code). Let $\mathscr{T}_n$ be the set of all spanning trees of K_n and let

$$\mathscr{S}_{n-2} := \{(s_1, s_2, \dots, s_{n-2}) : 1 \leq s_i \leq n\} = [n]^{n-2}$$

be the set of all strings of length $n-2$ over the alphabet $[n]$. A bijection (known as Prüfer code)

$$\begin{aligned} f : \mathscr{T}_n &\to \mathscr{S}_{n-2} \\ T &\mapsto f(T) := s = (s_1, s_2, \dots, s_{n-2}) \end{aligned}$$

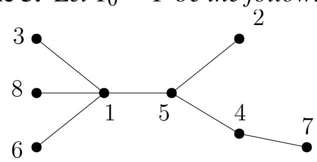
is defined successively as follows: We start with $T_0 = T$. Suppose the tree T_{i-1} has already been constructed for $i \geq 1$. We

- prune the smallest (labelled) leaf in T_{i-1};
- remove it from T_{i-1} together with the edge incident to it;
- call the resulting graph T_i;
- define the i-th term s_i of the sequence s as the *neighbour* of the leaf that is removed from T_{i-1}.

Doing this successively for $i = 1, 2, \dots, n-2$ we obtain the sequence $s = (s_1, s_2, \dots, s_{n-2})$.

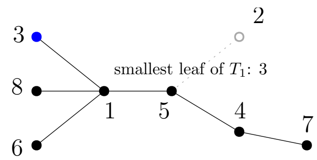
Example 3. *Let $T_0 = T$ be the following three with 8 vertices:*

T_0



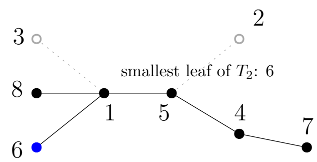
5 *(smallest of the leaves of T_0: 2)*

T_1



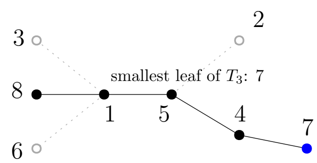
$s_1 = 5$, *neighbour of* 2

T_2



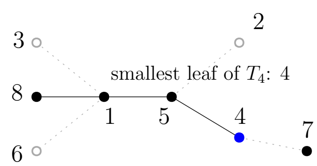
$s_2 = 1$, *neighbour of* 3

T_3



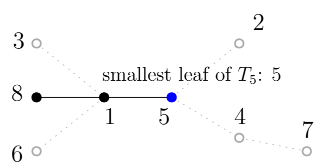
$s_3 = 1$, *neighbour of* 6

T_4



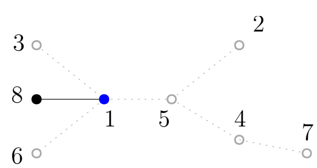
$s_4 = 4$, *neighbour of* 7

T_5



$s_5 = 5$, *neighbour of* 4

T_6



$s_6 = 1$, *neighbour of* 5

$s = (5, 1, 1, 4, 5, 1)$.

Conversely, we shall show how to construct the original tree $T = f^{-1}(s)$ from the sequence $s = (s_1, \ldots, s_{n-2})$ (that arose from some $T \in \mathscr{T}_n$).
Let $s = (s_1, \ldots, s_{n-2}) \in [n]^{n-2}$ be given. Let l_1 denote the leaf of T that was removed first. Then l_1 cannot occur anywhere in s and any vertex v that is not contained in s has to be a leaf of the tree T_0. For otherwise we would remove a leaf w adjacent to v at some step of "removal" process and therefore it would appear in s at that moment. According to the removal rule, l_1 must be

$$l_1 = \min\left(\{1, 2, \ldots, n\} \setminus \{s_1, s_2, \ldots, s_{n-2}\}\right)$$

Now draw the vertices l_1 and s_1 and connect them by an edge. We shall proceed this process successively as follows: Suppose we have reconstructed the leaves $l_1, \ldots, l_{i-1}$ removed from $T_0 = T$. To determine the leaf l_i we observe as before that $l_i \notin \{s_i, s_{i+1}, \ldots, s_{n-2}\}$ and also $l_i \notin \{l_1, \ldots, l_{i-1}\}$. Therefore,

$$l_i = \min\left([n] \setminus \{s_i, s_{i+1}, \ldots, s_{n-2}, l_1, \ldots, l_{i-1}\}\right).$$

If l_i and s_i have not been drawn yet, then draw them. Connect l_i and s_i by an edge e_i. After $n-2$ steps, we have reconstructed $n-2$ edges $e_1, e_2, \ldots, e_{n-2}$ of T (those edges that were removed in order to construct the sequence s). It remains to deduce the last edge e_{n-1}. One of its end must be s_{n-2}. The other end is

$$l_{n-1} = [n] \setminus \{l_1, \ldots, l_{n-2}, s_{n-2}\}.$$

This produces a subgraph G of K_n with $e_1, \ldots, e_{n-1}$ edges.

