Discrete and algebraic structures, Winter Semester 2016/17

K. Baur

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Contents

Part II

Chapter 4

Multilinear algebra

[Lecture 1, 22.11.2016]
In this chapter, we discuss multilinear maps, tensor maps, tensor products and alternating maps. Chapter 4 is important for differential geometry. Multilinear maps can be viewed as a generalization of linear maps between vector spaces.

Some notations are explained at the very end of the document (just before the bibliography).

Unless mentioned otherwise, all vector spaces are finite dimensional vector spaces over some field K. Most of the time we will tacitly assume $K = \mathbb{C}$.

An important idea of this chapter is to convert a multilinear map on a product space into a linear map on the tensor product of its factor spaces and to study the relationship between these two maps.

4.1 Multilinear maps and tensor maps

Definition 4.1. Let $V_1, V_2, \dots, V_m$ and W be vector spaces over a field K. A map $\varphi : V_1 \times \cdots \times V_m \to W$ is *m-multilinear* or *multilinear* if it is linear in every argument, i.e. if

$$\varphi(v_1, \dots, v_i + \lambda v_i', \dots, v_m) = \varphi(v_1, \dots, v_i, \dots, v_m) + \lambda\varphi(v_1, \dots, v_i', \dots, v_m)$$

holds for all $i = 1, \dots, m$, and for all $\lambda \in K$.

A linear map $f \in \operatorname{Hom}(V, W)$ can be viewed as a *1-multilinear* map.

Remark. Linear map and multilinear maps are not the same!
Consider the linear map $f \in \operatorname{Hom}(V_1 \times V_2, W)$ and a multilinear map $\varphi : V_1 \times V_2 \to W$.
f is linear and so,

$$f(v_1 + v_1', v_2 + v_2') = f(v_1, v_2) + f(v_1', v_2') = f(v_1, 0) + f(0, v_2) + f(v_1', 0) + f(0, v_2')$$

φ is multilinear, and so,

$$\varphi(v_1+v_1', v_2+v_2') = \varphi(v_1, v_2+v_2')+\varphi(v_1', v_2+v_2') = \varphi(v_1, v_2)+\varphi(v_1, v_2')+\varphi(v_1', v_2)+\varphi(v_1', v_2')$$

Note that $\varphi(v_1, 0) = 0 = \varphi(0, v_2)$ but that $f(v_1, 0)$ and $f(0, v_2)$ are not necessarily 0.

Example 4.2. The following maps are multilinear (please convince yourself of this!)

(a) $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $f(x, y) = xy$.

(b) $\varphi : V^* \times V \to \mathbb{C}$ defined by $\varphi(f, v) = f(v)$.

(c) $\varphi : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}$ defined by $\varphi(x, y) = x^T Ay$ for some matrix $A \in \mathbb{C}_{m\times n}$.

(d) $\otimes : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}_{m\times n}$ defined by $\otimes(x, y) = xy^T$. This is a tensor map (see below).

(e) $\det : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ defined by $\det(x_1, \dots, x_n) = \det A$ where A is the $n \times n$-matrix $[x_1, \dots, x_n]$ formed by the column vectors x_i.

(f) $f : V_1 \times \cdots \times V_m \to \mathbb{C}$ defined by $f(v_1, \dots, v_m) = \prod_{i=1}^m f_i(v_i)$ for given $f_i \in V_i^*$, $i = 1, \dots, m$.

(g) $g : V_1^* \times \cdots \times V_m^* \to W$ defined by $g(f_1, \dots, f_m) = \prod_{i=1}^m f_i(v_i)$ for $v_i \in V_i$ given, $i = 1, \dots, m$.

(h) Let $\varphi : V_1 \times \cdots \times V_m \to \mathbb{C}$ and $\psi : V_1 \times \cdots \times V_m \to W$ be multilinear. Then $\alpha\varphi + \beta\psi$ is also multilinear (for all $\alpha, \beta \in K$). So the set $M(V_1, \dots, V_m, W)$ of multilinear maps from $V_1 \times \cdots \times V_m$ to W is a vector space.

Let $\psi : V_1\times\cdots\times V_m \to W$ be a multilinear map. If we describe how ψ acts on a basis of $V_1 \times \cdots \times V_m$, ψ is determined. Let $n_i = \dim V_i$ for $i = 1, \dots, m$, let $E_i := \{e_{i1}, \dots, e_{i,n_i}\}$ be a basis of V_i. Then each $v_i \in V_i$ can be written as $v_i = \sum_{j=1}^{n_i} a_{ij}e_{ij}$ $(i = 1, \dots, m)$.

To describe the basis of a product of m vector spaces of dimensions $n_1, \dots, n_m$, we use the following notation.

$$\Gamma := \Gamma(n_1, \dots, n_m) := \{\gamma : \gamma = (\gamma(1), \dots, \gamma(m)) \mid 1 \le \gamma(i) \le n_i, i = 1, \dots, m\}.$$

with $|\Gamma| = \prod_{i=1}^m n_i$. For an example, take $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2$, with

$$\Gamma = \Gamma(3, 2, 2) = \left\{ \begin{array}{l} (1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), \\ (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2), \\ (3, 1, 1), (3, 1, 2), (3, 2, 1), (3, 2, 2) \end{array} \right\}$$

We can order Γ according to the lexicographic order, as in the example. Moreover, we have (please check):

$$\prod_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} = \sum_{\gamma\in\Gamma} \prod_{i=1}^{m} a_{i\gamma(i)} \tag{4.1}$$

as will be used later.

From the set-up, we have

$$\begin{aligned}\psi(v_1,\ldots,v_m) &= \psi(\sum_{j_1=1}^{n_1} a_{1,j_1}e_{1,j_1},\ldots,\sum_{j_m}^{n_m} a_{m,j_m}e_{m,j_m})\\ &= \sum_{j_1}^{n_1}\cdots\sum_{j_m}^{n_m} a_{1,j_1}\cdots a_{m,j_m}\psi(e_{1,j_1},\ldots,e_{m,j_m})\\ &= \sum_{\gamma\in\Gamma} a_{1,\gamma(1)}\cdots a_{m,\gamma(m)}\psi(e_{1,\gamma(1)},\ldots,e_{m,\gamma(m)})\\ &= \sum_{\gamma\in\Gamma} a_\gamma\psi(e_\gamma) \qquad (4.2)\end{aligned}$$

where for $\gamma\in\Gamma$, we write

$$a_\gamma := \prod_{i=1}^m a_{i,\gamma(i)} \in \mathbb{C} \qquad (4.3)$$

$$e_\gamma := (e_{1,\gamma(1)},\ldots,e_{m,\gamma(m)}) \in V_1\times\cdots\times V_m \qquad (4.4)$$

so e_γ is the basis vector of $V_1\times\cdots\times V_m$ with label γ.
Then the $\psi(e_\gamma)$ in (4.2) completely determine ψ.

Theorem 4.3 (Multilinear extension)**.** *Let $E_i=\{e_{i,1},\ldots,e_{i,n_i}\}$ be a basis of V_i, $i=1\ldots,m$. Let e_γ be as in (4.4). Let w_γ in W, for all $\gamma\in\Gamma=\Gamma(n_1,\ldots,n_m)$ be arbitrary. Then there exists a unique multilinear map $\varphi: V_1\times\cdots\times V_m\to W$ such that $\varphi(e_\gamma)=w_\gamma$ for all γ.*

Proof. Since we want $\varphi(e_\gamma)=w_\gamma$ for all $\gamma\in\Gamma$, we need to define (by (4.2)

$$\varphi(v_1,\ldots,v_m)=\sum_{\gamma\in\Gamma} a_\gamma w_\gamma$$

for a_γ as in (4.3) and $v_i=\sum_{j=1}^{n_i} a_{ij}e_{ij}$. Let $v_i'=\sum_{j=1}^{n_i} a_{ij}'e_{ij}$ for $i\in\{1,\ldots,m\}$ arbitrary. From the definition of φ, for $c\in K$,

$$\begin{aligned}&\varphi(v_1,\ldots,v_i+cv_i',\ldots,v_m)\\ =&\sum_{\gamma\in\Gamma} a_{1,\gamma(1)}\cdots(a_{i,\gamma(i)}+ca_{i,\gamma(i)}')\cdots a_{m,\gamma(m)}w_\gamma\\ =&\sum_{\gamma\in\Gamma} a_{1,\gamma(1)}\cdots a_{i,\gamma(i)}\cdots a_{m,\gamma(m)}w_\gamma + c\sum_{\gamma\in\Gamma} a_{1,\gamma(1)}\cdots a_{i,\gamma(i)}'\cdots a_{m,\gamma(m)}w_\gamma\\ =&\,\varphi(v_1,\ldots,v_i,\ldots,v_m)+c\varphi(v_1,\ldots,v_i',\ldots,v_m)\end{aligned}$$

i.e. φ is multilinear.

Now we show that $\varphi(e_\alpha) = w_\alpha$ for all $\alpha \in \Gamma$.
For $\alpha \in \Gamma$, write

$$e_{i,\alpha(i)} = \sum_{j=1}^{n_j} \delta_{\alpha(i),j} e_{ij}.$$

From the definition of φ and $e_\alpha = (e_{1,\alpha(1)}, \ldots, e_{m,\alpha(m)})$, we have

$$\varphi(e_\alpha) = \varphi(e_{1,\alpha(1)}, \ldots, e_{m,\alpha(m)}) = \sum_{\gamma \in \Gamma} \delta_{\alpha(1),1} \ldots \delta_{\alpha(m),m} w_\gamma = \sum_{\gamma \in \Gamma} \delta_{\alpha,\gamma} w_\gamma = w_\alpha$$

(with $\delta_{\alpha,\gamma}$ the obvious generalization of the Kronecker delta). So we have established the existence.

