DISCRETE AND ALGEBRAIC STRUCTURES

Master Study Mathematics

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Edited by Mihyun Kang

TU Graz

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Part I

Lectures by Mihyun Kang

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Standard Methods of Enumerative Combinatorics

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1.2.1 Recursions

Example 1. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0 \end{cases}$$

The sequence begins with $0, 1, 3, 7, 15, 31, \ldots$

We add 1 in both side of the latter equation to obtain

$$a_{n+1} + 1 = 2(a_n + 1), \quad n \geq 0.$$

Then we have $a_n + 1 = 2^n(a_0 + 1) = 2^n$ *and therefore the sequence is explicitly given by*

$$a_n = 2^n - 1, \quad n \geq 0.$$

Example 2. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 &= 0 \\ a_1 &= 1 \\ a_{n+2} &= a_{n+1} + a_n, \quad n \geq 0. \end{cases}$$

This sequence is called Fibonacci sequence and begins with $0, 1, 1, 2, 3, 5, 8, 13, \ldots$.

How can we derive the explicit form of the sequence?

1.2.2 Linear recurrences

Definition 1. We say that a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfies a k-th order recurrence (or recursion) if a_n can be written as

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}),, \quad n \geq k$$

for a function $f : \mathbb{C}^k \to \mathbb{C}$.

First-order linear recurrences

Theorem 1. *A first-order linear recurrence with constant coefficients of the form*

$$\begin{cases} a_0 = 0 \\ a_n = c_n \cdot a_{n-1} + d_n, \quad n \geq 1, \end{cases}$$

with $c_i \neq 0$ *has an explicit solution,*

$$a_n = d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n.$$

Proof. Divide both sides by $c_n c_{n-1} \cdots c_1$:

$$\frac{a_n}{c_n c_{n-1} \cdots c_1} = \frac{a_{n-1}}{c_{n-1} c_{n-2} \cdots c_1} + \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Change of variables: let

$$b_n = \frac{a_n}{c_n c_{n-1} \cdots c_1}.$$

We get the difference relation

$$b_n = b_{n-1} + \frac{d_n}{c_n c_{n-1} \cdots c_1},$$

that is,

$$b_n - b_{n-1} = \frac{d_n}{c_n c_{n-1} \cdots c_1}.$$

Summing up, we get

$$b_n = \sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1}.$$

Thus, for a_n we have

$$\begin{aligned} a_n &= c_n c_{n-1} \cdots c_1 \cdot \left(\sum_{i=1}^{n} \frac{d_i}{c_i c_{i-1} \cdots c_1} \right) \\ &= d_n + \sum_{i=1}^{n-1} d_i c_{i+1} c_{i+2} \cdots c_n. \end{aligned}$$

□

Example 3. *Returning to Example 1 we have*

$$\begin{cases} a_0 &= 0 \\ a_{n+1} &= 2a_n + 1, \quad n \geq 0, \end{cases}$$

so taking $c_i = 2, d_i = 1$ in Theorem 1 we have

$$a_n = 1 + \sum_{i=1}^{n-1} 2^{n-i} = 1 + 2^n \left(1 - \left(\frac{1}{2} \right)^{n-1} \right) = 2^n - 1,$$

because $\sum_{i=1}^{n} r^i = \frac{r(1-r^n)}{1-r}$ for any $r \neq 1$.

Higher order linear recurrences with constant coefficients

Theorem 2. *All solutions to the k-th order linear recurrence with constant coefficients*

$$a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k}, \quad n \geq k,$$

can be expressed as linear combinations of terms of the form

$$\alpha^n, n\alpha^n, n^2\alpha^n, \ldots, n^{m-1}\alpha^n$$

where α is a root of order m of the characteristic polynomial

$$q(z) = z^k - c_1 \cdot z^{k-1} - c_2 \cdot z^{k-2} - \ldots - c_k.$$

Remark 1. *If $q(z) = (z-\alpha)^m \cdot p(z)$ for some polynomial $p(z)$ where $(z-\alpha) \nmid p(z)$, i.e. α is a root of multiplicity m, then $q(\alpha) = q'(\alpha) = \ldots = q^{(m-1)}(\alpha) = 0$.*

Proof. Let α be a simple root of $q(z)$ (i.e. the multiplicity is 1) and $a_n = \alpha^n$. We want to check that a_n is a solution to the recurrence, i.e.

$$\begin{aligned} 0 &\stackrel{!}{=} \alpha^n - c_1\alpha^{n-1} - c_2\alpha^{n-2} - \ldots - c_k\alpha^{n-k} \\ &= \alpha^{n-k} \cdot \left(\alpha^k - c_1\alpha^{k-1} - \ldots - c_k\right) \\ &= \alpha^{n-k} \cdot q(\alpha) = 0. \end{aligned}$$

Let α now be a root of multiplicity of m of $q(z)$. Then, for $0 \le i \le m-1$, $a_n = n^i\alpha^n$ is a solution to the recurrence because

$$\begin{aligned} 0 &\stackrel{!}{=} a_n - c_1 \cdot a_{n-1} - c_2 \cdot a_{n-2} - \ldots - c_k a_{n-k} \\ &= n^i\alpha^n - c_1(n-1)^i\alpha^{n-1} - c_2(n-2)^i\alpha^{n-2} - \ldots c_k(n-k)^i\alpha^{n-k} \\ &= \alpha^{n-k} \cdot \left(n^i\alpha^k - c_1(n-1)^i\alpha^{k-1} - \ldots - c_k(n-k)^i\right) \\ &= \alpha^{n-k} \cdot \big((n-k)^i q(\alpha) + \alpha\left((n-i)^i - (n-k-1)^i\right)q'(\alpha) + \\ &\qquad \alpha^2\left(b_0(n-k)^i + b_1(n-k-1)^i + b_2(n-k-2)^i\right)q''(\alpha) + \ldots\big) \\ &= \alpha^{n-k} \cdot \left(\sum_{0\le j\le i} \cdot \left(\sum_{0\le \ell\le j} b_{j,0}(n-k-\ell)^i\right) q^{(j)}(\alpha)\right) = 0, \end{aligned}$$

because of the previous remark; the $b_{j,\ell}$ are constants.

Furthermore, a linear combination of $\alpha^n, n\cdot\alpha^n, \ldots, n^{m-1}\alpha^n$ is also a solution to the recurrence.

If $q(z)$ has distinct roots $\alpha_1, \ldots, \alpha_s$ with multiplicities $m_1, \ldots, m_j$ where $m_1 + \ldots + m_s = k$, then a linear combination of all these $n^j \cdot \alpha_i^n$ for $0 \le j \le m-1$, $1 \le i \le s$, i.e.

$$a_n = \sum_{i=1}^{s} \left(b_{i,0} + b_{i,1}n + \ldots + b_{i,m-1}n^{m_i-1}\right) \cdot \alpha_i^n$$

is also a solution to the recurrence.

We claim the opposite is also true. Let s be the set of sequences $\{a = (a_n \in \mathbb{R})_{n\ge 0}\}$ and R be the set of solutions to the recurrence. Then R is closed under addition and scalar multiplication, and $R \neq \emptyset$. Therefore, R is a vector space.

We claim that R has dimension k. Consider a map f from R to $\mathbb{R}^k$:

$$f: R \to \mathbb{R}^k, \quad a \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix}.$$

This is a linear map and also an isomorphism, because any solution to the recurrence is uniquely determined by the k initial values. So, R has dimension k.

Therefore, R is given (generated) by linear combinations of any k linearly independent solutions to the recurrence. Now it suffices to show that the set of k solutions $\{n^j \alpha_i^n \mid 1 \le j \le m_i - 1, 1 \le i \le s\}$ to the recurrence is linearly independent. But this is true because these solutions have different orders of growth (in particular at ∞). $\square$

Example 4. *Returning to Example 2 we consider the Fibonacci sequence*

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, & n \ge 2 \\ a_0 = 0, a_1 = 1. \end{cases}$$

Its characteristic polynomial is

$$q(z) = z^2 - z - 1 = \left(z - \frac{1+\sqrt{5}}{2}\right) \cdot \left(z - \frac{1-\sqrt{5}}{2}\right),$$

and the solution to the recurrence for a_n is

$$\begin{cases} a_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \\ a_0 = 0, a_1 = 1. \end{cases}$$

From the initial conditions, we get $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = -\frac{1}{\sqrt{5}}$. The explicit solution to (**??**) *is*

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Example 5. *Consider the second order linear recurrence*

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad n \ge 2$$

with the initial conditions $a_0 = 0, a_1 = 1$.