To complete the proof it remains to show that firstly G is a tree, secondly the Prüfer code "encoding" applied to G yields the original sequence s, and thirdly that f is surjective.

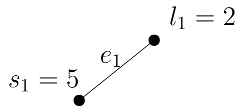
Let $E_i = \{e_i, e_{i+1}, \ldots, e_{n-1}\}$ for $1 \le i \le n-1$ and $G_i = ([n], E_i)$. To show that G is a tree, note that none of the edges $e_{i+1}, \ldots, e_{n-1}$ can be incident to l_i and therefore l_i is a leaf of G_i and by Lemma 26, G_i is a tree (up to isolated vertices) for $1 \le i \le n-1$. In particular, $G_1 = G$ is a tree.

To show that the encoding applied to G yields the original sequence s, it suffices to verify that l_i is the smallest leaf of G_i for $1 \le i \le n-2$. Observe that by definition of $l_i = \min\left([n] \setminus \{s_i, s_{i+1}, \ldots, s_{n-2}, l_1, \ldots, l_{i-1}\}\right)$, a smaller leaf of G_i can occur among $l_1, \ldots, l_{i-1}$ or $s_i, s_{i+1}, \ldots, s_{n-2}$. Because $l_1, l_2, \ldots, l_{i-1}$ are isolated vertices in G_i, the first case does not occur. So consider s_k for some $i \le k \le n-2$. In G_k, s_k is a neighbour of l_k and has another neighbour, so it is not a leaf of G_k. Because $G_k \subset G_i$, s_k is not a leaf of G_i. Therefore, none of $s_i, s_{i+1}, \ldots, s_{n-2}$ is a leaf of G_i.

For the surjectivity of $f : \mathscr{T}_n \to \mathscr{S}_{n-2}$ note that the above algorithm associates a tree to every string $(s_1, \ldots, s_{n-2})$ in $\mathscr{S}_{n-2}$. $\square$

Example (Example 3 cont.)**.** $S = (s_1, s_2, \ldots, s_6) = (5, 1, 1, 4, 5, 1)$. We show how to get a tree from this. (Try the same on an arbitrary word, say in $\mathscr{S}_5$ to illustrate the surjectivity of f)

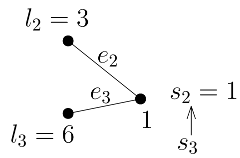
Step 1



$$\begin{aligned} l_1 &= \text{min of } [8] \setminus \{1,4,5\} \\ &= \min\{1,3,6,7,8\} \\ &= 2 \end{aligned}$$

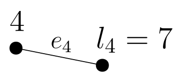
Step 2

Step 3



$$\begin{aligned} l_2 &= \text{min of } [n] \setminus \{s_2, \ldots, s_6, l_1\} \\ &= \min\{3,6,7,8\} \\ &= 3 \\ l_3 &= \text{min of } [n] \setminus \{s_3, \ldots, s_6, l_1, l_2\} \\ &= \min\{6,7,8\} \\ &= 6 \end{aligned}$$

Step 4

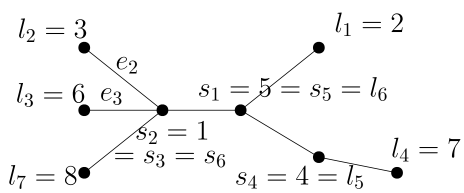


$$\begin{aligned} l_4 &= \text{min of } [n] \setminus \{s_4, \ldots, s_6, l_1, l_2, l_3\} \\ &= \min\{7,8\} \\ &= 7 \end{aligned}$$

Step 5

Step 6

Step 7



$$\begin{aligned} l_5 &= \text{min of } [n] \setminus \{s_5, s_6, l_1, l_2, l_3, l_4\} \\ &= \min\{4,8\} \\ &= 4 \\ l_6 &= \text{min of } [n] \setminus \{s_6, l_1, \ldots, l_5\} \\ &= \min\{5,8\} = 5 \\ l_7 &= \text{min of } [n] \setminus \{\ldots\} = 8 \end{aligned}$$

Chapter 3

Stochastic Aspects – Random Graphs

Lecture 10

Suggested reading for Chapter 3: [3] and [4].