Uniqueness: Suppose there is another multilinear map $\psi : V_1 \times \cdots \times V_m \to W$ such that $\psi(e_\gamma) = w_\gamma$ for all $\gamma \in \Gamma$. Then from (4.2) we have (for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$):

$$\psi(v_1, \ldots, v_m) = \sum_{\gamma \in \Gamma} a_\gamma \psi(e_\gamma) = \sum_{\gamma \in \Gamma} a_\gamma w_\gamma = \varphi(v_1, \ldots, v_m)$$

So $\psi = \varphi$. $\square$

Let us point out some *differences between linear and multilinear maps*. When $T : V \to W$ is linear, T is completely determined by the $n := \dim V$ images $T(e_1), \ldots, T(e_n)$ where $E = \{e_1, \ldots, e_n\}$ is a basis of V. But to determine a multilinear map φ, we need $|\Gamma| = \prod_{i=1}^m \dim V_i$ images. In general, this is much more than $\dim(V_1 \times \cdots \times V_m) = \sum_{i=1}^m \dim V_i$.

Recall Example 4.2(d) with $m = n = 2$, i.e. $\otimes : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}_{2\times 2}$ defined by $\otimes(x, y) = xy^T$. Since $\operatorname{rk}(xy^T) \le \min\{\operatorname{rk} x, \operatorname{rk} y^T\} \le 1$, we have $\det(\otimes(x, y)) = 0$. But if we take $x_1 = (1, 0)^T$ and $x_2 = (0, 1)^T$, then

$$\det(\otimes(x_1, x_1) + \otimes(x_2, x_2)) = \det I_2 = 1$$

(for I_2 the identity matrix in $\mathbb{C}_{2,2}$).
Hence $\otimes(x_1, x_1) + \otimes(x_2, x_2)$ is not in the image of $\otimes$ and $\operatorname{im} \otimes$ is not a subspace.
In general, the image $\operatorname{im} \varphi = \{\varphi(v_1, \ldots, v_m) \mid v_i \in V_i, i_1, \ldots, m\}$ is not necessarily a subspace of W. But we can consider the span $\langle \operatorname{im} \varphi \rangle$ of the image of φ. Clearly, $\dim\langle \operatorname{im} \varphi \rangle \le \prod_{i=1}^m \dim V_i$.

Definition 4.4. The *rank* of φ is defined to be

$$\operatorname{rk} \varphi = \dim\langle \operatorname{im} \varphi \rangle$$

The multilinear map φ is called a *tensor map* if $\operatorname{rk} \varphi = \prod_{i=1}^m \dim V_i$. In other words, a tensor map is a multilinear map with *maximal image span*.

Example 4.2(a) is a tensor map. Example 4.2(d) is a tensor map.

Theorem 4.5. *The multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is a tensor map if and only if the set $\{\varphi(e_\gamma) \mid \gamma \in \Gamma\}$ is linearly independent (for e_γ as in (4.4)).*

Proof. From (4.2), $\langle \varphi(e_\gamma) \mid \gamma \in \Gamma \rangle = \langle \operatorname{im} \varphi \rangle$ and $|\Gamma| = \prod_{i=1}^m \dim V_i$. $\square$

Theorem 4.6. *Tensor maps exist, i.e. for $V_1, \ldots, V_m$ there exist W and $\varphi : V_1 \times \cdots \times V_m \to W$ such that φ is a tensor map.*

Proof. By Theorem 4.3, pick W a vector space with $\dim W = \prod_{i=1}^m \dim V_i$ and let $\{w_\gamma : \gamma \in \Gamma\}$ be a basis so that the w_γ $(\gamma \in \Gamma)$ determine the multilinear map φ which is obviously a tensor map. $\square$

Clearly, tensor maps on $V_1 \times \cdots \times V_m$ are not unique. What makes them useful is the fact that every multilinear map can be factored as a tensor map composed with a linear map: The study of multilinear maps is reduced to the study of some linear map (not unique) via a tensor map.

Definition 4.7. A multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is said to have the *universal factorization property* if for any multilinear map $\psi : V_1 \times \cdots \times V_m \to W$, there is $T \in \operatorname{Hom}(P, W)$ such that $\psi = T \circ \varphi$. [1]

$$\begin{tikzcd} V_1 \times \cdots \times V_m \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & P \arrow[dl, "\exists\, T"] \\ & W & \end{tikzcd}$$

[Lecture 2, 28.11.2016]

Theorem 4.8. *The multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ is a tensor map if and only if φ has universal factorization property.*

Proof. Suppose that φ is a tensor map. Then $\{\varphi(e_\gamma) \mid \gamma \in \Gamma\}$ is a basis of $\langle \operatorname{im} \varphi \rangle$. There is a unique[2] $T \in \operatorname{Hom}(\langle \operatorname{im} \varphi \rangle, W)$ such that $T\varphi(e_\gamma) = \psi(e_\gamma)$ for all $\gamma \in \Gamma$. Since $T\varphi$ and ψ are multilinear maps $V_1 \times \cdots \times V_m \to W$ (Exercise 2 below or Problem 7.1 (a)), Theorem 4.3 implies $T\varphi = \psi$.

Conversely, suppose that φ has the universal factorization property. In particular consider a tensor map ψ on $V_1 \times \cdots \times V_m$, i.e. , $\dim\langle \operatorname{im} \psi \rangle = \prod \dim V_i$. (Such a map ψ exists by Theorem 4.6).

Then $T\varphi = \psi$ for some linear map T. Thus, $T(\langle \operatorname{im} \varphi \rangle) = \langle \operatorname{im} \psi \rangle$. Hence

$$\prod_i \dim V_i \overset{\psi \text{ tensor map}}{=} \dim\langle \operatorname{im} \psi \rangle \overset{T\varphi = \psi}{\leq} \dim\langle \operatorname{im} \varphi \rangle \leq \prod_i \dim V_i$$

So $\operatorname{rk} \varphi = \prod \dim V_i$ and φ is a tensor map. $\square$

[1] note that T may depend on ψ.

[2] This is a fact from linear algebra: if V and W are K-vector spaces with a basis $\{v_1, \ldots, v_n\}$ of V and if $w_1, \ldots, w_n$ are vectors in W. Then there exists a unique homomorphism $T : V \to W$ such that $T(v_i) = w_i$ for all i.

Definition 4.9. A multilinear map $\varphi : V_1 \times \cdots \times V_r \to K$ is called a *multilinear form* or an *r-form.* If $V_1 = \cdots = V_r =: V$ (and $W = K$), then φ is called an *r-form on V* or a *multilinear form on V.*

Exercises for Section 4.1

1. Let $V_1, \ldots, V_m$, $W_1, \ldots, W_m$ and W be $\mathbb{C}$-vector spaces.
   Let $\varphi : W_1 \times \cdots \times W_m \to W$ be multilinear and $T_i : V_i \to W_i$ linear for all i.
   Define $\psi : V_1 \times \cdots \times V_m \to W$ by $\psi(v_1, \ldots, v_m) = \varphi(T_1 v_1, \ldots, T_m v_m)$.
   Show that ψ is multilinear.

2. Let $V_1, \ldots, V_m$, W and W' be $\mathbb{C}$-vector spaces.
   Prove that if $\varphi : V_1 \times \cdots \times V_m \to W$ is multilinear and $T : W \to W'$ is linear, then $T \circ \varphi$ is multilinear.

3. Show that for $n > 1$, the determinant function $\det : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ is not a tensor map.

4. Suppose that the multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ has the universal factorization property. Show that the linear map T (from the defn of the universal factorization property) is unique if and only if $\langle \operatorname{im} \varphi \rangle = P$.

4.2 Tensor products and unique factorization

Let P be a vector space. If there is a tensor map $\otimes : V_1 \times \cdots \times V_m \to P$ such that $\langle \operatorname{im} \otimes \rangle = P$, then P is said to be a *tensor product* of $V_1, \ldots, V_m$ or a *tensor space.* It is written as $\otimes_{i=1}^m V_i$ or as $V_1 \otimes \cdots \otimes V_m$. If $V_1 = \cdots = V_m =: V$, we also write $V^{\otimes m}$. We will see below that all tensor product spaces of $V_1, \ldots, V_m$ are isomorphic and hence we will just say that $\otimes_{i=1}^m V_i$ is *the* tensor product of $V_1, \ldots, V_m$. We have

$$\dim(\otimes_{i=1}^m V_i) = \dim \langle \operatorname{im} \otimes \rangle = \prod_{i=1}^m \dim V_i$$

The elements of $\otimes_{i=1}^m V_i$ are *tensors.* The tensors of the form

$$\otimes(v_1, \ldots, v_m) =: v_1 \otimes \cdots \otimes v_m$$

are the *decomposable tensors* (or pure tensors), i.e., tensors in $\operatorname{im} \otimes$ are decomposable. The decomposable tensors span $\otimes_{i=1}^m V_i$, so we can find a basis of decomposable tensors for $\otimes_{i=1}^m V_i$. The tensors which are not decomposable, i.e., the tensors in $\langle \operatorname{im} \otimes \rangle \setminus \operatorname{im} \otimes$ are called *indecomposable tensors.* From Theorem 4.6, we get the following.

Corollary 4.10. *Let P be a vector space with $\dim P = \prod_{i=1}^m \dim V_i$. Then there exists $\otimes : V_1 \times \cdots \times V_m \to P$, so that P is the tensor space.*

Theorem 4.11. *Tensor spaces of $V_1, \dots, V_m$ are isomorphic:*
If $P = \otimes_{i=1}^m V_i$ and $Q = \boxtimes_{i=1}^m V_i$ are tensor products of $V_1, \dots, V_m$, then there exists an invertible $T \in \mathrm{Hom}(P, Q)$ such that $T \circ \otimes = \boxtimes$.