The characteristic polynomial is

$$q(z) = z^2 - 5z + 6 = (z-2)(z-3).$$

The solution will be of the form

$$a_n = c_1 2^n + c_2 3^n.$$

Due to the initial conditions, $c_1 = -1$ and $c_2 = 1$ and the solution is

$$a_n = -2^n + 3^n.$$

This can be solved in Maple as follows:

```
rsolve({a(n) = 5 * a(n-1) - 6 * a(n-2), a(0) = 0, a(1) = 1}, a(n));
```

In Mathematica:

```
RSolve[{a[n] == 5 * a[n-1] - 6 * a[n-2], a[0] == 0, a[1] == 1}, a[n], n]
```

Exercise 1. *Find initial conditions a_0, a_1, a_2 for which the growth rate of the solution to the recurrence*

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n \ge 3$$

is (a) constant, (b) exponential, and (c) fluctuating in sign.

Higher order linear recurrences with non-constant coefficients

Example 6. *Consider the recurrence*

$$\begin{cases} a_n = na_{n-1} + n(n-1)a_{n-2}, & n \geq 2 \\ a_0 = 0, a_1 = 1. \end{cases}$$

Divide by $n!$ to receive

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + \frac{a_{n-2}}{(n-2)!}.$$

Change variables: let $b_n = \frac{a_n}{n!}$. Then we get the recurrence

$$\begin{cases} b_n = b_{n-1} + b_{n-2}, & n \geq 2 \\ b_0 = 0, b_1 = 1. \end{cases}$$

for the Fibonacci sequence. From Example 4 we have

$$b_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

This yields

$$a_n = \frac{n!}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Exercise 2. *Solve the recurrence*

$$\begin{cases} n(n-1)a_n = (n-1)a_{n-1} + a_{n-2}, & n \geq 2 \\ a_0 = 0, \ a_1 = 1. \end{cases}$$

1.2.3 Non-linear recurrences

Example 7. *Consider the second order non-linear recurrence*

$$\begin{cases} a_n = \sqrt{a_{n-1} \cdot a_{n-2}}, & n \geq 2 \\ a_0 = 1, \ a_1 = 2. \end{cases}$$

We use the logarithm function, since

$$\log_a xy = \log_a x + \log_a y.$$

Let $b_n = \log a_n$. We get

$$\begin{cases} b_n = \frac{1}{2}\left(b_{n-1} + b_{n-2}\right), & n \geq 2 \\ b_0 = 0, \ b_1 = 1. \end{cases}$$

This can be solved similar to the previous linear recurrences.

1.3 Generating Functions

1.3.1 Formal Power Series

Now let us return to Example 1.

Example 8. *Let a sequence* $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ *of numbers be given by a recursion*

$$\begin{cases} a_0 & = \ 0 \\ a_{n+1} & = \ 2a_n + 1, \quad n \geq 0 \end{cases} \tag{1.1}$$

and let z *be a formal indeterminate variable.*

Multiplying the both side of (1.1) *by* z^{n+1} *and summing over* $n \in \mathbb{N}\cup\{0\}$, *we obtain*

$$\sum_{n=0}^{\infty} a_{n+1} z^{n+1} \ = \ \sum_{n=0}^{\infty} (2a_n + 1)\ z^{n+1}.$$

If it were true that $\sum_{n=0}^{\infty}(2a_n+1)\ z^{n+1} \ = \ 2z\ \sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n$, *we would obtain*

$$\sum_{n=1}^{\infty} a_n z^n \ = \ 2z\ \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n$$

Defining $A(z) := \sum_{n=0}^{\infty} a_n\ z^n$ *we have*

$$A(z) \ = \ 2zA(z) + z\sum_{n=0}^{\infty} z^n \quad \textit{equiv.} \quad (1-2z)A(z) \ = \ z\sum_{n=0}^{\infty} z^n$$

because $a_0 = 0$. *From this we we would further obtain*

$$\begin{aligned} A(z) \ &\stackrel{?}{=} \ \frac{z}{1-2z} \sum_{n=0}^{\infty} z^n \ \stackrel{?}{=} \ \frac{z}{1-2z} \cdot \frac{1}{1-z} \ = \ 2z\ \frac{1}{1-2z} \ - \ z\ \frac{1}{1-z} \\ &\stackrel{?}{=} \ 2z\ \sum_{n=0}^{\infty} (2z)^n \ - \ z\ \sum_{n=0}^{\infty} z^n \\ &\stackrel{?}{=} \ \sum_{n=1}^{\infty} (2z)^n \ - \ \sum_{n=1}^{\infty} z^n \\ &\stackrel{?}{=} \ \sum_{n=1}^{\infty} (2^n - 1)\ z^n \\ &= \ \sum_{n=0}^{\infty} (2^n - 1)\ z^n, \end{aligned}$$

if the following were true:

$$(1-2z)A(z) = z\sum_{n=0}^{\infty} z^n \quad \overset{?}{\Longleftrightarrow} \quad A(z) = \frac{z}{1-2z}\sum_{n=0}^{\infty} z^n \tag{1.2}$$

$$\sum_{n=0}^{\infty}(az)^n \overset{?}{=} \frac{1}{1-az}, \quad a \in \mathbb{K} \tag{1.3}$$

$$\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} \overset{?}{=} 2z\sum_{n=0}^{\infty} a_n z^n + z\sum_{n=0}^{\infty} z^n. \tag{1.4}$$

Summing up, we would have $\sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty}(2^n-1)\, z^n$ *and therefore* $a_n = 2^n - 1$ *for every* $n \in \mathbb{N} \cup \{0\}$*, by comparing the coefficients, if* (1.2)–(1.4) *were true. When are they true?*

Definition 2 (Formal Power Series)**.** Let $\mathbb{K}$ denote a commutative ring, usually we take $\mathbb{K} = \mathbb{C}$. Given a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ with $a_n \in \mathbb{K}$ and a formal variable z, we call an infinite sum of the form

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{n\geq 0} a_n\, z^n$$

a *formal power series*. We call a_n the coefficient of z^n in $f(z)$ and use the notation

$$[z^n]\, A(z) := a_n.$$

The ring of formal power series is denoted by $\mathbb{K}[[z]]$ and endowed with the operations of addition and product:

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) + \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty}(a_n+b_n)\, z^n \tag{1.5}$$

$$\left(\sum_{n=0}^{\infty} a_n\, z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\, z^n\right) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \tag{1.6}$$

We often skip $\cdot$ for the product of two formal power series. As a special case of (1.6) we have, for any $a \in \mathbb{K}$,

$$a\left(\sum_{n=0}^{\infty} b_n\, z^n\right) = \sum_{n=0}^{\infty} a\, b_n\, z^n. \tag{1.7}$$

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ we also define the following algebraic operations:

- Differentiation: $A'(z) := \sum_{n\geq 1} n a_n\, z^{n-1} = \sum_{n\geq 0}(n+1)a_{n+1}\, z^n$.
- Integration: $\int_0^z A(t)\,\mathrm{d}t := \sum_{n\geq 0} \frac{a_n}{n+1}\, z^{n+1} = \sum_{n\geq 1} \frac{a_{n-1}}{n}\, z^n$.

A formal power series $B(z) := \sum_{n=0}^{\infty} b_n\, z^n$ is called a *reciprocal* of a formal power series $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$ (and vice versa) if $A(z)B(z) = B(z)A(z) = 1$.