3.1 Basics

Lemma 27 (Union bound)**.**

For a finite or countable set of events $A_1, A_2, \ldots$,

$$\mathbb{P}\left[\bigcup A_i\right] \leq \sum \mathbb{P}[A_i].$$

In words, "the probability that at least on of the events $A_1, A_2, \ldots$ happens is no greater than the sume of the probabilities of individual events".

Proof. Exercise!

- For the finite case induction on n
- For the countable case σ-subadditivity of (probabilistic) measure

□

Lemma 28 (Markov's inequality)**.**

For any non-negative random variable X and any constant $t > 0$,

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Proof. Exercise! □

Remark 19 (First moment method)**.** For a non-negative integral random variable X and $t = 1$, Markov's inequality implies that

$$\mathbb{P}[X \neq 0] = \mathbb{P}[X \geq 1] \leq \mathbb{E}[X].$$

In particular, if $\mathbb{E}[X] \to 0$, then $\mathbb{P}[X \neq 0] \to 0$. Therefore it is useful to prove the non-existence of certain events.

Remark 20 (Variant of Markov's inequality)**.**

Markov's inequality in terms of deviation from the expectation:

$$\mathbb{P}[X \geq t\mathbb{E}[X]] \leq \frac{1}{t}$$

Definition 39 (Variance and covariance)**.** The *variance* of a random variable X is defined as

$$\mathrm{Var}[X] := \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X,Y] := \mathbb{E}\left[(X-\mathbb{E}[X])\cdot(Y-\mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Lemma 29 (Chebyshev's inequality)**.** *For any random variable X and any constant $t>0$,*

$$\mathbb{P}\left[\,|X-\mathbb{E}[X]| \geq t\,\right] \leq \frac{\mathrm{Var}[X]}{t^2}.$$

Proof. Apply Markov's inequality! □

Remark 21 (Second moment method)**.** Let X be a random variable with $\mathbb{E}[X]>0$. Then Chebyshev's inequality implies

$$\mathbb{P}[X=0] \leq \mathbb{P}\left[|X-\mathbb{E}[X]| \geq \mathbb{E}[X]\right] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} = \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]^2} - 1.$$

Therefore, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, equivalently $\mathbb{E}[X^2] = (1+o(1))\mathbb{E}[X]^2$, then $\mathbb{P}[X=0] \to 0$, in other words, $\mathbb{P}[X>0] \to 1$. Therefore it is useful to prove the existence of certain events.

Remark 22 (Application of Chebyshev's inequality)**.** Lemma 29 implies that for any $\varepsilon > 0$,

$$\mathbb{P}\left[\,|X-\mathbb{E}[X]| \geq \varepsilon\mathbb{E}[X]\,\right] \leq \frac{\mathrm{Var}[X]}{\varepsilon^2\mathbb{E}[X]^2}.$$

Thus, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, then $\mathbb{P}\left[\,|X-\mathbb{E}[X]| \geq \varepsilon\mathbb{E}[X]\,\right] \to 0$, equivalently, if $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$, then $\mathbb{P}\left[\,|X-\mathbb{E}[X]| \leq \varepsilon\mathbb{E}[X]\,\right] \to 1$.

Let $X = X_1 + X_2 + \ldots$ be a sum of indicator random variables[1]. Then

$$\begin{aligned}
\mathrm{Var}[X] = \mathbb{E}\left[(X-\mathbb{E}[X])^2\right] &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \mathbb{E}\left[\left(\sum_i X_i\right)\left(\sum_i X_i\right)\right] - \mathbb{E}\left[\sum_i X_i\right]\mathbb{E}\left[\sum_i X_i\right] \\
&= \sum_i \left(\mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2\right) + \sum_{i\neq j}\left(\mathbb{E}\left[X_iX_j\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right]\right) \\
&= \sum_i \mathrm{Var}\left[X_i\right] + \sum_{i\neq j}\mathrm{Cov}\left[X_i, X_j\right]
\end{aligned}$$