Proof. Since P and Q are tensor spaces of $V_1, \dots, V_m$, $\dim P = \dim Q$. The sets

$$\{\otimes e_\gamma \mid \gamma \in \Gamma\}, \quad \{\boxtimes e_\gamma \mid \gamma \in \Gamma\}$$

are bases of P and Q. We get T by sending basis elements to basis elements.
By Theorem 4.8 (universal factorization property) there exists $T \in \mathrm{Hom}(P, Q)$ such that $T \otimes (e_\gamma) = \boxtimes e_\gamma$, for all $\gamma \in \Gamma$. Thus T is invertible and $T\otimes = \boxtimes$. □

Theorem 4.12 (Unique factorization property)**.** *Let $\psi : V_1 \times \cdots \times V_m \to W$ be multilinear. Then there exists a unique linear map $T : \otimes_{i=1}^m V_i \to W$ such that $\psi = T \circ \otimes$, i.e.,*

$$\psi(v_1, \dots, v_m) = T \otimes (v_1, \dots, v_m) = T(v_1 \otimes \cdots \otimes v_m).$$

Proof. Use Exercise 4.1.4 or Problem 8.2 (a) □

In other words, the tensor map on $\otimes_{i=1}^m V_i$ has the *unique factorization property.*

Corollary 4.13. *Let $\varphi : V_1 \times \cdots \times V_m \to P$ be multilinear. Then φ is a tensor map and $\langle \operatorname{im} \varphi \rangle = P$ if and only if φ has the unique factorization property.*

Exercises for Section 4.2

1. Show that if some $v_i = 0$, then $v_1 \otimes \cdots \otimes v_m = 0$.
2. Let $z \in U \otimes V$ so that z can be represented as $z = \sum_{i=1}^k u_i \otimes v_i$. Prove that if k is the smallest number among all such representations, then $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets.
3. Suppose that $e_1, e_2 \in V$ are linearly independent. Prove that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes V$ is indecomposable.
4. Let $P = \otimes_{i=1}^m V_i$ with tensor map φ and let $T \in \mathrm{Hom}(P, Q)$ be invertible. Prove that $\psi = T \circ \varphi$ is also a tensor map and that Q is the tensor space with tensor map ψ.

4.3 Basic properties of tensors and induced inner products

As before, let $V_1, \dots, V_m$ be vector spaces over $K = \mathbb{C}$.

Theorem 4.14. *Consider the tensor product $\otimes_{i=1}^m V_i$ and tensors $u_1 \otimes \cdots \otimes u_m, \dots, w_1 \otimes \cdots \otimes w_m \in \otimes_i V_i$.*
If $u_1 \otimes \cdots \otimes u_m + \cdots + w_1 \otimes \cdots \otimes w_m = 0$ then $\varphi(u_1, \dots, u_m) + \cdots + \varphi(w_1, \dots, w_m) = 0$ for any multilinear map $\varphi : V_1 \times \cdots \times V_m \to W$.

Proof. By Theorem 4.12 we can write $\varphi = T \circ \otimes$ for some $T \in \operatorname{Hom}(\otimes_{i=1}^m V_i, W)$. So

$$\varphi(u_1, \ldots, u_m) = T \otimes (u_1, \ldots, u_m) = T(u_1 \otimes \cdots \otimes u_m)$$

and similarly for all decomposable tensors in the sum. Hence

$$\begin{aligned}
&\varphi(u_1, \ldots, u_m) + \cdots + \varphi(w_1, \ldots, w_m) \\
&= T(u_1 \otimes \cdots \otimes u_m) + \cdots + T(w_1 \otimes \cdots \otimes w_m) \\
&= T(\underbrace{u_1 \otimes \cdots \otimes u_m + \cdots + w_1 \otimes \cdots \otimes w_m}_{=0}) \\
&= 0
\end{aligned}$$

□

Theorem 4.15. *Let $v_i \in V_i$, $i = 1, \ldots, m$. Then $v_1 \otimes \cdots \otimes v_m = 0$ if and only if there exists i with $v_i = 0$.*

Proof. $\Longleftarrow$ is clear.
$\Longrightarrow$: Suppose $v_1 \otimes \cdots \otimes v_m = 0$ but $v_i \neq 0$ for all i. Then for each i there exists $f_i \in V_i^*$ such that $f_i(v_i) = 1$. Define

$$\varphi := \prod_{i=1}^m f_i : V_1 \times \cdots \times V_m \to \mathbb{C}$$

This is a multilinear map (cf. Example 4.2 (f)). Then

$$\varphi(v_1, \ldots, v_m) = \prod_{i=1}^m f_i(v_i) = 1$$

But by Theorem 4.14, $\varphi(v_1, \ldots, v_m) = 0$, a contradiction. □

Theorem 4.16. *Let $u_1 \otimes \cdots \otimes u_m$ and $v_1 \otimes \cdots \otimes v_m$ be in $V_1 \otimes \cdots \otimes V_m$.*
$u_1 \otimes \cdots \otimes u_m = v_1 \otimes \cdots \otimes v_m \neq 0 \iff v_i = c_i u_i$ for all $i = 1, \ldots, m$ and $\prod_{i=1}^m c_i = 1$.

Proof. $\Longleftarrow$ is clear.
$\Longrightarrow$: Suppose $u_1 \otimes \cdots \otimes u_m = v_1 \otimes \cdots \otimes v_m \neq 0$. From Theorem 4.15, all u_i and v_i are non-zero. From Theorem 4.14 (applied to $u_1 \otimes \cdots \otimes u_m - v_1 \otimes \cdots \otimes v_m$) we know that

$$\varphi(u_1, \ldots, u_m) = \varphi(v_1, \ldots, v_m).$$

holds for any multilinear map φ from $V_1 \times \cdots \times V_m$ to some $\mathbb{C}$-vector space W.
Suppose that u_k and v_k are not linearly dependent for some k (i.e. assume that there is an index k such that u_k is not a (non-zero) multiple of v_k). Then there is $f_k \in V_k^*$ such that $f_k(v_k) = 1$ and $f_k(u_k) = 0$. For $i \neq k$ choose $f_i \in V_i^*$ such that $f_i(v_i) = 1$. Set $\varphi := \prod_{i=1}^m f_i$. (This is multilinear, cf. Example 4.2 (f)). Then

$$\varphi(v_1, \ldots, v_m) = \prod_{i=1}^m f_i(v_i) = 1$$

and since $f_k(u_k) = 0$

$$\varphi(u_1, \dots, u_m) = \prod_{i=1}^{m} f_i(u_i) = 0,$$

contradicting $\varphi(u_1, \dots, u_m) = \varphi(v_1, \dots, v_m)$. Hence for all k there exists $c_k \neq 0$ such that $v_k = c_k u_k$. From

$$0 \neq u_1 \otimes \cdots \otimes u_m = v_1 \otimes \cdots \otimes v_m = (\prod_{i=1}^{m} c_i) u_1 \otimes \cdots \otimes u_m$$

we have $\prod_{i=1}^{m} c_i = 1$. □

Since the decomposable elements span the tensor space $\otimes_{i=1}^m V_i$, each $z \in \otimes_{i=1}^m V_i$ is a linear combination of decomposable tensors. Let k be the smallest number of decomposable tensors in all such linear combinations for z. We call k the *rank* or the *smallest length* of z. The rank of $z \neq 0$ is one if and only if z is decomposable.

Theorem 4.17. *Assume that $z \in U \otimes V$ can be represented as $z = \sum_{i=1}^r u_i \otimes v_i$ for some $r \geq 1$. Then the smallest length of z is k if and only if $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets, respectively.*

Proof. The implication $\Longrightarrow$ is Exercise 2 of Section 4.2 or Problem 8.1 (b).
$\Longleftarrow$: Assume that $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets. Let $z = \sum_{j=1}^r x_j \otimes y_j$, we show that $k \leq r$. Let $l \in \{1, \dots, k\}$ be arbitrary. Since the v_i are linearly independent, there is $g \in V^*$ such that $g(v_l) = 1$ and $g(v_j) = 0$ for $j \neq l$. Let $f \in U^*$ be arbitrary. Then the map $\varphi := fg : U \times V \to \mathbb{C}$ is bilinear (cf. Example 4.2 (f)). By assumption,

$$\sum_{i=1}^{k} u_i \otimes v_i = \sum_{j=1}^{r} x_j \otimes y_j$$

Using Theorem 4.14 for $\varphi = fg$, we have (first equality by the choice of g)

$$f(u_l) = \sum_{i=1}^{k} f(u_i) g(v_i) = \sum_{j=1}^{r} f(x_j) g(y_j) = f(\sum_{j=1}^{r} g(y_i) x_j)$$

Since f is arbitrary, this holds for all $f \in U^*$. And so we get $u_l = \sum_{j=1}^r g(y_i) x_j$, so u_l is in $\langle x_1, \dots, x_r \rangle$ for all $l = 1, \dots, k$. (choosing other linear maps g when l varies). Since $\{u_1, \dots, u_k\}$ is linearly independent, we have $k \leq r$. □

[Lecture 3, 29.11.2016]

We now consider an induced inner product (a reminder on inner products is at the end of the chapter, cf. Subsection 4.7.1) of $\otimes_{i=1}^m V_i$. Suppose that $(\cdot, \cdot)_i$ is an inner product on V_i and $E_i = \{e_{i1}, \dots, e_{in_i}\}$ is an orthonormal basis of V_i for all $i = 1, \dots, m$. We know that

$$E := \{e_\gamma^\otimes := e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)} : \gamma \in \Gamma\}$$

is a basis of $\otimes_{i=1}^m V_i$, where $\Gamma = \Gamma(n_1, \dots, n_m)$. We would like to have an inner product $(\cdot, \cdot)$ on the tensor product $\otimes_{i=1}^m V_i$ such that E is an orthonormal basis, i.e.

$$(e_\alpha^\otimes, e_\beta^\otimes) = (e_{1\alpha(1)} \otimes \cdots \otimes e_{m\alpha(m)}, e_{1\beta(1)} \otimes \cdots \otimes e_{m\beta(m)}) = \delta_{\alpha,\beta}$$

($\delta_{\alpha,\beta}$ is the generalization of the Kronecker delta to m-tuples).
Such an inner product is unique (Section 4.7.1).

We define

$$(u, v) := \sum_{\gamma \in \Gamma} a_\gamma \overline{b_\gamma} \tag{4.5}$$

where $u = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma^\otimes$, $v = \sum_{\gamma \in \Gamma} b_\gamma e_\gamma^\otimes \in \otimes_{i=1}^m V_i$. With this defintion, E becomes an ONB (cf. Problem 8.4 (a)). The definition appears to depend on the choice of the basis, but it does not (cf. Problem 7.3).

Theorem 4.18. *Let $V_1, \dots, V_m$ be inner product spaces with orthonormal bases $E_i = \{e_{i1}, \dots, e_{i,n_i}\}$, $i = 1, \dots, m$. The inner product obtained from (4.5) satisfies*

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^m (u_i, v_i)_i \tag{4.6}$$

(where $u_i, v_i \in V_i$ for all i and $(\cdot, \cdot)_i$ is the inner product on V_i).