Propostion 1. *A formal power series $A(z) := \sum_{n=0}^{\infty} a_n\ z^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case, the reciprocal is unique and so we denote the reciprocal of $A(z)$ by $\frac{1}{A(z)}$.*

Proof. Exercise! □

In Example 8 we wished to have

$$(1-2z)A(z) = 1 \quad \overset{?}{\Longleftrightarrow} \quad A(z) = \frac{1}{1-2z}$$

$$\sum_{n=0}^{\infty} z^n \overset{?}{=} \frac{1}{1-z},$$

$$\sum_{n=0}^{\infty} (2a_n+1)\ z^{n+1} \overset{?}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n.$$

From definition, equality $(1-2z)A(z) = 1$ and Proposition 1, $A(z)$ is the (unique) reciprocal of $1-2z$ and vice versa, so

$$(1-2z)A(z) = 1 \quad \Longleftrightarrow \quad A(z) = \frac{1}{1-2z}.$$

For the second equality $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, we shall show that the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} z^n$ is $B(z) := 1-z$ and vice versa. Letting $A(z) := \sum_{n=0}^{\infty} a_n\ z^n$ with $a_n = 1$ for $n \geq 0$ and $B(z) := \sum_{n=0}^{\infty} b_n\ z^n$ with $b_0 = 1, b_1 = -1$ and $b_n = 0$ for $n \geq 2$, we have

$$A(z)B(z) \overset{(1.6)}{=} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n = a_0 b_0 + \sum_{n=1}^{\infty} (a_{n-1} b_1 + a_n b_0)\ z^n = 1.$$

By proposition 1, $A(z)$ is the unique reciprocal of $B(z)$ and so $A(z) = \frac{1}{B(z)} = \frac{1}{1-z}$, that is, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

in the ring $\mathbb{K}[[z]]$.

Analogously one can show that for any $a \in \mathbb{K}$, the reciprocal of the formal power series $A(z) := \sum_{n=0}^{\infty} (az)^n$ is $B(z) := 1-az$ and vice versa. In other words, for any $a \in \mathbb{K}$,

$$\sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az} \tag{1.8}$$

in the ring $\mathbb{K}[[z]]$.

For the third equality $\sum_{n=0}^{\infty} (2a_n+1)\ z^{n+1} = 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n$, we use (1.5) and (1.6) to have

$$\sum_{n=0}^{\infty} (2a_n+1)\ z^{n+1} \overset{(1.5)}{=} \sum_{n=0}^{\infty} 2a_n\ z^{n+1} + \sum_{n=0}^{\infty} z^{n+1} \overset{(1.6)}{=} 2z \sum_{n=0}^{\infty} a_n z^n + z \sum_{n=0}^{\infty} z^n.$$

That is, $\sum_{n=0}^{\infty}(2a_n+1)\, z^{n+1} = 2z\, \sum_{n=0}^{\infty} a_n z^n + z\, \sum_{n=0}^{\infty} z^n$ in the ring $\mathbb{K}[[z]]$.

Given a formal power series $A(z) = \sum_{n\geq 0} a_n\, z^n$ and a constant $\beta \in \mathbb{K}$ we have the following properties:

- $A(\beta z) = \sum_{n\geq 0} a_n \beta^n\, z^n, \quad$ so $[z^n]\, A(\beta z) = a_n\beta^n = \beta^n a_n = \beta^n\, [z^n]\, A(z)$ (scaling)
- $(A(z)-a_0)/z = \sum_{n\geq 1} a_n\, z^{n-1} = \sum_{n\geq 0} a_{n+1}\, z^n$ (left shift)
- $z\, A(z) = \sum_{n\geq 0} a_n\, z^{n+1} = \sum_{n\geq 1} a_{n-1}\, z^n$ (right shift)
- $A(z)/(1-z) = A(z)\cdot \sum_{n\geq 0}\, z^n = \sum_{n\geq 0}\left(\sum_{0\leq k\leq n} a_k\right) z^n$ (partial sum)

Example 9. *Let us study the recursion for the Fibonacci sequence*

$$\begin{cases} a_0 & = \; 0 \\ a_1 & = \; 1 \\ a_{n+2} & = a_{n+1} + a_n, \quad n \geq 0, \end{cases}$$

using the generating function $A(z) := \sum_{n=0}^{\infty} a_n\, z^n$. Multiplying the both side of the recursion by z^{n+2} and summing over $n \in \mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}\, z^{n+2} \;&=\; \sum_{n=0}^{\infty} (a_{n+1}+a_n)\, z^{n+2} \\
\iff \quad \sum_{n=2}^{\infty} a_n z^n \;&=\; z \sum_{n=1}^{\infty} a_n\, z^n \;+\; z^2 \sum_{n=0}^{\infty} a_n\, z^n \\
\iff \quad A(z) - z \;&=\; z\, A(z) \;+\; z^2\, A(z) \\
\iff \quad (1-z-z^2)\, A(z) \;&=\; z \\
\iff \quad A(z) \;=\; \frac{z}{1-z-z^2} \;&=\; \frac{1}{\beta_1-\beta_2}\left(\frac{1}{1-\beta_1 z} - \frac{1}{1-\beta_2 z}\right) \\
&\overset{(1.8)}{=}\; \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}(\beta_1 z)^n - \sum_{n=0}^{\infty}(\beta_2 z)^n\right) \\
&\overset{(1.7)}{=}\; \left(\sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\beta_1^n\, z^n - \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\beta_2^n\, z^n\right) \\
&\overset{(1.5)}{=}\; \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\, (\beta_1^n - \beta_2^n)\; z^n,
\end{aligned}$$

where $\beta_1 = \frac{1+\sqrt{5}}{2}$ and $\beta_2 = \frac{1-\sqrt{5}}{2}$. Therefore we have

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \quad n \geq 0.$$

1.3.2 Ordinary and Exponential Generating Functions

Throughout the lecture we take $\mathbb{K} = \mathbb{C}$.

Definition 3 (Generating Functions)**.** The *ordinary generating function* of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n.$$

The *exponential generating function* of a sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ is the formal power series

$$A(z) := \sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!}.$$

Definition 4 (Combinatorial Class, Counting Sequence and Generating Function)**.**

- A combinatorial class $\mathscr{A}$ is a finite or denumerable set on which a size function is defined such that the size $|\alpha|$ of an element α is a non-negative integer and the number of elements of any given size is finite.
- Given a combinatorial class $\mathscr{A}$ we denote by $\mathscr{A}_n$ the set of elements in $\mathscr{A}$ of size n for any $n \in \mathbb{N}\cup\{0\}$.
- The counting sequence of a combinatorial class $\mathscr{A}$ is the sequence $(a_n)_{n\in\mathbb{N}\cup\{0\}}$ where a_n is the number of elements in $\mathscr{A}_n$, which is often denoted by $|\mathscr{A}_n|$.
- The ordinary generating function of a combinatorial class $\mathscr{A}$ is the ordinary generating function of the sequence of numbers $a_n = |\mathscr{A}_n|$. Therefore we have

  $$A(z) := \sum_{n=0}^{\infty} a_n\, z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}.$$

  Notation: $[z^n]A(z) := a_n$.
- The exponential generating function of a combinatorial class $\mathscr{A}$ is the exponential generating function of the sequence of numbers $a_n = |\mathscr{A}_n|$. Therefore we have

  $$A(z) := \sum_{n=0}^{\infty} a_n\, \frac{z^n}{n!} = \sum_{\alpha\in\mathscr{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

  Notation: $[z^n]A(z) := \frac{a_n}{n!}$.

We say the variable z marks the size in the generating function $A(z)$.

Two combinatorial classes are said to be combinatorially equivalent if their counting sequences are identical.