Since the X_i's are indicator variables, we have $\mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[X_i\right]$, and therefore

$$\begin{aligned}
\mathrm{Var}\left[X_i\right] = \mathbb{E}\left[X_i^2\right] - \mathbb{E}[X_i]^2 \leq \mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[X_i\right] \\
\sum_i \mathrm{Var}\left[X_i\right] \leq \sum_i \mathbb{E}\left[X_i\right] = \mathbb{E}\left[\sum_i X_i\right] = \mathbb{E}[X]
\end{aligned}$$

[1] A random variable associated with the occurence of an event; X_A has value 1 if A occurs and 0 otherwise.

and

$$\mathrm{Cov}\,[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \leq \mathbb{E}[X_i X_j].$$

Note that if X_i and X_j are independent, then

$$\mathrm{Cov}\,[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] = 0.$$

Hence,

$$\mathrm{Var}[X] = \sum\nolimits_i \mathrm{Var}\,[X_i] + \sum\nolimits_{i \neq j} \mathrm{Cov}\,[X_i, X_j] \leq \mathbb{E}[X] + \sum\nolimits_{i \neq j,\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j]. \quad (3.1)$$

Remark 23 (Sum of indicator random variables)**.** Let $X = X_1 + X_2 + \ldots$ be a sum of indicator random variables with $\mathbb{E}[X] > 0$. Lemma 29 and (3.1) imply that

$$\mathbb{P}\,[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{i \neq j} \mathbb{E}[X_i X_j]}{\mathbb{E}[X]^2}$$

and Remark 22 and (3.1) imply that

$$\mathbb{P}\,[\,|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]\,] \leq \frac{\mathrm{Var}[X]}{\varepsilon^2 \mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{i \neq j\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j]}{\varepsilon^2 \mathbb{E}[X]^2}.$$

Therefore, if $\mathbb{E}[X] \to \infty$ and $\sum_{i \neq j\ X_i, X_j \text{ dependent}} \mathbb{E}[X_i X_j] = o(\mathbb{E}[X]^2)$, then

$$\mathbb{P}\,[X \geq 1] \to 1 \quad \text{and} \quad \mathbb{P}\,[\,|X - \mathbb{E}[X]| \leq \varepsilon \mathbb{E}[X]\,] \to 1.$$

(Cf. Remark 22).

Definition 40 (Erdős–Rényi random graph)**.** Let $p \in [0,1]$ be a constant independent of n or let $p = p\,(n) \in [0,1]$ be a function in n.

The *binomial random graph* $G\,(n,p)$, also known as the *Erdős–Rényi random graph*, is a graph with vertex set $[n] := \{1, 2, \ldots, n\}$, in which each pair of vertices is joined by an edge with probability p, independently of each other. Equivalently, given $p \in [0,1]$, let Ω be the set of all graphs with vertex set $[n]$ and define, for each $G \in \Omega$,

$$\mathbb{P}\,[G] := p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|},$$

where $E\,(G)$ is the edge set of G.

Example 4 (Expected number of edges in $G\,(n,p)$)**.**

Let $G = G(n,p)$. Let X be the number of edges in G. This can be written as the sum of indicator random variables

$$X := |E(G)| = \sum X_{\{u,v\}},$$

where

$$X_{\{u,v\}} = \begin{cases} 1, & \{u,v\} \in E(G) \\ 0, & \{u,v\} \notin E(G) \end{cases}$$

Using the linearity of expectation we obtain

$$\mathbb{E}[X] = \sum \mathbb{E}\left[X_{\{u,v\}}\right] = \sum \mathbb{P}\left[\{u,v\} \in E(G)\right] = \sum p = \binom{n}{2} p.$$

Therefore,

$$\mathbb{E}[X] = \binom{n}{2} p \rightarrow \begin{cases} 0, & p \ll n^{-2} \\ \infty, & p \gg n^{-2} \end{cases}$$

By Markov's inequality (cf. Remark 19), if $p \ll n^{-2}$, then

$$\mathbb{P}[X \neq 0] = \mathbb{P}[X \geq 1] \leq \mathbb{E}[X] \rightarrow 0.$$

3.2 Ramsey Number and First Moment Method

Definition 41 (Ramsey number)**.** The Ramsey number $R(k,l)$ is defined as the smallest integer n such that any graph G on n vertices contains either a clique of order k or an independent set of order l, i.e.

$$R(k,l) := \min\{n \in \mathbb{N} : \forall\, G \text{ on } n \text{ vertices } \omega(G) \geq k \vee \alpha(G) \geq l\}.$$

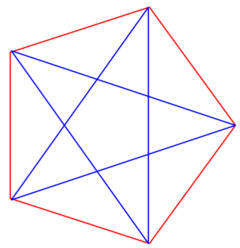
Equivalently, $R(k,l)$ is the smallest integer n such that in any two-coloring of the edges of the complete graph K_n by red and blue, there exists a either a red K_k or a blue K_l.