Proof. We write u_i and v_i in terms of the basis E_i,

$$u_i = \sum_{j=1}^{n_i} a_{ij} e_{ij}, \quad v_i = \sum_{j=1}^{n_i} b_{ij} e_{ij}, \quad i = 1, \dots, m.$$

Since $\otimes$ is multilinear, from (4.2) we get

$$u^\otimes := u_1 \otimes \cdots \otimes u_m = \otimes(u_1, \dots, u_m) = \sum_{\gamma \in \Gamma} \Big(\overbrace{\prod_{i=1}^m a_{i\gamma(i)}}^{a_\gamma} \Big) e_\gamma^\otimes$$
$$v^\otimes := v_1 \otimes \cdots \otimes v_m = \otimes(v_1, \dots, v_m) = \sum_{\gamma \in \Gamma} \Big(\underbrace{\prod_{i=1}^m b_{i\gamma(i)}}_{b_\gamma} \Big) e_\gamma^\otimes$$

By the definition (4.5) of the inner product and since $\prod_{i=1}^m \sum_{j=1}^{n_i} c_{ij} \overset{(4.1)}{=} \sum_{\gamma\in\Gamma} \prod_{i=1}^m c_{i\gamma(i)}$,

$$\begin{aligned}(u^\otimes, v^\otimes) &= \sum_{\gamma\in\Gamma} \prod_{i=1}^m a_{i\gamma(i)} \overline{b_{i\gamma(i)}} \\ &\overset{(4.1)}{=} \prod_{i=1}^m \sum_{j=1}^{n_i} a_{ij}\overline{b_{ij}} \\ &= \prod_{i=1}^m (\sum_{j=1}^{n_i} a_{ij}e_{ij}, \sum_{j=1}^{n_i} b_{ij}e_{ij})_i \\ &= \prod_{i=1}^m (u_i, v_i)_i.\end{aligned}$$

(where $(\cdot,\cdot)_i$ denotes the inner product on V_i). $\square$

There are many bilinear maps from $(\otimes_{i=1}^k V_i) \times (\otimes_{i=k+1}^m V_i)$ to $\otimes_{i=1}^m V_i$ since

$$\dim \otimes_{i=1}^m V_i = \prod_{i=1}^m n_i = \prod_{i=1}^k n_i \prod_{i=k+1}^m n_i = \dim(\otimes_{i=1}^k V_i)\dim(\otimes_{i=k+1}^m V_i). \tag{4.7}$$

What we like is one that maps the pair $(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m)$ to $v_1 \otimes \cdots \otimes v_m$.

Theorem 4.19. *There is a unique multilinear map* $\boxtimes : (\otimes_{i=1}^k V_i) \times (\otimes_{i=k+1}^m V_i) \to \otimes_{i=1}^m V_i$ *such that*

$$\boxtimes(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_m \tag{4.8}$$

and

$$(V_1 \otimes \cdots \otimes V_k) \boxtimes (V_{k+1} \otimes \cdots \otimes V_m) = V_1 \otimes \cdots \otimes V_m \tag{4.9}$$

Proof. By (4.7), the dimensions are ok, so the tensor map $\boxtimes$ satisfying (4.8) exists (Theorem 4.10) and is unique (Theorem 4.12). From $\langle \mathrm{im}\, \boxtimes\rangle = \langle v_1 \otimes \cdots \otimes v_m \mid v_i \in V_i\rangle = \otimes_{i=1}^m V_i$, (4.9) follows. (See Exercise no. 4 in 4.3 for details.) $\square$

We also write $\otimes$ for $\boxtimes$ in Theorem 4.18. So we can write (4.9) as

$$(V_1 \otimes \cdots \otimes V_k) \otimes (V_{k+1} \otimes \cdots \otimes V_m) = V_1 \otimes \cdots \otimes V_m$$

and (4.8) can be written as

$$(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes v_m.$$

Exercises for Section 4.3

1. Suppose that $v_1, \dots, v_k \in V$ are linearly independent and let $u_1, \dots, u_k \in U$. Prove that $\sum_{i=1}^k u_i \otimes v_i = 0$ if and only if $u_1 = \cdots = u_k = 0$.

2. Let $v_1, \dots, v_k \in V$ and $A \in \mathbb{C}_{k\times k}$. Suppose $AA^T = I_k$ and $u_j = \sum_{i=1}^k a_{ij} v_i$, $j = 1, \dots, k$. Prove that $\sum_{i=1}^k u_i \otimes u_i = \sum_{i=1}^k v_i \otimes v_i$.

3. Define $\otimes : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}_{k\times n}$ by $x \otimes y := xy^T$. Let $\mathbb{C}^k$ and $\mathbb{C}^n$ be equipped with the standard inner products. Prove that for any $A, B \in \mathbb{C}_{k\times n} = \mathbb{C}^k \otimes \mathbb{C}^n$, the induced inner product is given by $(A, B) = \operatorname{tr}(B^* A)$. (notation: $\operatorname{tr}(B^*A)$ is the trace of the square matrix B^*A)

4. Let $E_i = \{e_{i1}, \dots, e_{in_i}\}$ be a basis of V_i, $i = 1, \dots, m$. Define

$$\varphi : (V_1 \otimes \cdots \otimes V_k) \times (V_{k+1} \otimes \cdots \otimes V_m) \to V_1 \otimes \cdots \otimes V_m$$

by $\varphi(e_{1i_1} \otimes \cdots \otimes e_{ki_k}, e_{k+1 i_{k+1}} \otimes \cdots \otimes e_{mi_m}) = e_{1i_1} \otimes \cdots \otimes e_{mi_m}$ (with bilinear extension). Show that φ is the tensor map satisfying

$$\varphi(v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_m) = v_1 \otimes \ldots \otimes v_m$$

5. Let $z = \sum_{i=1}^k u_i \otimes v_i \otimes w_i \in U \otimes V \otimes W$. Prove that if $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent and $w_i \neq 0$ for all i, then k is the smallest length of z.

4.4 Induced maps

In this section, we study $\operatorname{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$. Let $T_i \in \operatorname{Hom}(V_i, W_i)$ for all i. Then we define a multilinear map from $V_1 \times \cdots \times V_m$ to $\otimes_{i=1}^m W_i$ by

$$\varphi(v_1, \dots, v_m) := T_1 v_1 \otimes \cdots \otimes T_m v_m$$

By Theorem 4.12 (with $\otimes_{i=1}^m W_i$ in the role of W) there is a unique $T \in \operatorname{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$ such that ($\varphi = T \circ \otimes$, i.e.)

$$T(v_1 \otimes \cdots \otimes v_m) = T_1 v_1 \otimes \cdots \otimes T_m v_m$$

We denote this T by $T_1 \otimes \cdots \otimes T_m$ and call it the *induced map* of $T_1, \dots, T_m$, i.e.

$$(\otimes_{i=1}^m T_i)(v_1 \otimes \cdots \otimes v_m) = T_1 v_1 \otimes \cdots \otimes T_m v_m.$$

(Drawing the corresponding diagram with vector spaces might be helpful).

One can show that $T_1 \otimes \cdots \otimes T_m$ is a tensor (an element) of the tensor product $\otimes_{i=1}^m \operatorname{Hom}(V_i, W_i)$.

The main focus of this section is to study the map $T_1\otimes\cdots\otimes T_m \in \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$ as a linear map. We will see how it behaves with respect to composition, how to determine its rank and its adjoint.

Theorem 4.20. *Let $S_i \in \mathrm{Hom}(W_i, U_i)$, $T_i \in \mathrm{Hom}(V_i, W_i)$, $i = 1, \ldots, m$. Then*

$$(\otimes_{i=1}^m S_i) \circ (\otimes_{i=1}^m T_i) = \otimes_{i=1}^m (S_i \circ T_i).$$

(It might be helpful to draw the diagrams with the vector spaces to illustrate the statement).

Proof. It is enough to show the claim on decomposable elements, since $\otimes_{i=1}^m V_i$ is spanned by them. This is what we do.

$$\begin{aligned}(\otimes_{i=1}^m S_i)(\otimes_{i=1}^m T_i)(v_1 \otimes \cdots \otimes v_m) &= (\otimes_{i=1}^m S_i)(T_1 v_1 \otimes \cdots \otimes T_m v_m)\\ &= S_1 T_1 v_1 \otimes \cdots \otimes S_m T_m v_m\\ &= \otimes_{i=1}^m (S_i T_i)(v_1 \otimes \cdots \otimes v_m).\end{aligned}$$

□

Theorem 4.21. *Let $T_i \in \mathrm{Hom}(V_i, W_i)$, $i = 1, \ldots, m$. Then*

$$\mathrm{rk}\,(T_1 \otimes \cdots \otimes T_m) = \prod_{i=1}^m \mathrm{rk}\, T_i$$

Proof. Let $\mathrm{rk}\, T_i = k_i$ for all i. So there is a basis $\{e_{i1}, \ldots, e_{ik_i}, e_{ik_i+1}, \ldots, e_{in_i}\}$ for V_i such that $T_i e_{i1}, \ldots, T_i e_{ik_i}$ are linearly independent in W_i and $T_i e_{ik_i+1} = \cdots = T_i e_{in_i} = 0$, $i = 1, \ldots, m$. The set $\{e_\gamma^\otimes : \gamma \in \Gamma(n_1, \ldots, n_m)\}$ is a basis for $\otimes_{i=1}^m V_i$.

Moreover,

$$(\otimes_{i=1}^m T_i) e_\gamma^\otimes = T_1 e_{1\gamma(1)} \otimes \cdots \otimes T_m e_{m\gamma(m)}$$

so that if $\gamma \notin \Gamma(k_1, \ldots, k_m)$, then $(\otimes_{i=1}^m T_i) e_\gamma^\otimes = 0$ (as for some i, $\gamma(i) > k_i$). Since $T e_{i1}, \ldots, T e_{i,k_i}$ are linearly independent in W_i for all i, the vectors

$$(\otimes_{i=1}^m T_i) e_\gamma^\otimes, \quad \gamma \in \Gamma(k_1, \ldots, k_m)$$

are linearly independent in $\otimes_{i=1}^m W_i$ (why?), hence

$$\mathrm{rk} \otimes_{i=1}^m T_i = |\Gamma(k_1, \ldots, k_m)| = \prod_{i=1}^m k_i = \prod_{i=1}^m \mathrm{rk}\, T_i.$$

□

The next result describes the adjoint of $T_1 \otimes \cdots \otimes T_m$ viewed as a linear map. For notation/reminder: see Subsection 4.7.2 at the end of this chapter.