Propostion 2. *For the product of two exponential generating functions we have*

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{z^n}{n!}.$$

Proof. Exercise! □

Example 10 (Triangulations of Convex Polygon)**.** *Let $\mathscr{T}$ denote the class of all triangulation of convex polygons with one distinguished edge, in which the size of a triangulation in $\mathscr{T}$ is defined as the number of triangles it is composed of. Then $\mathscr{T}$ is a combinatorial class. For $n \geq 1$ we let $\mathscr{T}_n$ denote the class of all triangulation of convex polygons of size n and let $t_n := |\mathscr{T}_n|$. Set $t_0 := 1$. That is, $\mathscr{T}_n$ is the set of all triangulations of convex $(n+2)$-gons (i.e. with n triangles) with one distinguished edge.*

The sequence $(t_n)_{n\in\mathbb{N}\cup\{0\}}$ begins with $1,1,2,5,14,42,\dots$

By deleting the triangle incident to the distinguished edge we obtain the recursion

$$t_n = \sum_{k=0}^{n-1} t_k t_{n-1-k}, \quad n \geq 1.$$

Let $T(z) := \sum_{n=0}^{\infty} t_n z^n$ be the ordinary generating function of $\mathscr{T}$. Multiplying the both side of $t_n = \sum_{k=0}^{n-1} t_k\, t_{n-1-k}$ by z^n and summing over $n \geq 1$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} t_n z^n &= \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n-1} t_k\, t_{n-1-k}\Big) z^n \\ \iff \quad T(z) - t_0 &= z \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{n-1} t_k\, t_{n-1-k}\Big) z^{n-1} \\ \iff \quad T(z) - 1 &= z \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} t_k\, t_{n-k}\Big) z^n \\ T(z) - 1 &\overset{(1.7)}{=} z\, T(z)^2. \end{aligned}$$

Therefore $T(z)$ satisfies the quadratic equation

$$z\, T(z)^2 - T(z) + 1 = 0,$$

among whose two solutions we choose

$$T(z) = \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative.

From this, we get for $n \geq 1$:

$$\begin{aligned} t_n = [z^n]T(z) &= -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} \\ &\overset{\textit{rescaling}}{=} -\frac{1}{2}(-4)^{n+1}\, [z^{n+1}](1+z)^{1/2}. \end{aligned}$$

Using the generalised binomial theorem

$$[z^{n+1}](1+z)^{1/2} = \binom{\frac{1}{2}}{n+1} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!} = \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}(n+1)!},$$

we obtain

$$\begin{aligned} t_n &= -\frac{1}{2}(-4)^{n+1}\ \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}(n+1)!} = \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2^n}{(n+1)!} \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2\cdot 4\cdot 6\cdots(2n)}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Using Stirling's formula

$$n! = \left(1+O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of binary trees

$$t_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\ \frac{(2n)!}{n!\,n!} \sim \frac{1}{\sqrt{\pi}}4^n n^{-3/2}.$$

1.4 Symbolic method

1.4.1 Unlabelled combinatorial objects

For a given combinatorial class $\mathscr{A}$, denote by $\mathscr{A}_n$ the set of elements of size n in $\mathscr{A}$, and let $a_n = |\mathscr{A}_n|$. We say the OGF

$$A(z) = \sum_{n\geq 0} a_n z^n = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|}$$

enumerates $\mathscr{A}$.

Basic constructions and OGF's

(1) $\mathscr{E}$ is the neutral class that consists of a single element of size 0. The OGF of $\mathscr{E}$ is 1.

(2) $\mathscr{Z}$ is the atomic class that consists of a single element of size 1. The OGF of $\mathscr{Z}$ is z.

(3) Combinatorial sum (disjoint union) $\mathscr{A}+\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A}\cap\mathscr{B}=\emptyset$ is the set of objects consisting of two disjoint copies of $\mathscr{A}$ and $\mathscr{B}$, in which the size of an element $\alpha\in\mathscr{A}+\mathscr{B}$ is defined as $|\alpha|_{\mathscr{A}}$ if $\alpha\in\mathscr{A}$ and $|\alpha|_{\mathscr{B}}$ if $\alpha\in\mathscr{B}$ (i.e. the size of an element in $\alpha\in\mathscr{A}+\mathscr{B}$ is inherited from its size in its class of origin). In order to formalise $\mathscr{A}+\mathscr{B}$ we introduce red marker to $\mathscr{A}$ and blue marker to $\mathscr{B}$. The combinatorial sum $\mathscr{A}+\mathscr{B}$ is a well-defined combinatorial class. Its OGF satisfies

$$\sum_{\alpha\in\mathscr{A}+\mathscr{B}} z^{|\alpha|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|} + \sum_{\alpha\in\mathscr{B}} z^{|\alpha|}.$$

(4) Cartesian product $\mathscr{A}\times\mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A}\times\mathscr{B} = \{(\alpha,\beta) \;\big|\; \alpha\in\mathscr{A}, \beta\in\mathscr{B}\},$$

in which the size of a pair (α,β) is defined as $|\alpha|+|\beta|$. The Cartesian product $\mathscr{A}\times\mathscr{B}$ is a well-defined combinatorial class. Its OGF satisfies

$$\sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta)\in\mathscr{A}\times\mathscr{B}} z^{|\alpha|+|\beta|} = \sum_{\alpha\in\mathscr{A}} z^{|\alpha|} + \sum_{\beta\in\mathscr{B}} z^{|\beta|} = A(z)\cdot B(z).$$

For any $n\geq 1$, we define $\mathscr{A}^n = \mathscr{A}^{n-1}\times\mathscr{A} = \mathscr{A}\times\cdots\times\mathscr{A}$ recursively.

(5) Sequence $\mathrm{SEQ}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0=\emptyset$ that consists of sequences of elements from $\mathscr{A}$ is the infinite sum

$$\mathscr{E}+\mathscr{A}+\mathscr{A}^2++\mathscr{A}^3+\ldots.$$

Note that the condition $\mathscr{A}_0 = \emptyset$ (equiv. $a_0 = 0$) guarantees the finiteness condition for sizes, and therefore the multiset construction $\text{MSET}(\mathscr{A})$ is a well-defined combinatorial class. Its OGF satisfies

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)},$$

where the latter equality is because $[z^0]A(z) = 0$.

(6) Multiset $\text{MSET}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite multisets (i.e. repetition allowed) of elements from $\mathscr{A}$, more precisely, we define

$$\text{MSET}(\mathscr{A}) := \text{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is the equivalence class of sequences defined by

$$(\alpha_1, \ldots, \alpha_n) \sim_{\text{R}} (\beta_1, \ldots, \beta_n)$$

iff there is a permutation σ of $[1 \ldots n]$ such that for all $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$.

For a finite set $\mathscr{A}$, we let $\alpha_1, \alpha_2, \ldots$ be a canonical listing of the elements of $\mathscr{A}$. Then any multiset can be sorted in such a way that it can be viewed as formed by a sequence of repeated elements of α_1, followed by a sequence of repeated elements of α_2, and so on. It follows that

$$\text{MSET}(\mathscr{A}) = \prod_{\alpha \in A} \text{SEQ}(\alpha).$$

Therefore, the OGF of $\mathscr{C} = \text{MSET}(\mathscr{A})$ satisfies

$$C(z) = \prod_{\alpha \in \mathscr{A}} \frac{1}{1 - z^{|\alpha|}} = \prod_{n \geq 1} \left(\frac{1}{1 - z^n} \right)^{a_n},$$

where the latter equality holds because $a_0 = 0$.

Consider the formal power series

$$\exp(z) := \sum_{n \geq 0} \frac{z^n}{n!}, \quad \ln(z) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (z-1)^n.$$

Applying the exp-ln transformation we have

$$\begin{aligned} C(z) &= \exp\left(\ln\left(\prod_{n \geq 1} (1 - z^n)^{-a_n} \right) \right) = \exp\left(\sum_{n \geq 1} a_n \ln\left(\frac{1}{1 - z^n} \right) \right) \\ &= \exp\left(\sum_{n \geq 1} a_n \sum_{k \geq 1} \frac{(z^n)^k}{k} \right) = \exp\left(\sum_{k \geq 1} \frac{1}{k} \sum_{n \geq 1} a_n (z^k)^n \right) = \exp\left(\sum_{k \geq 1} \frac{1}{k} A(z^k) \right) \\ &= \exp\left(A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \ldots \right). \end{aligned}$$

The case of infinite class $\mathscr{A}$ follows by a limit argument.

(7) Power set $\mathrm{PSET}(\mathscr{A})$ of of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is the collection of all finite subsets of $\mathscr{A}$ (without repetition).