Remark 24 (Ramsey number)**.**

$$\begin{array}{ll} (0) & R(k,1) = R(1,k) = 1 \\ (1) & R(k,l) = R(l,k) \\ (2) & R(k,2) = R(2,k) = k \\ (3) & R(k,l) \leq R(k-1,l) + R(k,l-1) \\ (4) & R(k,l) \leq \binom{k+l-2}{k-1} \end{array}$$

Proof. Exercise! □

Example. $R(3,3)=6$: one can show that $R(3,3)\le 6$ (exercise). To see that $R(3,3)>5$, consider the colouring of K_5 where all the outer edges are of one colour, all the crossing edges of the inner star of the other colour:



Theorem 8 (Erdős–Szekeres 1935)**.**

$$R(k,k) \;\le\; (1+o(1))\,\frac{4^{k-1}}{\sqrt{\pi k}}$$

Proof. Using Remark 24 (4) and applying Stirling's formula we have

$$\begin{aligned}
R(k,k) \le \binom{2k-2}{k-1} &= \frac{(2k-2)!}{((k-1)!)^2}\\
&= \frac{\left(1+O\left(\frac{1}{2(k-1)}\right)\right)\sqrt{4\pi(k-1)}\left(\frac{2(k-1)}{e}\right)^{2(k-1)}}{\left(\left(1+O\left(\frac{1}{k-1}\right)\right)\sqrt{2\pi(k-1)}\left(\frac{k-1}{e}\right)^{k-1}\right)^2}\\
&= (1+o(1))\,\frac{2^{2(k-1)}}{\sqrt{\pi(k-1)}}\\
&= (1+o(1))\,\frac{4^{k-1}}{\sqrt{\pi k}}
\end{aligned}$$

□

Theorem 9 (Erdős 1947)**.**

$$R(k,k) \ge (1+o(1))\,\frac{k}{e\sqrt{2}}\sqrt{2}^k$$

Proof. We shall apply the first moment method.

Consider a random graph $G=G\left(n,\frac{1}{2}\right)$. For each subset $S\subseteq[n]$ of size k,

$$\mathbb{P}[S \text{ forms a clique}] = \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}}.$$

So, using the union bound (Lemma 27) we have

$$\begin{aligned}
\mathbb{P}[\omega(G) \geq k] &= \mathbb{P}[G \text{ contains a clique of size } k] \\
&= \mathbb{P}\left[\bigcup\nolimits_{S \in \binom{[n]}{k}} [S \text{ forms a clique}]\right] \\
&\leq \sum\nolimits_{S \in \binom{[n]}{k}} \mathbb{P}[S \text{ forms a clique}] \\
&= \binom{n}{k} 2^{-\binom{k}{2}}.
\end{aligned}$$

Similarly, for each subset $S \subseteq [n]$ of size k,

$$\mathbb{P}[S \text{ is an independent set}] = \left(1 - \frac{1}{2}\right)^{\binom{k}{2}} = 2^{-\binom{k}{2}},$$

and thus we have

$$\begin{aligned}
\mathbb{P}[\alpha(G) \geq k] &= \mathbb{P}\left[\bigcup\nolimits_{S \in \binom{[n]}{k}} [S \text{ is an independent set}]\right] \\
&\leq \sum\nolimits_{S \in \binom{[n]}{k}} \mathbb{P}[S \text{ is an independent set}] \\
&= \binom{n}{k} 2^{-\binom{k}{2}}.
\end{aligned}$$

Summing up, we have

$$\mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] \leq \mathbb{P}[\omega(G) \geq k] + \mathbb{P}[\alpha(G) \geq k] \leq \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Therefore, if

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1,$$

then

$$\mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] < 1.$$

In other words,

$$\mathbb{P}[\omega(G) < k \wedge \alpha(G) < k] = 1 - \mathbb{P}[\omega(G) \geq k \vee \alpha(G) \geq k] > 0.$$