Theorem 4.22. *Let $T_i \in \mathrm{Hom}(V_i, W_i)$, where V_i and W_i are inner product spaces for $i = 1, \dots, m$. Then*

$$(\otimes_{i=1}^m T_i)^* = \otimes_{i=1}^m T_i^*.$$

Proof. We use the notation $(\cdot, \cdot)$ for the inner products on all the spaces V_i, W_i.

$$\begin{aligned}
((\otimes_{i=1}^m T_i)v^\otimes, w^\otimes) &= (T_1 v_1 \otimes \cdots \otimes T_m v_m, w_1 \otimes \cdots \otimes w_m) && \text{(inner product on } \otimes_i W_i) \\
&= \prod_{i=1}^m (T_i v_i, w_i) && \text{(defin.: inner prod on } \otimes_i W_i \text{ in terms of inner prod on } W_i\text{'s)} \\
&= \prod_{i=1}^m (v_i, T_i^* w_i) && \text{(adjoints of } T_i^*\text{, now on } V_i\text{'s)} \\
&= (v_1 \otimes \cdots \otimes v_m, T_1^* w_1 \otimes \cdots \otimes T_m^* w_m) && \text{(defin. of inner prod on } \otimes_i V_i) \\
&= (v^\otimes, \otimes_{i=1}^m T_i^* w^\otimes)
\end{aligned}$$

Since $\otimes_{i=1}^m V_i$ is spanned by decomposable tensors, we have the desired result. □

[Lecture 4, 5.12. 2016]

Exercises for Section 4.4

1. Prove that

   (a) Prove that $T_1 \otimes \cdots \otimes T_m = 0$ if and only if some $T_i = 0$,

   (b) $T_1 \otimes \cdots \otimes T_m$ is invertible if and only if all T_i are invertible.

2. Let $S_i, T_i \in \mathrm{Hom}(V_i, W_i)$ for $i = 1, \dots, m$. Prove that $\otimes_{i=1}^m T_i = \otimes_{i=1}^m S_i \neq 0$ if and only if $T_i = c_i S_i \neq 0$ for $i = 1, \dots, m$ and $\prod_{i=1}^m c_i = 1$.

3. Let $T_i \in \mathrm{End}\, V_i$ for $i = 1, \dots, m$. By 1 (b), $\otimes_{i=1}^m T_i$ is invertible if and only if T_i is invertible for every i. Prove that in this case, $(\otimes_{i=1}^m T_i)^{-1} = \otimes_{i=1}^m T_i^{-1}$.

4. Let $T_i \in \mathrm{Hom}(V_i, W_i)$ for $i = 1, \dots, m$. Define

   $$\varphi : \mathrm{Hom}(V_1, W_1) \times \cdots \times \mathrm{Hom}(V_m, W_m) \to \mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$$

   by $\varphi(T_1, \dots, T_m) = T_1 \otimes \cdots \otimes T_m$. Prove that φ is multilinear.

Exercise 4 can be used to show that $T_1 \otimes \cdots \otimes T_m$ is an element of a tensor product, namely of $\mathrm{Hom}(\otimes_{i=1}^m V_i, \otimes_{i=1}^m W_i)$. And then exercises 1-3 follow directly.

4.5 Some models of tensor products

We now consider the special case $V_1 = \cdots = V_m$ where all the spaces V_i are equal and where the target space is the field $\mathbb{C}$.

Recall that $M(V, \ldots, V; \mathbb{C})$ denotes the $\mathbb{C}$-vector space of all m-multilinear maps $f : V^m := \underbrace{V \times \cdots \times V}_{m \text{ factors}} \to \mathbb{C}$.

The goal of this section is to see that the tensor product $\overbrace{V \otimes \cdots \otimes V}^{m \text{ copies}}$ can be viewed as $M(\overbrace{V^*, \ldots, V^*}^{m \text{ copies}}; \mathbb{C})$ or as $M(V, \ldots, V; \mathbb{C})$ and that the tensor product $V^* \otimes \cdots \otimes V^*$ can be understood as $M(\underbrace{V, \ldots, V}_{m \text{ copies}}; \mathbb{C})$.

Let $E = \{e_1, \ldots, e_n\}$ be a basis of V and let $E^* = \{f_1, \ldots, f_n\}$ be the dual basis of V^*, i.e.,

$$f_i(e_j) = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

In this situation we abbreviate the set $\Gamma(\overbrace{n, \ldots, n}^{m \text{ times}})$ as

$$\Gamma(n : m) = \{\gamma = (\gamma(1), \ldots, \gamma(m)) \mid 1 \leq \gamma(i) \leq n, i = 1, \ldots, m\}.$$

We write $V^{\otimes m}$ or also $\otimes^m(V)$ for the m-fold tensor product $V \otimes \cdots \otimes V$.

Theorem 4.23. *1. The set* $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha \in \Gamma(n : m)\}$ *is a basis of* $M(V, \ldots, V; \mathbb{C})$.

2. $M(V, \ldots, V; \mathbb{C}) = (V^*)^{\otimes m}$, *in other words: there is a tensor map* $\otimes : V^* \times \cdots \times V^* \to M(V, \ldots, V; \mathbb{C})$ *and* $\dim M(V, \ldots, V; \mathbb{C}) = n^m$.

3. $M(V^*, \ldots, V^*; \mathbb{C}) = V^{\otimes m}$, *in other words: there is a tensor map* $\otimes : V \times \cdots \times V \to M(V^*, \ldots, V^*; \mathbb{C})$ *and* $\dim M(V^*, \ldots, V^*; \mathbb{C}) = n^m$.

Proof. (1) We first show that the set $S := \{\prod_{i=1}^m f_{\alpha(i)} : \alpha \in \Gamma(n : m)\}$ spans $M(V, \ldots, V; \mathbb{C})$. For this, first observe that for each $e_\beta = (e_{\beta(1)}, \ldots, e_{\beta(m)}) \in V \times \cdots \times V$ and $\beta \in \Gamma(n : m)$,

$$(\prod_{i=1}^m f_{\alpha(i)}) e_\beta = \prod_{i=1}^m f_{\alpha(i)}(e_{\beta(i)}) = \delta_{\alpha,\beta} \tag{4.10}$$

Let $f \in M(V, \ldots, V; \mathbb{C})$. Then we claim we can write f as follows

$$f = \sum_{\alpha \in \Gamma(n:m)} f(e_\alpha) \prod_{i=1}^m f_{\alpha(i)}$$

where $e_\alpha = (e_{\alpha(1)}, \dots, e_{\alpha(m)})$. This works since by (4.10),

$$\left(\sum_{\alpha \in \Gamma(n:m)} f(e_\alpha) \prod_{i=1}^m f_{\alpha(i)} \right)(e_\beta) = \sum_{\alpha \in \Gamma(n:m)} f(e_\alpha)\delta_{\alpha,\beta} = f(e_\beta), \qquad \beta \in \Gamma(n:m)$$

So f can be written as a linear combination of elements of S.

It remains to show that S is a linearly independent set. Assume

$$\sum_{\alpha \in \Gamma(n:m)} c_\alpha \prod_{i=1}^m f_{\alpha(i)} = 0.$$

Then

$$0 = \left(\sum_{\alpha \in \Gamma(n:m)} c_\alpha \prod_{i=1}^m f_{\alpha(i)} \right)(e_\beta) = \Big(\sum_{\alpha \in \Gamma(n:m)} c_\alpha \underbrace{\prod_{i=1}^m f_{\alpha(i)}(e_\beta)}_{\delta_{\alpha,\beta}} \Big) = c_\beta \quad \text{for every } \beta \in \Gamma(n:m)$$

(2) It is easy to see that the map $\otimes : (V^*)^m \to M(V, \dots, V; \mathbb{C})$ defined by

$$\otimes(g_1, \dots, g_m) = \prod_{i=1}^m g_i$$

is multilinear. From part (1), $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha \in \Gamma(n:m)\}$ is a basis of $M(V, \dots, V; \mathbb{C})$. So

$$\dim M(V, \dots, V; \mathbb{C}) = |\Gamma(n:m)| = n^m = (\dim V)^m = (\dim V^*)^m.$$

Therefore, $\otimes$ is a tensor map and $\langle \operatorname{im} \otimes \rangle = M(V, \dots, V; \mathbb{C})$, i.e. $M(V, \dots, V; \mathbb{C}) = \otimes^m V^*$.

(3) Analoguous to the proof of (2): We can define $\otimes : V^m \to M(V^*, \dots, V^*; \mathbb{C})$ by $\otimes(v_1, \dots, v_m) = \prod_{i=1}^m v_i$ where the latter is defined as follows:

$$\left(\prod_{i=1}^m v_i \right)(g_1, \dots, g_m) := \prod_{i=1}^m g_i(v_i)$$

□

By Theorem 4.23 (2) and (3), we can say that $M(V, \dots, V; \mathbb{C})$ *is a model for* $(V^*)^{\otimes m}$ and that $M(V^*, \dots, V^*, \mathbb{C})$ is a model for $V^{\otimes m}$.
Another model for $V^{\otimes m}$ is $M(V, \dots, V; \mathbb{C})^*$, the dual space of $M(V, \dots, V; \mathbb{C})$, as we will see now.