For a finite set $\mathscr{A}$, we have

$$\mathrm{PSET}(\mathscr{A}) = \prod_{\alpha \in \mathscr{A}} (\mathscr{E} + \{\alpha\}),$$

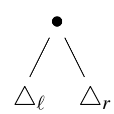
because the distributing the products in all possible ways forms all possible combinations of elements of $\mathscr{A}$ (i.e. the sets of elements from $\mathscr{A}$ without repetition). So, the OGF of $\mathscr{C} = \mathrm{PSET}(\mathscr{A})$ satisfies

$$\begin{aligned} C(z) &= \prod_{\alpha \in \mathscr{A}} (1 + z^{|\alpha|}) = \prod_{n \geq 1} (1 + z^n)^{a_n} = \exp\Big(\sum_{n \geq 1} a_n \ln(1 + z^n)\Big) \\ &= \exp\Big(\sum_{n \geq 1} a_n \sum_{k \geq 1} (-1)^{k-1} \frac{z^{nk}}{k}\Big) = \exp\Big(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} A(z^k)\Big) \\ &= \exp\left(A(z) - \frac{A(z^2)}{2} + \frac{A(z^3)}{3} \pm \dots\right). \end{aligned}$$

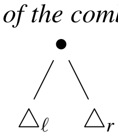
Applications

Example 11. *A binary tree is a combinatorial structure that is recursively defined such that*

- *It is either a single external node* $\circ$*, or*
- *it consists of an internal node (the root* $\bullet$*) and two binary trees attached to the root (left tree* $\triangle_\ell$ *and right tree* $\triangle_r$*),*

$\circ$ *or*  .

Let $\mathscr{B}$ *denote the class of all binary trees, in which the size of a binary tree is defined as the number of internal nodes. Then* $\mathscr{B}$ *is a combinatorial class. For* $n \geq 0$ *we let* $\mathscr{B}_n$ *denote the class of all binary trees of size* n *and let* $b_n := |\mathscr{B}_n|$*. (Note that if a binary tree has* n *internal nodes, then it has* $n+1$ *external nodes. Thus* b_n *counts the number of binary trees with* $n+1$ *external nodes.)*

The sequence $(b_n)_{n \in \mathbb{N} \cup \{0\}}$ *begins with* $1, 1, 2, 5, 14, 42, \dots$ *Let* $B(z) := \sum_{n=0}^{\infty} b_n z^n$ *be the ordinary generating function of the combinatorial class* $\mathscr{B}$*. We have* $b_0 = 1$*, since the only tree is* $\circ$ *and for* $n \geq 1$*,*  .

Thus its OGF satisfies

$$B(z) = 1 + zB(z)^2.$$

The solution for the quadratic equation

$$zB(z)^2 - B(z) + 1 = 0$$

is

$$B(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Since $b_n \geq 0$, $B(z)$ increases along the real axis and therefore it increases. Thus the right solution of $B(z)$ is

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

As in Example 10 we get for $n \geq 1$:

$$b_n = [z^n]B(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 3 (Number of binary trees)**.** *The number b_n of binary trees with n internal nodes (equiv. $n+1$ external nodes) is given by the so-called Catalan number*

$$b_n = \frac{1}{n+1}\binom{2n}{n}.$$

Remark 2. *Using Stirling's formula*

$$n! = \left(1 + O\left(\frac{1}{n}\right)\right)\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n,$$

we can derive the asymptotic number of binary trees

$$b_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}}4^n n^{-3/2}.$$

Example 12. *Let us return to triangulations of convex polygons we saw in Example 10.*

Let $\mathcal{T}_n$ be the set of all triangulations of convex $(n+2)$-gons (n triangles) with one edge distinguished. Using the basic constructions described above we have

$$\mathcal{T} = \biguplus_{n\geq 0}\mathcal{T}_n = \sum_{n\geq 0}\mathcal{T}_n, \qquad \mathcal{T}_0 = \mathcal{E}, \quad \mathcal{T}_1 = \mathcal{Z}.$$

$\mathcal{T}_0$ contains only one edge, $\mathcal{T}_1$ one triangle. By removing one edge of the convex $(n+2)$-gon, we end up with two separate convex triangulations (sharing one node), so

$$\mathcal{T} \; = \; \mathcal{E} + \mathcal{T} \times \mathcal{Z} \times \mathcal{T}$$

$$T(z) \; = \; 1 + zT^2(z),$$

among whose two solutions we choose

$$T(z) \;=\; \frac{1-\sqrt{1-4z}}{2z},$$

because the coefficients of $T(z)$ are non-negative.

As in the proof of Theorem 3,

$$t_n = [z^n]T(z) = -\frac{1}{2}[z^{n+1}]\sqrt{1-4z} = \ldots = \frac{1}{n+1}\binom{2n}{n}.$$

This suggests a bijection between binary trees and triangulations, where each node corresponds to one triangle (or alternatively, the root node of a binary tree B corresponds to the distinguished edge of a triangulation T, and each internal node of B to a diagonal edge of T, and each external node to the external edges of T except the distinguished edge of T.

Example 13. *Let $\mathscr{S}$ be the set of binary strings with no two consecutive 0 bits; for example,* $\emptyset, 0, 1, 01, 10, 11, 010, 011, \ldots$.

$$\mathscr{S} = \mathscr{E} + \{0\} + \{1\} \times \mathscr{S} + \{01\} \times \mathscr{S}.$$

From the theorem, we get

$$S(z) = 1 + z + zS(z) + z^2 S(z)$$

and thus

$$S(z) = \frac{1+z}{1-z-z^2}.$$

Exercise 3. *Let $S(z)$ be as above.*

- *Find the closed solution form of $s_n = |\mathscr{S}_n|$.*
- *Find a recurrence of s_n that leads to $S(z)$.*

1.4.2 Labelled combinatorial objects

A labelled combinatorial class $\mathscr{A}$ is a combinatorial class, where each atom carries an integer label in such a way that all the labels occurring in an object of size n are distinct and the collection of its labells is the complete integer interval $[n]$.

Given a labelled combinatorial class $\mathscr{A}$, we consider the EGF

$$A(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!} = \sum_{\alpha \in \mathscr{A}} \frac{z^{|\alpha|}}{|\alpha|!}.$$

Basic constructions and EGFs

(1) Neutral class: $\mathscr{E}$. The EGF of $\mathscr{E}$ is 1.

(2) Atomic class: $\mathscr{Z}$. The EGF of $\mathscr{Z}$ is z.

(3) Disjoint union: $\mathscr{A} + \mathscr{B}$. The EGF of $\mathscr{A} + \mathscr{B}$ is $A(z) + B(z)$.

(4) Labelled product $\mathscr{A} * \mathscr{B}$ of two combinatorial classes $\mathscr{A}$ and $\mathscr{B}$ is defined as

$$\mathscr{A} * \mathscr{B} := \sum_{\alpha \in \mathscr{A}, \beta \in \mathscr{B}} \alpha * \beta.$$

Given $\alpha \in \mathscr{A}, \beta \in \mathscr{B}$, let $\alpha * \beta$ denote the set of all pairs (α', β') where the atoms of (α', β') get distinct labels from $[n]$ where $n = |\alpha| + |\beta|$, such that the labelling preserves the relative order of labels of α, β. There are $\binom{|\alpha|+|\beta|}{|\alpha|}$ possibilities for such labelling (choose which labels go to the first substructure, then the previous relative order determines which label belongs to which element). In other words, if $\mathscr{C} = \mathscr{A} * \mathscr{B}$, $C(z) = A(z) \cdot B(z)$.