This means that there *exists* a graph G on n vertices with

$$\omega(G) < k \wedge \alpha(G) < k.$$

This implies

$$R(k,k) > n.$$

To find the asymptotically largest n such that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, we observe the following using the binomial coefficient approximation:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

and so,

$$\binom{n}{k}2^{1-\binom{k}{2}} \leq \left(\frac{en}{k}\right)^k \cdot 2 \cdot 2^{-\frac{k(k-1)}{2}} = \left(\frac{en}{k} \cdot 2^{\frac{1}{k}} \cdot 2^{-\frac{k-1}{2}}\right)^k = \left(n \cdot \frac{e\sqrt{2}}{k\sqrt{2}^k} 2^{\frac{1}{k}}\right)^k \overset{!}{=} 1$$

if

$$n = \left(\frac{e\sqrt{2}}{k\sqrt{2}^k} \cdot 2^{\frac{1}{k}}\right)^{-1} = \frac{k\sqrt{2}^k}{e\sqrt{2}} \underbrace{2^{-\frac{1}{k}}}_{\to 1 \text{ as } k\to\infty} = (1+o(1))\,\frac{k}{e\sqrt{2}}\sqrt{2}^k.$$

□

Lecture 11

3.3 Independence Number and Deletion Method

Theorem 10 (Weak Turán Theorem)**.** *For any graph G with vertex set $[n]$,*

$$\alpha(G) \geq \frac{n}{2d(G)},$$

where $d(G)$ is the average degree of G.

Proof. We shall apply the so-called *Deletion Method*.

Let $S \subset [n]$ be a random set defined by

$$\mathbb{P}[v \in S] := \frac{1}{d(G)},$$

the events $v \in S$ being mutually independent.

Let $X = |S|$ and Y be the number of edges in the induced subgraph $G[S]$. For each $\{u,v\} \in E(G)$, let

$$Y_{\{u,v\}} = \begin{cases} 1, & \{u,v\} \in S \\ 0, & \{u,v\} \notin S \end{cases}$$

Obviously we have

$$Y = \sum_{\{u,v\} \in E(G)} Y_{\{u,v\}}$$

and using the linearity of expectation we have

$$\mathbb{E}[Y] = \sum_{\{u,v\} \in E(G)} \mathbb{E}[Y_{\{u,v\}}] = |E(G)| \left(\frac{1}{d(G)}\right)^2 = \frac{nd(G)}{2d(G)^2} = \frac{n}{2d(G)},$$

where the penultimate equality follows from $d(G) := \frac{1}{n} \sum_{v \in [n]} d(v) = \frac{2|E(G)|}{n}$. On the other hand, we have

$$\mathbb{E}[X] = \mathbb{E}[|S|] = \frac{n}{d(G)}.$$

By the linearity of expectation we have

$$\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] = \frac{n}{d(G)} - \frac{n}{2d(G)} = \frac{n}{2d(G)}. \tag{3.2}$$

This implies that there exists a set $S_0 \subset [n]$, for which the number of vertices in the induced subgraph $G[S_0]$ minus the number of edges in $G[S_0]$ satisfies

$$|V(G[S]) - E(G[S])| \geq \frac{n}{2d(G)},$$

for otherwise (i.e. if for all $S \subset [n]$, $|V(G[S]) - E(G[S]) < n/2d(G))$

$$\mathbb{E}[X - Y] < \frac{n}{2d(G)},$$

which contradicts (3.2).

Now select one vertex from each edge of $G[S_0]$ and delete it from S_0. This results in a set S^* which contains at least $\frac{n}{2d(G)}$ vertices and no edges, in other words, S^* is an independent set of size at least $\frac{n}{2d(G)}$, which implies that $\alpha(G) \geq \frac{n}{2d(G)}$ as desired. □

3.4 Subgraphs and Second Moment Method

Definition 42 (Monotone family)**.**

Let Γ be the set of all 2-element subsets of $[n]$ and 2^Γ be the set of all subsets of Γ. A family $\mathscr{F} \subseteq 2^\Gamma$ (i.e. a family of subsets of Γ) is called *increasing* if

$$A \subseteq B \wedge A \in \mathscr{F} \quad \Rightarrow \quad B \in \mathscr{F}.$$

It is called *decreasing* if the family of complements in Γ is increasing. It is called *monotone* if it is increasing or decreasing.