Theorem 4.24. *$M(V, \dots, V; \mathbb{C})^*$ is a model for $\otimes^m V$, i.e., there is a tensor map $\otimes : V \times \cdots \times V \to M(V, \dots, V; \mathbb{C})^*$ and $\dim M(V, \dots, V; \mathbb{C})^* = n^m$.*

Proof. For $v_1, \ldots, v_m \in V$ define $\prod_{i=1}^m v_i \in M(V, \ldots, V; \mathbb{C})^*$ by

$$(\prod_{i=1}^m v_i) f = f(v_1, \ldots, v_m), \quad f \in M(V, \ldots, V; \mathbb{C}).$$

From Theorem 4.23 (1), the set $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha \in \Gamma_{m,n}\}$ is a basis of $M(V, \ldots, V; \mathbb{C})$. Now from (4.10), $\{\prod_{i=1}^m e_{\alpha(i)} : \alpha \in \Gamma(n : m)\}$ is the dual basis of $\{\prod_{i=1}^m f_{\alpha(i)} : \alpha \in \Gamma(n : m)\}$ and thus is a basis of $M(V, \ldots, V; \mathbb{C})^*$. Then define $\otimes : V \times \cdots \times V \to M(V, \ldots, V; \mathbb{C})^*$ by $\otimes(v_1, \ldots, v_m) = \prod_{i=1}^m v_i$. □

Elements of the vector space $M(V, \ldots, V; \mathbb{C})$ are called *contra-variant tensors*; elements of $M(V^*, \ldots, V^*, \mathbb{C})$ are called *covariant tensors*.[3]

The tensor space

$$V_q^p := \overbrace{V \otimes \cdots \otimes V}^{p} \otimes \overbrace{V^* \otimes \cdots \otimes V^*}^{q}$$

is called a *tensor space of type* (p, q) (with *covariant type of degree* p *and with contra-variant type of degree* q). Analoguous to the previous treatment, under some tensor map, $M(V^*, \ldots, V^*, V, \ldots, V; \mathbb{C})$ (p copies of V and q copies of V^*, compare this with Theorem 4.23) is a model of V_q^p as we will indicate now:

Let $E = \{e_1, \ldots, e_n\}$ be a basis of V and let $E^* = \{f_1, \ldots, f_n\}$ be the dual basis of V^*. Then

$$\{\prod_{i=1}^p e_{\alpha(i)} \prod_{j=1}^q f_{\beta(j)} \mid \alpha \in \Gamma(n : p), \beta \in \Gamma(n : q)\}$$

is a basis for

$$M(V^*, \ldots, V^*, V, \ldots, V; \mathbb{C}) \quad (p \text{ copies of } V \text{ and } q \text{ copies of } V^*)$$

Define $\otimes : V \times \cdots \times V \times V^* \times \cdots \times V^* \to M(V^*, \ldots, V^*, V, \ldots, V; \mathbb{C})$ by

$$\otimes(e_{\alpha(1)}, \ldots, e_{\alpha(p)}, f_{\beta(1)}, \ldots, f_{\beta(q)}) = \prod_{i=1}^p e_{\alpha(i)} \prod_{j=1}^q f_{\beta(j)}$$

Then

$$\begin{aligned} &\{e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(p)} \otimes f_{\beta(1)} \otimes \cdots \otimes f_{\beta(q)} : \alpha \in \Gamma_{p,n}, \beta \in \Gamma_{q,n}\} \\ =&\{e_\alpha^\otimes \otimes f_\beta^\otimes : \alpha \in \Gamma_{p,n}, \beta \in \Gamma_{q,n}\} \end{aligned}$$

is a basis of V_q^p.

[3]contravariant vectors: under a base change using the invertible matrix A, the entries of a contravariant vector change with the inverse A^{-1} of the matrix of the base change. Under this base change, covariant vectors change with A.

Exercises for Section 4.5

1. Define a simple tensor map $\boxtimes : (V^*)^m \to (\otimes^m V)^*$ such that $\boxtimes^m V^* = (\otimes^m V)^*$.

2. Let $M(V_1, \ldots, V_m; W)$ be the set of all multilinear maps from $V_1 \times \cdots \times V_m$ to W. Prove that $\dim M(V_1, \ldots, V_m; W) = \dim W \cdot \prod_{i=1}^m \dim V_i$.

4.6 Exterior spaces

In this section, we consider some concrete examples of multilinear maps. The starting point is the space of multilinear maps (forms) $M(V, \ldots, V; K)$ from m copies of the K-vector space V to the field K.

We start by considering a specific notion of multiplication, namely one that relates to area and volume. The basic idea is as follows: Given two vectors v and w, we can form the parallelogram that they span, and write $v \wedge w$ for something as the "area" of the parallelogram. This is not quite the usual notion of area, however, because we want to think of it as not just a single number (in general) but also as having a "two-dimensional direction" (the same way a single vector v both has a size and a direction). That is, if we had a parallelogram pointing in a "different direction", i.e. in a different plane, we would think of it as different.

What would the properties of $v \wedge w$ be. Scaling the entry v or the entry w scales the parallelogram, so it should scale its area. So we expect, for scalars c, to have $(cv) \wedge w = c(v \wedge w) = v \wedge (cw)$. So the operation $\wedge$ should be bilinear. Another property of $\wedge$ would be, that for any vector v, $v \wedge v$ should be 0 - if the vectors point in the same direction, the "parallelogram" they span is just a line segment and has no area. These are the only two properties we really need.

As before, vector spaces are finite dimensional and defined are over some field K - mostly over $\mathbb{C}$ or over the real numbers $\mathbb{R}$.

Definition 4.25. Let V be a vector space. Then the *exterior square (product)* $\bigwedge^2(V)$ of V is the quotient of $V \otimes V$ by the subspace U spanned by the elements $v \otimes v$ for all $v \in V$. We write $v \wedge w$ for the image of $v \otimes w$ under the quotient map $V \otimes V \to \bigwedge^2(V)$.

What does $\bigwedge^2(V)$ look like? First observation (consequence of $v \wedge v = 0$):

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v.$$

So for any v and w, $v \wedge w = -w \wedge v$. The operation $\wedge$ is thus anti-commutative or *alternating.*

In terms of a basis of V, say $\{e_i\}_i$, we have that the set $\{e_i \otimes e_j\}_{ij}$ is a basis of $V \otimes V$. In $\bigwedge^2(V)$, we have $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. So $\bigwedge^2(V)$ can be spanned by the elements $e_i \wedge e_j$ for $i < j$. We can show that they are linearly independent:

Theorem 4.26. *Suppose $\{e_i\}_{i=1,\ldots,n}$ is a basis for V. Then $\{e_i \wedge e_j\}_{1 \le i < j \le n}$ is a basis for $\bigwedge^2(V)$. In particular,* $\dim \bigwedge^2(V) = \binom{n}{2}$.

Proof. The idea behind the proof is that $\bigwedge^2(V)$ is the "free" (or "universal") vector space in which you can multiply two elements of V in an anti-commutative way, so to show that the $e_i \wedge e_j$ (for $i < j$) are linearly independent, you have to construct some vector space with such a multiplication in which they are linearly independent.

We define a vector space E as follows: an element of E is a formal linear combination of symbols e_{ij} for $i < j$. So the set $\{e_{ij}\}_{i<j}$ is a basis of E.
We define a map $T : V \otimes V \to E$ by $T(e_i \otimes e_j) = e_{ij}$ if $i < j$, $T(e_i \otimes e_j) = -e_{ji}$ if $i > j$, and $T(e_i \otimes e_i) = 0$ and extend it linearly to all of $V \otimes V$. So T is a linear map. We want to show that T gives a map $S : \bigwedge^2(V) \to E$; it suffices to show that $T(v \otimes v) = 0$ for all $v \in V$. Let $v = \sum c_i e_i$; then

$$v \otimes v = \sum_{ij} c_i c_j e_i \otimes e_j = \sum_i c_i^2 e_i \otimes e_i + \sum_{i<j} c_i c_j (e_i \otimes e_j + e_j \otimes e_i).$$

We thus see that $T(v \otimes v) = 0$. Hence T gives a map $S : \bigwedge^2(V) \to E$ (also a linear map) which sends $e_i \wedge e_j$ to e_{ij}. Since the e_{ij} (for $i < j$) are linearly independent in E by construction, this implies that the $e_i \wedge e_j$ (for $i < j$) are linearly independent, and hence a basis. □

[Lecture 5, 6.12. 2016]

One thing to note about $\bigwedge^2(V)$ (as we know about $\bigotimes^2(V)$) is that *not* every element is of the form $v \wedge w$. For example, if $\{e_i\}$ is a basis of V, $\dim V \geq 4$, then $e_1 \wedge e_2 + e_3 \wedge e_4$ cannot be simplified to a single $v \wedge w$.

Definition 4.27. Let $r \geq 0$ be an integer, and V be a vector space. Then the *rth exterior power of* V, $\bigwedge^r(V)$ is the quotient of $V \otimes \cdots \otimes V$ (with r factors) by the subspace U spanned by all tensors $v_1 \otimes \cdots \otimes v_r$ for which two of the v_i are equal.
Elements of $\bigwedge^r(V)$ are *alternating* r*-tensors* or *alternating tensors of degree* r. Elements of the form $u_1 \wedge \cdots \wedge u_r$ with $u_i \in V$ are called *simple* or *decomposable alternating* r*-tensors*.
The *exterior algebra* $\bigwedge(V)$ is the direct sum $\bigoplus_{r \geq 0} \bigwedge^r(V)$. It is also called the *Grassmann algebra* or the *alternating algebra*.

Remark 4.28. (1) The exterior algebra is an algebra over the field K. This means, $\bigwedge(V)$ is a K-vector space with a bilinear product, it is a set with multiplication, addition and scalar multiplication by elements of the field. (Note that it is finite dimensional, since we assumed that V is finite dimensional, cf. Theorem 4.31).

(2) All through Chapter 4, we could have worked with modules over a commutative ring instead (and this is also done when introducing tensor products, exterior product, etc.), so replacing V and K with a module E over a commutative ring R and considered r-multilinear maps from E^r to the field K. The tensor product $E^{\otimes m}$ and the exterior product $\bigwedge^r(E)$ are also modules for R.

By definition, there is an r-multilinear map $\bigwedge^r : V^r \to \bigwedge^r(V)$ (called canonical (multilinear map)) obtained from the composition of $\otimes$ with the quotient:

$$V^r \longrightarrow V^{\otimes r} \longrightarrow V^{\otimes r}/U = \bigwedge^r(V). \tag{4.11}$$

The map is alternating (should be clear). We also have:

Theorem 4.29 (Unique factorization property of the rth exterior power of a vector space)**.** *Let V and W be vector spaces. If $f : V^r \to W$ is r-multilinear and alternating, there exists a unique linear map $f' : \bigwedge^r(V) \to W$ such that $f = f' \circ \wedge^r$, i.e. making the diagram*

$$\begin{tikzcd} V^r \arrow[r, "\wedge^r"] \arrow[dr, "f"'] & \bigwedge^r(V) \arrow[d, "f'"] \\ & W \end{tikzcd}$$

commutative.