(5) Labelled sequence $\mathrm{SEQ}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\mathrm{SEQ}(\mathscr{A}) = \mathscr{E} + \mathscr{A} + \mathscr{A} * \mathscr{A} + \ldots = \sum_{k \geq 0} \mathrm{SEQ}_k(\mathscr{A}),$$

where $\mathrm{SEQ}_k(\mathscr{A}) = \mathscr{A} * \cdots * \mathscr{A}$ is the labelled product of k copies of $\mathscr{A}$. The EGF of $\mathrm{SEQ}(\mathscr{A})$ satisfies

$$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \frac{1}{1 - A(z)}.$$

(6) Set $\mathrm{SET}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\mathrm{SET}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{R},$$

where $\mathscr{R}$ is an equivalence relation such that $(\alpha_1, \ldots, \alpha_n) \sim_{\mathscr{R}} (\beta_1, \ldots, \beta_n)$ if there exists a permutation σ of $[n]$ such that for each $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\mathrm{SET}(\mathscr{A})$ satisfies

$$1 + A(z) + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \ldots = \exp(A(z)).$$

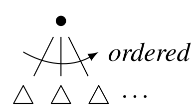
(7) Cylces $\textsc{Cyc}(\mathscr{A})$ of a combinatorial class $\mathscr{A}$ with $\mathscr{A}_0 = \emptyset$ is defined as

$$\textsc{Cyc}(\mathscr{A}) := \mathrm{SEQ}(\mathscr{A}) \Big/ \mathscr{S},$$

where $\mathscr{S}$ is the equivalence relation such that $(\alpha_1, \ldots, \alpha_n) \sim_{\mathscr{R}} (\beta_1, \ldots, \beta_n)$ if there exists a cyclic permutation σ of $[n]$ such that for each $1 \leq i \leq n$, $\beta_i = \alpha_{\sigma(i)}$. The EGF of $\textsc{Set}(\mathscr{A})$ satisfies

$$1 + A(z) + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \ldots = \log \frac{1}{1 - A(z)}.$$

Example 14 (Plane trees, ordered trees)**.** *Plane trees are the trees embedded in the plane, so that subtrees attached to each vertex are ordered, say from left to right. There is a root vertex, implicitly defined; to a root vertex, subtrees are attached in a specified order.*



So,

$$\mathscr{P} = \mathscr{Z} * \textsc{Seq}(\mathscr{P}).$$

The generating function is then

$$P(z) = z \cdot \frac{1}{1 - P(z)}.$$

This is a quadratic equation. As previously, since $P(z)$ increases along the real axis, the negative solution is the correct one,

$$P(z) = \frac{1 - \sqrt{1 - 4z}}{2} = z \cdot B(z),$$

where $B(z)$ is the OGF of binary trees.

Let p_n be the number of plane trees on n vertices, and b_n the number of binary trees on $n+1$ external nodes, so $p_{n+1} = b_n$. $P(z) = zB(z)$ suggests that there is a combinatorial bijection between plane trees on $n+1$ vertices and binary trees on $n+1$ external nodes.

Exercise 4. *Find such a bijection between plane trees and binary trees.*

Example 15 (2-regular graphs)**.** *Let $\mathscr{R}$ be the set of all 2-regular labelled graphs, i.e. each vertex has exactly 2 neighbours. Note that connected 2-regular graphs are undirected cycles of length ≥ 3.*

$$\mathscr{R} = \textsc{Set}(\textit{connected 2-regular graphs}) = \textsc{Set}(\textsc{UCyc}_{\geq 3}(\mathscr{Z})),$$

where UCYC *is the undirected cyclic construction. Then, the generating function is*

$$R(z) = \exp\left(\mathrm{UCYC}_{\geq 3}(z)\right) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}},$$

since the generating function for $\mathrm{UCYC}_{\geq 3}(\mathscr{Z})$ *is* $\frac{1}{2}\left(\log(\frac{1}{1-z}) - z - \frac{z^2}{2}\right)$.

How can we derive $[z^n]R(z)$?

1.5 Analytic Methods

1.5.1 Analytic functions

Definition 5. Let $\Omega \subset \mathbb{C}$ be a region (i.e. open and connected). A function $f : \Omega \to \Omega$ is called complex differentiable at $z_0 \in \Omega$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. It is called holomorphic in Ω if f is holomorphic for every point in Ω.

Definition 6. Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \Omega \to \Omega$ is called analytic at $z_0 \in \Omega$ if $\exists \varepsilon > 0$ such that $\forall z \in B_\varepsilon(z_0)$, $f(z)$ is representable by a *convergent* power series expansion around z_0

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n, \quad \text{for some } c_n \in \mathbb{C}.$$

It is called analytic in Ω if f is analytic for every point in Ω.

Theorem 4. *Let $\Omega \subset \mathbb{C}$ be a region. A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in Ω iff it is analytic in Ω.*

Definition 7. Let $\Omega \subset \mathbb{C}$ be a region and $f : \Omega \to \Omega$ be holomorphic at $z_0 \in \Omega$. Set $f^{(0)}(z_0) := f(z_0)$. Then the power series

$$f(z) = f(z_0) + f'(z_0)z + \frac{f''(z_0)}{2!}z^2 + \ldots = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} z^n$$

is called the Taylor series expansion of $f(z)$ around z_0.

Exercise 5. *Find the maximum regions, in which the functions $z \mapsto e^z$, $z \mapsto \frac{1}{1-z}$, $z \mapsto (1+z)^m$ (for fixed $m \in \mathbb{N}$) are analytic. Show that their Taylor series expansions around the origin are given by*

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}, \qquad \frac{1}{1-z} = \sum_{n \geq 0} z^n, \qquad (1+z)^m = \sum_{n \geq 0} \binom{m}{n} z^n.$$

Exercise 6. *Find the Taylor series expansions of*

$$\frac{1}{\sqrt{1-4z}}, \quad \sin z, \quad \cos z, \quad z \cdot e^z, \quad \ln z, \quad \frac{1}{1-z} \ln \frac{1}{1-z}, \quad \frac{1}{\sqrt{1-z}} \ln \frac{1}{1-z}$$

around the origin and find the maximum regions where these power series converge.

Exercise 7. *Prove*

$$\sum_{n \geq m} \binom{n}{m} z^n = \frac{z^m}{(1-z)^{m+1}}.$$

Exercise 8. *Prove*

$$\sum_{n\geq 0} H_n z^n = \frac{1}{1-z}\ln\frac{1}{1-z}.$$

Exercise 9. *Find OGF for each of the following sequences:*

$$\{2^{n+1}\}_{n\geq 0}, \qquad \{n2^{n+1}\}_{n\geq 0}, \qquad \{nH_n\}_{n\geq 1}, \qquad \{n^3\}_{n\geq 2}.$$

1.5.2 Cauchy's coefficient formula

Theorem 5 (Cauchy's coefficient formula, 1st version)**.** *Let $\Omega \subset \mathbb{C}$ be a region, f be analytic in Ω and γ be a simple loop in Ω. Then*

$$\int_\gamma f = 0.$$

Theorem 6 (Cauchy's coefficient formula, 2nd version)**.** *Let $\Omega \subset \mathbb{C}$ be a region and f be analytic in Ω. Let $z_0 \in \Omega$ and γ be a simple loop encircling z_0 such that the image of γ is contained in Ω. Then*

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

In general, for $n \geq 1$

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

1.5.3 Lagrange Inversion Theorem

Theorem 7 (Langrange Inversion, 1st version)**.** *Let $\phi(u) = \sum_{k\geq 0}\phi_k u^k$ be a power series in $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then the equation*

$$A = z\phi(A)$$

admits a unique solution in $\mathbb{C}[[z]]$ and the coefficients of

$$A(z) = \sum_{n\geq 0} a_n z^n$$

are given by

$$a_n = \frac{1}{n}[u^{n-1}]\phi(u)^n.$$

Furthermore, for any function H,

$$[z^n]H(A(z)) = \frac{1}{n}[u^{n-1}]\left(H'(u)\phi(u)^n\right).$$

In particular, for $k \geq 1$,

$$[z^n]A(z)^k = \frac{k}{n}[u^{n-k}]\phi(u)^n.$$

Theorem 8 (Langrange Inversion, 2nd version)**.** *Let the generating function*

$$A(z) = \sum_{n\geq 0} a_n z^n$$

satisfy the functional equation

$$z = \psi(A(z)),$$

where $\psi(0) = 0$ but $\psi'(0) \neq 0$ (i.e. ψ is the compositional inverse of A).