Remark 25 (Monotone property)**.** We identify a property of subsets of Γ with the corresponding family of all subsets having that property.

In case of graph properties, a property is increasing (resp. decreasing) if adding (resp. deleting) edges does not violate the property.

Example 5 (Monotone properties)**.** *(1) Being connected (increasing property)*

(2) Containing a triangle (increasing property)

(3) Having an isolated vertex (decreasing property)

(4) Being planar (decreasing property)

Theorem 11 (Increasing property)**.** *For any increasing property $\mathscr{F} \subseteq 2^\Gamma$ and $0 \leq p_1 \leq p_2 \leq 1$,*

$$\mathbb{P}[G(n, p_1) \in \mathscr{F}] \leq \mathbb{P}[G(n, p_2) \in \mathscr{F}].$$

Proof. Define $p_0 \in [0, 1]$ by

$$p_1 + (1 - p_1)p_0 = p_2.$$

Then

$$G(n, p_0) \cup G(n, p_1) = G(n, p_2)$$

(”two-round exposure”). Because $\mathscr{F}$ is increasing, we have

$$G(n,p_1) \in \mathscr{F} \quad \Rightarrow \quad G(n,p_0) \cup G(n,p_1) \in \mathscr{F}$$

and therefore

$$\mathbb{P}[G(n,p_1) \in \mathscr{F}] \leq \mathbb{P}[G(n,p_0) \cup G(n,p_1) \in \mathscr{F}] \leq \mathbb{P}[G(n,p_2) \in \mathscr{F}].$$

□

Exercise 2. *Is it true that for any decreasing property $\mathscr{F} \subseteq 2^\Gamma$ and $0 \leq p_1 \leq p_2 \leq 1$,*

$$\mathbb{P}[G(n,p_1) \in \mathscr{F}] \geq \mathbb{P}[G(n,p_2) \in \mathscr{F}]?$$

Definition 43 (Threshold)**.**

For a monotone property $\mathscr{F} \subseteq 2^\Gamma$, a sequence $(f(n))_{n\geq 1}$ is a threshold if

$$\mathbb{P}[G(n,p) \in \mathscr{F}] \to \begin{cases} 0, & \text{if } p \ll f(n) \\ 1, & \text{if } p \gg f(n) \end{cases} \quad \text{as} \quad n \to \infty$$

Note that a threshold is a family of functions (defined up to constant).

Theorem 12 (Bollobás-Thomason 1987)**.** *Every monotone property has a threshold.*

Theorem 13 (Triangle threshold)**.**

$$\mathbb{P}[G(n,p) \text{ contains a triangle}] \to \begin{cases} 0, & \textit{if } p \ll \frac{1}{n} \\ 1, & \textit{if } p \gg \frac{1}{n} \end{cases}$$

In other words, $f(n) = \frac{1}{n}$ is a threshold for the property of containing a triangle.

Proof. Let X denote the number of triangles contained in $G(n,p)$. Define indicator variables

$$X_S = \begin{cases} 1, & \text{if } S \text{ forms a triangle} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$X = \textstyle\sum_{S \in \binom{[n]}{3}} X_S$$

and

$$\mathbb{E}[X] = \sum_{S \in \binom{[n]}{3}} \mathbb{E}[X_S] = \sum_{S \in \binom{[n]}{3}} \mathbb{P}[S \text{ forms a triangle}] = \binom{n}{3} p^3 = \Theta\left(n^3 p^3\right).$$

For the first statement of the theorem, we apply Markov's inequality (the first moment method) to obtain

$$\mathbb{P}\left[G(n,p)\,\text{contains a triangle}\right] \;=\; \mathbb{P}[X \geq 1] \;\leq\; \mathbb{E}[X] \to 0, \quad \text{if } p \ll \frac{1}{n}.$$

In order to prove the second part we observe that

$$\mathbb{E}[X] = \Theta\left(n^3 p^3\right) \to \infty, \quad \text{if } p \gg \frac{1}{n}.$$

But does it imply what we want? Not necessarily! We shall use the second moment method to prove

$$P[X=0] \to 0, \quad \text{if } p \gg \frac{1}{n}.$$

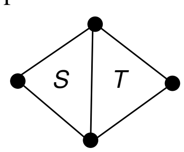
Assume that $p \gg \frac{1}{n}$ (p fixed). Then $\mathbb{E}[X] \to \infty$ and by Chebyshev's inequality, more precisely Remark 23, we have

$$\mathbb{P}[X=0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \leq \frac{\mathbb{E}[X] + \sum_{S \neq T\ \in \binom{[n]}{3}\ X_S, X_T \text{ dependent}} \mathbb{E}[X_S X_T]}{\mathbb{E}[X]^2}.$$