The proof of this works analoguously as the proofs for the statements about tensor products (Theorems 4.8 and 4.12), so we could omit it.

Proof. By the the universal property of tensor maps of Theorem 4.8 and by Theorem 4.12, there is a unique linear map $\tilde{f} : V^{\otimes r} \to W$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_r) = f(v_1, \ldots, f_r)$. Since f is alternating, $\tilde{f}$ vanishes on the subspace U from Definition 4.27. Hence it induces $f' : V^{\otimes r}/U \to W$ is as desired.

$$\begin{tikzcd} V^r \arrow[r, "\otimes"] \arrow[dr, "f"'] & V^{\otimes r} \arrow[r, "\text{projection}"] \arrow[d, "\tilde{f}"] & \bigwedge^r(V) \arrow[dl, "f'"] \\ & W & \end{tikzcd}$$

This induced map vanishes on the subspace U (spanned by all the tensors with repeated entry), hence $f = f' \circ \wedge^r$.

□

Corollary 4.30. *The space of r-linear alternating maps $\mathrm{Alt}^r(V; W) = \{f : V^r \to W : f \text{ is alternating}\}$ is isomorphic to $\mathrm{Hom}(\wedge^r(V), W)$. In particular, if $W = K$, this yields an isomorphism $(\wedge^r(V))^* \cong \mathrm{Alt}^r(V; K)$.*

Similarly as for $r = 2$, one can show the following:

Theorem 4.31. *Let $\{e_i\}_{i=1,\ldots,n}$ be a basis for V and $r \geq 0$. Then $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}\}_{i_1 < \cdots < i_r}$ is a basis for $\bigwedge^r(V)$. In particular, $\dim \bigwedge^r(V) = \binom{n}{r}$.*

Proof. The claims can be shown analoguosly as the statements of Theorem 4.26. □

Example 4.32. Let V be a vector space with basis $\{v_1, v_2, v_3, v_4\}$. Then

	basis
$\bigwedge^0(V)$	$\{1\}$
$\bigwedge^1(V)$	$\{v_1, v_2, v_3, v_4\}$
$\bigwedge^2(V)$	$\{v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4, v_3 \wedge v_4\}$
$\bigwedge^3(V)$	$\{v_1 \wedge v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_4, v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4\}$
$\bigwedge^4(V)$	$\{v_1 \wedge v_2 \wedge v_3 \wedge v_4\}$

We now consider special values of r more generally (using Theorem 4.31):

- $\bigwedge^0(V) = K$, since the "empty" tensor product is K.
- $\bigwedge^1(V) = V$.
- $\bigwedge^n(V)$ is 1-dimensional, spanned by $e_1 \wedge \cdots \wedge e_n$ for any basis $\{e_i\}_i$ of V.
- $\dim \bigwedge^r(V) = 0$ if $r > n$.

Remark 4.33. Note that given any linear map $T : V \to W$ between two vector spaces, we get a linear map $\bigwedge^r T : \bigwedge^r(V) \to \bigwedge^r(W)$ by setting $\bigwedge^r T(v_1 \wedge \cdots \wedge v_r) = T(v_1) \wedge \cdots \wedge T(v_r)$ (and extending linearly to $\bigwedge^r(V)$). To see that this is well-defined, we note that this map is multilinear and vanishes if there exist $i \neq j$ with $v_i = v_j$. This can be extended to a map $\bigwedge(T) : \bigwedge(V) \to \bigwedge(W)$, by setting $\bigwedge(T)(x_1 \wedge \cdots \wedge x_t) = T(x_1) \wedge \cdots \wedge T(x_t)$ for any $x_1, \ldots, x_t$ in V (any $t \geq 0$). (The map $\bigwedge(T)$ is a homomorphism of graded K-algebras.)

Now in particular, we can consider the case $W = V$ and $r = n = \dim V$. In this case we have a map T from V to itself. What is T doing to (n-dimensional) volume in V? The space $\bigwedge^n(V)$ is 1-dimensional, so $\bigwedge^n(T)$ is a linear map from a 1-dimensional space to itself. Any such map is multiplication by some scalar, and this is independent of the choice of basis of V. This scalar is what T multiplies volumes by, from a geometric point of view.

Definition 4.34. Let $T : V \to V$ be a linear map and $n = \dim V$. Then the *determinant* $\det(T)$ is the scalar λ such that $\bigwedge^n T$ is multiplication by $\det(T) = \lambda$.

Recall that the exterior algebra of V is $\bigwedge(V) = \oplus_{m \geq 0} \bigwedge^m(V)$. It is graded by the degree of alternating tensors.

Since V is finite dimensional, say of dimension d, we actually have

$$\bigwedge(V) = \bigoplus_{m=0}^{d} \bigwedge^m(V)$$

and since each $\bigwedge^m(V)$ has dimension $\binom{d}{m}$, we deduce that

$$\dim(\bigwedge(V)) = \sum_{m=0}^{d} \binom{d}{m} = 2^d = 2^{\dim(V)}.$$

[Lecture 7, 16.12. 2016]

Remark 4.35. There is a linear map $\mu' : \bigwedge^r(V) \otimes \bigwedge^s(V) \to \bigwedge^{r+s}(V)$ induced by the (bilinear) multiplication map $\mu : \bigwedge^r(V) \times \bigwedge^s(V) \to \bigwedge^{r+s}(V)$ (cf. Definition 4.7), defined on pairs of decomposable alternating tensors as follows

$$\mu(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_s) = v_1 \wedge \cdots \wedge v_s \wedge w_1 \wedge \cdots \wedge w_s$$

and extended to all of $\bigwedge^r(V) \times \bigwedge^s(V)$. We write $\mu(x, y)$ (with $x \in \bigwedge^r(V)$, $y \in \bigwedge^s(V)$) as $x \wedge y$. :

The multiplication

$$\bigwedge^r(V) \times \bigwedge^s(V) \to \bigwedge^{r+s}(V)$$

is skew-symmetric in the following sense:

Theorem 4.36. *For all $\alpha \in \bigwedge^r(V)$ and for all $\beta \in \bigwedge^s(V)$, we have*

$$\beta \wedge \alpha = (-1)^{rs} \alpha \wedge \beta \qquad (in \bigwedge^{r+s}(V))$$

Proof. Since $v \wedge u = -u \wedge v$ for all $u, v \in V$, the claim follows by induction. □

Remark 4.37. Note that $\alpha \wedge \alpha = 0$ for every simple tensor $\alpha = u_1 \wedge \cdots \wedge u_n$. What about $\alpha \wedge \alpha = 0$ for an arbitrary element $\alpha \in \bigwedge(V)$?. If $\dim(V) \leq 3$, one checks that $\alpha \wedge \alpha = 0$ for all $\alpha \in \bigwedge(V)$ (please convince yourself). In dimension 4, this is not true anymore, e.g. if $\{u_1, u_2, u_3, u_4\}$ is a basis for V, and if $\alpha = u_1 \wedge u_2 + u_3 \wedge u_4$, we have

$$\begin{aligned} \alpha \wedge \alpha =& (u_1 \wedge u_2 + u_3 \wedge u_4) \wedge (u_1 \wedge u_2 + u_3 \wedge u_4) \\ =& \cdots = u_1 \wedge u_2 \wedge u_3 \wedge u_4 + u_3 \wedge u_4 \wedge u_1 \wedge u_2 \\ =& 2\, u_1 \wedge u_2 \wedge u_3 \wedge u_4 \end{aligned}$$

[Lecture 6, 12.12. 2016]

Pairings

Definition 4.38. Let V and W be vector spaces. A *pairing* is a bilinear map $\langle\, , \rangle : V \times W \to K$. The pairing is *non-degenerate* if

$$\begin{aligned} \langle v_0, w \rangle = 0 \ \forall w \ \in W \implies v_0 = 0 \\ \langle v, w_0 \rangle = 0 \ \forall v \ \in V \implies w_0 = 0 \end{aligned}$$

An example for a non-degenerate pairing is the so-called *evaluation map*:

$$V^* \times V \to K, \quad (f, v) \mapsto f(v)$$

Theorem 4.39. *If $b : V \times W \to K$, $(v, w) \mapsto b(v, w)$, is a non-degenerate pairing, then $V \cong W^*$ and $W \cong V^*$*

Proof. Define the map $f : V \to W^*$ as follows

$$(f(v))(w) = b(v, w)$$

This map is linear and

$$\ker f = \{v \in V : f(v) = 0\} = \{v \in V : b(v, w) = 0 \ \forall \ w \ \in W\} = \{0\}.$$

(Note that $f(v) = 0$ means that $f(v)$ is the zero map.) The last equality follows from the non-degeneracy of the pairing. So $\dim V \leq \dim W^* = \dim W$. Similarly, define a linear map $g : W \to V^*$ by $g(w)(v) = b(v, w)$. Its kernel is also 0. So $\dim W \leq \dim V^*$, therefore $\dim V^* = \dim V = \dim W^* = \dim W$ and we get that f and g are isomorphisms. ☐

Corollary 4.40. *Let* $1 \leq k \leq n$. *The map* $\langle \cdot, \cdot \rangle$*:* $\bigwedge^k(V^*) \times \bigwedge^k(V) \to K$ *given by*

$$\langle v_1^* \wedge \cdots \wedge v_k^*, v_1 \wedge \cdots \wedge v_k \rangle = \det\left(v_i^*(v_j)\right)$$

gives a non-degenerate pairing, hence

$$\bigwedge^k(V^*) \cong \bigwedge^k(V)$$

(The pairing is defined in simple alternating tensors - and then extended to all of $\bigwedge^k(V^*) \times \bigwedge^k(V)$)