If

$$\psi(A) = \frac{A}{\phi(A)},$$

i.e. $A = \psi(A) \cdot \phi(A) = z\phi(A)$, then

$$a_n = \frac{1}{n}[z^{n-1}]\phi(z)^n = \frac{1}{n}[z^{n-1}]\left(\frac{z}{\psi(z)}\right)^n.$$

Application of Lagrange Inversion to Binary Trees

Let $\tilde{\mathscr{B}}$ denote the class of all binary trees, in which the size of a binary tree is defined as the total number of all vertices (internal nodes and leaves alike). For $n \geq 1$ we let $\tilde{\mathscr{B}}_n$ denote the class of all binary trees of size n and let $\tilde{b}_n := |\tilde{\mathscr{B}}_n|$ and set $\tilde{b}_0 = 0$.

Let $\tilde{B}(z)$ be the ordinary generating function of the combinatorial class $\tilde{\mathscr{B}}$. Then

$$\tilde{B}(z) \;=\; z \,+\, z\,\tilde{B}(z)^2 \tag{1.9}$$

Let $\tilde{B} = \tilde{B}(z)$ and $\phi(u) := 1 + u^2$. Then (1.9) can be rewritten as

$$\tilde{B} \;=\; z\,\phi(\tilde{B}) \tag{1.10}$$

Note that $\phi(u) := 1 + u^2 = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.12) admits a unique solution in the ring $\mathbb{C}[[u]]$ whose coefficients are given by $\tilde{B}(z) =:= \sum_{n=0}^{\infty} \tilde{b}_n\, z^n$, where

$$\begin{aligned}
\tilde{b}_n \;&=\; \frac{1}{n}\,[u^{n-1}]\,\phi(u)^n \\
&=\; \frac{1}{n}\,[u^{n-1}]\,(1+u^2)^n \\
&=\; \frac{1}{n}\,[u^{n-1}]\,\left(\sum_{k=0}^{n}\binom{n}{k}\,u^{2k}\right) \\
&=\; \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n}\,\binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

In other words, we have, for any $n \in \mathbb{N}$,

$$\tilde{b}_{2n+1} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

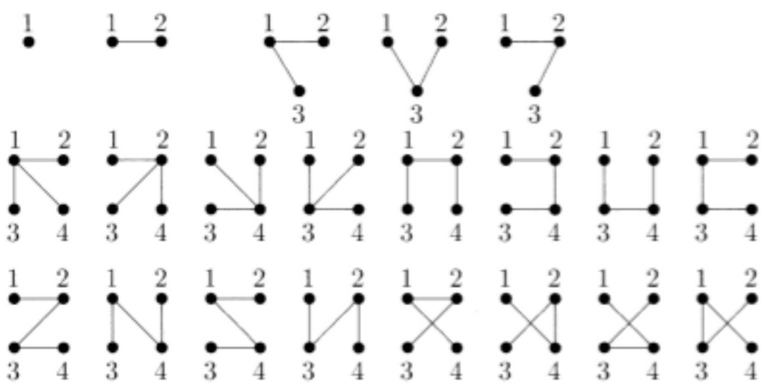
Remark 3. *Recall that b_n was defined in Theorem 3 as the number of binary trees of size n, in which the size of a binary tree is defined as the number of internal nodes. If a binary tree has n internal nodes, then it has $n+1$ external nodes and therefore $2n+1$ vertices in total. Thus we have $\tilde{b}_{2n+1} = b_n$.*

Application of Lagrange Inversion to Labelled Trees

Definition 8. A tree is a connected graph without cycles.

- A labelled tree (also called Cayley tree) is a tree whose vertices are distinctly labelled by numbers in $\mathbb{N}$. The vertices of a Cayley tree on n vertices are labelled by distinct numbers from $[n]$.
- A rooted labelled tree on n vertices is a labelled tree on n vertices, in which one vertex is distinguished by a mark from the other vertices.

Let $\mathscr{C}$ denote the class of all labelled trees, in which the size of a tree is defined as the number of vertices, and let $\mathscr{C}_n$ denote the class of all Cayley tree on n vertices. Then $\mathscr{C}$ is a combinatorial class. For $n \geq 1$ we let $c_n := |\mathscr{C}_n|$ and set $c_0 = 0$. Let $C(z) := \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial class $\mathscr{C}$. The sequence $(c_n)_{n\geq 0}$ begins with $0, 1, 1, 3, 16, 125\ldots$

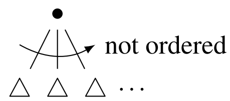


Theorem 9 (Cayley's formula (Cayley 1889))**.**

$$c_n = n^{n-2}, \quad n \geq 2.$$

Proof. Let $\mathscr{T}$ denote the class of all rooted labelled trees (i.e. the set of all Cayley trees in $\mathscr{C}$, in which one vertex is distinguished by a mark from the other vertices) and let $\mathscr{T}_n$ denote the class of all rooted labelled tree on n vertices. For $n \geq 1$ we let $t_n := |\mathscr{T}_n|$ and set $t_0 = 0$. Because there are n ways to choose a root vertex of a labelled tree on n vertices, the number of Cayley trees on n vertices is equal to $t_n = n\, c_n$.

Let $T(z) := \sum_{n=0}^{\infty} t_n \frac{z^n}{n!}$ be the exponential generating function of the combinatorial class $\mathscr{T}$.



$$\mathscr{T} = \mathscr{Z} * \text{SET}(\mathscr{T}).$$

Its EGF

$$T(z) \;=\; z\,\exp(T(z)). \tag{1.11}$$

So $T(z)$ is implicitly defined.

Let $T = T(z)$ and $\phi(u) := \exp(u) = \sum_{k\geq 0} \frac{u^k}{k!}$. Then (1.11) can be rewritten as

$$T \;=\; z\,\phi(T) \tag{1.12}$$

Note that $\phi(u) := \exp(u) = \sum_{k=0}^{\infty} \phi_k\, u^k$ is a formal power series in the ring $\mathbb{C}[[u]]$ with $\phi_0 = 1 \neq 0$. Therefore by Lagrange Inversion Theorem, the equation (1.12) admits a unique solution in the ring $\mathbb{C}[[u]]$ whose coefficients are given by $T(z) = \sum_{n=0}^{\infty} t_n\, z^n$, where

$$\begin{aligned} t_n \;&=\; \frac{1}{n}\,[u^{n-1}]\;\phi(u)^n \;=\; \frac{1}{n}\,[u^{n-1}]\;(\exp(u))^n \;=\; \frac{1}{n}\,[u^{n-1}]\;\exp(un) \\ &=\; \frac{1}{n}\,[u^{n-1}]\left(\sum_{k=0}^{\infty}\frac{(un)^k}{k!}\right) \;=\; \frac{1}{n}\,\frac{n^{n-1}}{(n-1)!} \;=\; \frac{n^{n-1}}{n!}. \end{aligned}$$

In other words, for any $n \in \mathbb{N}$,

$$t_n = n^{n-1}.$$

Therefore, we have

$$c_n = \frac{t_n}{n} = n^{n-2}.$$

□

Remark 4. *There are several interesting proofs for this in The Book by M. Aigner and G. Ziegler. We shall see one proof in Part III, Section 2.3.4.*

1.5.4 Singularities

Definition 9. Let f be an analytic function in a region Ω and z_0 be a point on the boundary of Ω.

We say f is analytically continuable at z_0 if there is an analytic function g defined in a region Ω' containing z_0 such that $g(z) = f(z)$ in $\Omega \cap \Omega'$.

Example 16. $f(z) = \frac{1}{1-z}$ *is analytic for* $|z| < 1$*, and is analytically continuable except for* $z_0 = 1$.

Definition 10. A function f is said to be singular at $z_0 \in \Omega$ or z_0 is called a singularity of f if f is not analytically continuable at z_0.