Thus it suffices to show

$$\sum_{S \neq T\ \in \binom{[n]}{3}\ X_S, X_T \text{ dependent}} \mathbb{E}[X_S X_T] = o(\mathbb{E}[X]^2).$$

Let S and T be two distinct 3-element subsets of $[n]$. Then they may share exactly two vertices or at most one vertex. In the latter case, S and T can not share any edge in common in $G(n,p)$, and so X_S and X_T are independent. So we consider only pairs of $S \neq T\ \in \binom{[n]}{3}$ that share two vertices. If S and T form triangles, then $S \cup T$ forms a graph with two triangles sharing one edge in common (as in the following picture).



In this case we have

$$\mathbb{E}[X_S X_T] = p^5.$$

Summing up we have

$$\sum_{S \neq T\ \in \binom{[n]}{3}\ X_S, X_T \text{ dependent}} \mathbb{E}[X_S X_T] \;=\; \Theta(n^4 p^5) \;\ll\; \Theta(n^6 p^6) = \mathbb{E}[X]^2,$$

as desired. □

Remark 26 (Subgraph threshold)**.** Given a fixed graph H with $|V(H)| = v_h$ and $|E(H)| = e_H$, what is the threshold for the property that $G(n,p)$ contains a subgraph isomorphic to H (a copy of H)?

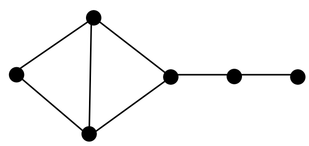
To answer this question we let X denote the number of copies of H. Then

$$\mathbb{E}[X] = \binom{n}{v_H} p^{e_H} = \Theta\left(n^{v_h} p^{e_H}\right) \to \begin{cases} 0, & \text{if } p \ll n^{-\frac{v_H}{e_H}} \\ \infty, & \text{if } p \gg n^{-\frac{v_H}{e_H}} \end{cases}$$

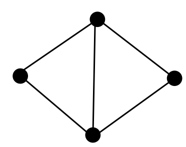
Is it true that

$$\mathbb{P}\left[X > 0\right] \to \begin{cases} 0, & \text{if } p \ll n^{-\frac{v_H}{e_H}} \\ 1, & \text{if } p \gg n^{-\frac{v_H}{e_H}} \end{cases}$$

This is indeed not true in general. To see this let us consider the kite graph H and the diamond graph D defined as follows.



H



D

Let X_H denote the number of kite graphs contained in $G(n,p)$ and X_D denote the number of diamond graphs contained in $G(n,p)$. Then we have

$$\mathbb{P}\left[X_D \geq 1\right] \leq \mathbb{E}\left[X_D\right] = O\left(n^4 p^5\right) \to 0, \quad \text{if } p \ll n^{-\frac{4}{5}}$$
$$\mathbb{P}\left[X_H \geq 1\right] \leq \mathbb{E}\left[X_H\right] = O\left(n^6 p^7\right) \to 0, \quad \text{if } p \ll n^{-\frac{6}{7}}$$

On one hand we have

$$n^{-\frac{v_H}{e_H}} = n^{-\frac{6}{7}} \ll n^{-\frac{4}{5}} = n^{-\frac{v_D}{e_D}},$$

but on the other hand we $\mathbb{P}\left[X_H \geq 1\right] \leq \mathbb{P}\left[X_D \geq 1\right]$, because D is a subgraph of H.

What's wrong here?

Definition 44 (Maximum density)**.** Let

$$m_H := \max\left\{ \frac{e_{H'}}{v_{H'}} : H' \subseteq H \quad \text{with} \quad v_{H'} > 0 \right\}$$

Theorem 14 (Bollobás 1981)**.**

For any graph H with $e_H, v_H > 0$,

$$\mathbb{P}\left[G(n,p) \text{ contains a copy of } H\right] \to \begin{cases} 0, & \text{if } p \ll n^{-\frac{1}{m_H}} \\ 1, & \text{if } p \gg n^{-\frac{1}{m_H}} \end{cases}$$

Proof. Follow the lines of the proof of Theorem 13. □

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