Proof. Let $b : (V^*)^k \times V^k \to K$ be (the $2k$-linear map) defined (on simple elements) by

$$b(v_1^*, \ldots, v_k^*, v_1, \ldots, v_k) = \det(v_i^*(v_j)_{ij})$$

For fixed $(v_1^*, \ldots, v_k^*)$ in $(V^*)^k$, b is alternating in the v_j's. So there is a map $\overline{b} : (V^*)^k \times \bigwedge^k(V) \to K$ with

$$(v_1^*, \ldots, v_k^*, v_1 \wedge \cdots \wedge v_k) \mapsto \det(l_i(v_j))$$

(Theorem 4.29). Similarly, for fixed $v_1 \wedge \cdots \wedge v_k$ in $\bigwedge^k(V)$, $\overline{b}$ is alternating in the v_i^*'s (and k-linear), so there exists a pairing $b' : \bigwedge^k(V^*) \times \bigwedge^k(V) \to K$.
To check non-degeneracy, evaluate the pairing on the respective bases. The claim then follows from Theorem 4.39. ☐

Remark 4.41. Explicitely, every element $v_1^* \wedge \cdots \wedge v_k^* \in \bigwedge^k(V^*)$ defines a k-linear alternating map by setting

$$v_1^* \wedge \cdots \wedge v_k^* \, (v_1, \ldots, v_k) := \det(l_i(v_j)_{ij})$$

for all $(v_1, \ldots, v_k) \in V^k$. In particular,

$$v_1^* \wedge v_2^* \, (v_1, v_2) = v_1^*(v_1)v_2^*(v_2) - v_1^*(v_2)v_2^*(v_1).$$

Exercises for Section 4.6

1. Identify $\bigwedge^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $e_1 \wedge e_2$ with e_3, $e_2 \wedge e_3$ with e_1 and $e_3 \wedge e_1$ with e_2. Show that under this identification, the exterior product $v \wedge w \in \bigwedge^2(\mathbb{R}^3) = \mathbb{R}^3$ is the same as the cross product $u \times w \in \mathbb{R}^3$.

2. Let V have basis $\{e_1, e_2\}$ and let $T : V \to V$ be given by $T(e_1) = ae_1 + ce_2$ and $T(e_2) = be_1 + de_2$. Compute $\bigwedge^2 T : \bigwedge^2(V) \to \bigwedge^2(V)$ in terms of this basis. What is $\det(T)$?

3. Let $\mathrm{Sym}^2(V)$ be the quotient of $V \otimes V$ by the subspace spanned by elements of the form $v \otimes w - w \otimes v$. We write vw for the image of $v \otimes w$ under the quotient map $V \otimes V \to \mathrm{Sym}^2(V)$. If $\{e_i\}$ is a basis for V, show that $\{e_i e_j\}_{i \leq j}$ is a basis for $\mathrm{Sym}^2(V)$.
   Hint: imitate the proof of theorem 4.26

4. Let $x \in \bigwedge^r(V)$, $y \in \bigwedge^s(V)$ and $z \in \bigwedge^t(V)$. Show that $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and that $x \wedge y = (-1)^{rs} y \wedge x$ where $x \wedge y$ is defined as in Lemma 4.35.

5. For any vector space V, the vectors $u_1, \ldots, u_n \in V$ are linearly independent iff $u_1 \wedge \cdots \wedge u_n \neq 0$.

4.7 Reminder from linear algebra

4.7.1 Reminder on inner products

Let V be a vector space over $\mathbb{C}$. An inner product on V is a function $(\cdot,\cdot)\colon V \times V \to \mathbb{C}$ such that
1. $(u, v) = \overline{(v, u)}$ for all $u, v \in V$.
2. $(\alpha_1 v_1 + \alpha_2 v_2, u) = \alpha_1 (v_1, u) + \alpha_2 (v_2, u)$ for all $v_i, u \in V$, $\alpha_i \in \mathbb{C}$.
3. $(v, v) \geq 0$ for all $v \in V$ and $(v, v) = 0$ if and only if $v = 0$.

V is then called an inner product space. The *norm* induced by the inner product is defined as

$$\|v\| = \sqrt{(v, v)}, \quad v \in V$$

Vectors v with $\|v\| = 1$ are *unit vectors.* Two vectors $u, v \in V$ are *orthogonal* if $(u, v) = 0$, denoted by $u \perp v$. A basis $E = \{e_1, \ldots, e_n\}$ is an *orthogonal basis*, if the vectors are pairwise orthogonal. It is *orthonormal*, if $(e_i, e_j) = \delta_{ij}$.
Fact:
Let $F := \{f_1, \ldots, f_n\}$ be a basis of the $\mathbb{C}$-vector space V. Then there exists a unique inner product $(\cdot,\cdot)$ on V such that F is an orthonormal basis. (cf. Problem 7.3 (a))

Proof of this fact: Let $(\cdot,\cdot)$ be an inner product with ONB $E = \{e_1, \ldots, e_n\}$. Define $S \in \mathrm{End}V$ by $Sf_i = e_i$. The endomorphism S is invertible. (Why? please give an argument - e.g. by definition, S has rank $n = \dim V$). Set $T := S^*S > 0$. (positive semi-definiteness, i.e. $S^*S \geq 0$, holds for any homomorphism between inner produc spaces. Positive definiteness, i.e. $S^*S > 0$, holds for invertible homomorphisms. View S as a square matrix. Then the adjoint S^* of S is the complex conjugate transpose of S.) So $\langle u, v\rangle := (Tu, v)$ is an inner product. (Fact: For $T \in \mathrm{End}\ V$, $\langle u, v\rangle := (Tu, v)$ defines an inner product if and only if T is pos. definite w.r.t. $(\cdot,\cdot)$). The elements $f_1, \ldots, f_n$ form an ONB w.r.t. $\langle\cdot,\cdot\rangle$. Uniqueness: straightforward.

Exercise:
Let $E = \{e_1, \ldots, e_n\}$ be a basis of V. For any $u = \sum_{i=1}^n a_i e_i$ and $v = \sum_{i=1}^n b_i e_i$, show that $(u, v) := \sum_{i=1}^n a_i \overline{b_i}$ is the unique inner product on V so that E is an orthonormal basis (cf. Problem 7.3 (b)).

4.7.2 Reminder on adjoints

Let V, W be inner product spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$ respectively. For each $T \in \mathrm{Hom}(V, W)$, the *adjoint* of T is $S \in \mathrm{Hom}(W, V)$ such that $(Tv, w)_W = (v, Sw)_V$ for all $v \in V$, $w \in W$ and is denoted by T^*. Clearly, $(T^*)^* = T$.
Fact:
Let W, V be inner product spaces. Each $T \in \mathrm{Hom}(V, W)$ has a unique adjoint.

: we can find an ONB w.r.t. $(\cdot,\cdot)_V$. Let $E = \{e_1, \ldots, e_n\}$ be an ONB of V. Then for $w \in W$, one defines $S \in \mathrm{Hom}(W, V)$ by

$$Sw := \sum_{i=1}^n (w, Te_i)_W e_i.$$

etc. And show uniqueness.

And then another fact:
If E and $F = \{f_1, \ldots, f_m\}$ are ONB's of the inner product spaces V and W and $T \in \mathrm{Hom}(V, W)$ with matrix A representing T w.r.t. the bases E and F, then T^* is represented by A^* (the complex conjugate transpose of the matrix A), a matrix w.r.t. the bases F and E.

Notation for Chapters 4 and 5

- Let V and W be vector spaces over the field K. We write $\mathrm{Hom}(V, W)$ for the space of K-linear maps between V and W (or *Homomorphisms between* V *and* W). We write $\mathrm{End}(V)$ for $\mathrm{Hom}(V, V)$, the space of *endomorphisms of* V.
- $\mathbb{C}_{m,n}$ the space of $m \times n$-matrices over $\mathbb{C}$. If R is a ring (with unit), $R_{m,n}$ is the space of $m \times n$-matrices over R.
- $x \in V$, V vector space: x^T the transpose of x (similar for matrices).
- $[x_1, \ldots, x_n]$ for the matrix formed by the column vectors $x_i \in \mathbb{C}^m$.
- $V^m = \times^m(V) = V \times \cdots \times V$ for the product of m factors of the vector space V.
- $M(V_1, \ldots, V_m, W)$ the set of multilinear maps from $V_1 \times \cdots \times V_m$ to W.
- The space of r-linear alternating maps, $\mathrm{Alt}^r(V; W) := \{f : V^r \to W : f \text{ is alternating}\}$.
- Let $V_1, \ldots, V_m$ be K-vector spaces of dimensions $n_1, \ldots, n_m$. Then we define $\Gamma(n_1, \ldots, n_m) := \{\gamma : \gamma = (\gamma(1), \ldots, \gamma(m)) \mid 1 \leq \gamma(i) \leq n_i, i = 1, \ldots, m\}$, sometimes abbreviated as Γ. This notation helps enumerating basis vectors of the tensor product of m vector spaces of dimensions $n_1, \ldots, n_m$.
  If $V_1 = \cdots = V_m =: V$ we have $n_1 = \cdots = n_m = n$ for $n := \dim V$; in this case we just write $\Gamma(n : m)$:
  $\Gamma(n : m) = \{\gamma : \gamma = (\gamma(1), \ldots, \gamma(m)), 1 \leq \gamma(i) \leq n, i = 1, \ldots, m\}$.
- Consider $V_1 \otimes \cdots \otimes V_m$ for $\mathbb{C}$-vector spaces V_i of dimension n_i, and where $E_i = \{e_{i1}, \ldots, e_{in_i}\}$ is a basis of V_i for $i = 1, \ldots, m$. Let $\gamma \in \Gamma = \Gamma(n_1, \ldots, n_m)$. We write
  $e_\gamma^\otimes$ for the basis vector $e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)}$.
  For $(u_1, \ldots, u_m) \in V_1 \times \cdots \times V_m$ we write
  $u^\otimes := u_1 \otimes \cdots \otimes u_m$ (which is, by definition, $\otimes(u_1, \ldots, u_m)$).
- $V^{\otimes m} = \otimes^m(V) = V \otimes \cdots \otimes V$ (m copies of V in the tensor product) and $(V^*)^{\otimes m} = \otimes^m(V^*) = V^* \otimes \cdots \otimes V^*$ (m copies of V^*).
- Let I be a set. We write $\mathcal{P}^*(I)$ for the set of all non-empty finite subsets of I.

- Let R be a ring (with unit), let M be a (left) R-module. We write $\mathcal{L}(M)$ for the lattice of submodules of M.
- $\mathbb{N} = \{0, 1, 2, 3, \dots\} = \mathbb{Z}_{\geq 0}$.

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