Theorem 10 (Boundary singularities)**.** *Let f be analytic at the origin and let R be the finite radius of convergence of a power series expansion of f at the origin. Then f has necessarily a singularity on the boundary of the disc of convergence.*

Theorem 11 (Pringsheim's Theorem)**.** *Let f be analytic at the origin and $f(z) = \sum_{n\geq 0} f_n z^n$ be its convergence power series expansion at the origin with the radius of convergence R. If $f_n \geq 0, \forall n \geq 0$, then the point $z = R$ is a singularity of f; in this case we call $z = R$ the dominant singularity of f.*

Remark 5. *The radius R of convergence of $\sum f_n z^n$ is*

$$R = \frac{1}{\limsup_{n\to\infty} |f_n|^{\frac{1}{n}}},$$

so we have

$$f_n = R^{-n}\theta(n),$$

where

$$\limsup_{n\to\infty} |\theta(n)|^{\frac{1}{n}} = 1.$$

1.5.5 Meromorphic functions

Definition 11. A function $f(z)$ is meromorphic at z_0 if $\forall z$ in a neighbourhood $B_\varepsilon(z_0)$ of z_0 with $z \neq z_0$, it can be represented as

$$f(z) = \frac{h(z)}{g(z)},$$

where $h(z)$ and $g(z)$ are analytic at z_0. In this case $f(z)$ admits an expansion of the form

$$f(z) = \sum_{n\geq -M} f_n (z - z_0)^n$$

for $z \in B_\varepsilon(z_0)$. If $f_{-M} \neq 0$, $M \geq 1$, we say $f(z)$ has a pole of order M at z_0. In this case, we have

$$\begin{aligned} f(z) &= f_{-M}(z - z_0)^{-M} + O((z - z_0)^{-M+1}) \\ &= f_{-M}(-z_0)^{-M}\left(1 - \frac{z}{z_0}\right)^{-M} + O\left(\left(1 - \frac{z}{z_0}\right)^{-M+1}\right). \end{aligned}$$

Remark 6. *Scaling rule: If $f(z)$ is singular at z_0, then $g(z) = f(z_0 z)$ is singular at 1. If $z_0 > 0$ is the dominant singularity of $f(z)$, then $f(z)$ admits a convergent series expansion*

$$f(z) = \sum_{n\geq 0} f_n z^n, \quad |z| < z_0.$$

Therefore, $g(z)$ admits a convergent series expansion

$$g(z) = \sum_{n\geq 0} g_n z^n, \quad |z| < 1, \quad g_n = f_n z_0^n.$$

1.5.6 Newton's generalised binomial theorem

For any $\alpha \in \mathbb{C}$,

$$[z^n](1-z)^{-\alpha} = \binom{-\alpha}{n} \stackrel{?}{=} \binom{n+\alpha-1}{\alpha-1} \stackrel{?}{=} \Theta(n^{\alpha-1}).$$

Theorem 12. *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.*

$$[z^n](1-z)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right),$$

where the Γ-function is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, \mathrm{d}t.$$

Properties of the Gamma function

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, \mathrm{d}t = \sqrt{\pi}.$$

$$\begin{cases} \Gamma(t+1) = t\,\Gamma(t) \\ \Gamma(1) = 1. \end{cases}$$

For $n \in \mathbb{N}$,

$$\Gamma(n+1) = n\Gamma(n) = \ldots = n!.$$

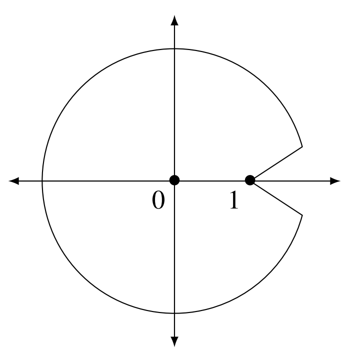
1.5.7 Transfer theorem

Definition 12. Given two numbers R, ϕ with $R > 1$ and $0 < \phi < \frac{\pi}{2}$, then the set

$$\Delta = \Delta(R, \phi) = \{z \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi\}$$

is called Δ-domain at 1.

A function f is called Δ-analytic at 1 if it is analytic at some Δ-domain at 1.



Theorem 13 (Transfer theorem)**.** *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $f(z)$ is Δ-analytic at 1.*
If $f(z) = O((1-z)^{-\alpha})$, then

$$[z^n]f(z) = O(n^{\alpha-1}).$$

If $f(z) = o((1-z)^{-\alpha})$, then

$$[z^n]f(z) = o\left(n^{\alpha-1}\right).$$

Therefore, if $f(z) = (1-z)^{-\alpha} + o((1-z)^{-\alpha})$, then

$$[z^n]f(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right) + o(n^{\alpha-1}).$$

Example 17 (2-regular graphs)**.** *Consider the EGF of 2-regular graphs,*

$$\mathscr{G} = \text{SET}(\text{UCYC}_{\geq 3}\mathscr{X}).$$

Then

$$G(z) = \exp\left(\frac{1}{2}\left(\log\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2}\right)\right) = \frac{1}{\sqrt{1-z}}e^{-\frac{z}{2}-\frac{z^2}{4}}.$$

$\frac{1}{\sqrt{1-z}}$ *is analytic in $\mathbb{C} \setminus [1,\infty)$. $G(z)$ is in particular Δ-analytic at 1.*

To derive the singular expansion of G near 1, we use Taylor expansion of $e^{-\frac{z}{2}-\frac{z^2}{4}}$ at 1.

$$\begin{aligned} h(z) = e^{-\frac{z}{2}-\frac{z^2}{4}} &= h(1) + h'(1)(z-1) + \frac{h''(1)}{2}(z-1)^2 + \dots \\ &= e^{-\frac{3}{4}} + e^{-\frac{3}{4}}(1-z) + e^{-\frac{3}{4}}(z-1)^2 + \dots \end{aligned}$$

The singular expansion of $G(z)$ near 1 is

$$\begin{aligned} G(z) &= \frac{e^{-\frac{3}{4}}}{\sqrt{1-z}} + e^{-\frac{3}{4}}(1-z)^{\frac{1}{2}} + e^{-\frac{3}{4}}(1-z)^{\frac{3}{2}} + \dots \\ &= e^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}} + o\left((1-z)^{-\frac{1}{2}}\right). \qquad (z \to 1) \end{aligned}$$

Using the transfer theorem, we get

$$[z^n]G(z) = e^{-\frac{3}{4}}\frac{n^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\left(1 + O\left(\frac{1}{n}\right)\right) + o\left(n^{-\frac{1}{2}}\right) = \frac{e^{-\frac{3}{4}}}{\sqrt{n\pi}}(1 + o(1)).$$

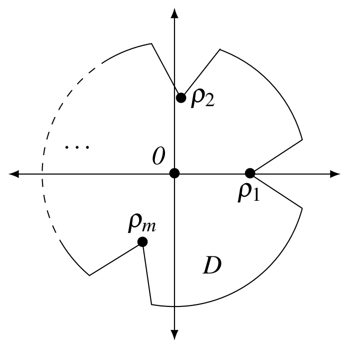
The first term corresponds to $h(1)$, the second is from the theorem.

1.5.8 Multiple singularities

Theorem 14. *Let $f(z)$ be analytic in $|z| < \rho$. Suppose that f has a finite number of singularities on the circle $|z| = \rho$, at points $\rho_j = \rho \cdot e^{i\theta_j}$ for $j = 1, \ldots, m$. Assume there exists a Δ-domain Δ_0 such that $f(z)$ is analytic in the region*

$$D = \bigcap_{j=1}^{m} \rho_j \Delta_0,$$

where $\rho_j\Delta_0$ is the dilation (rotation) of the Δ_0-domains for each ρ_j.



Let

$$S = \{(1-z)^{-\alpha} \mid \alpha \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}\}.$$

Assume there exists m functions $g_1(z), \ldots, g_m(z)$ and each is a linear combination of the functions in S, and there exists a function $h(z) = (1-z)^{-\alpha_0}$ such that

$$f(z) = g_j\left(\frac{z}{\rho_j}\right) + O\left(h\left(\frac{z}{\rho_j}\right)\right) \qquad \text{as } z \to \rho_j \text{ in } D.$$

Then

$$[z^n]f(z) = \sum_{j=1}^{m} \left([z^n]g_j\left(\frac{z}{\rho_j}\right) + O\left([z^n]h\left(\frac{z}{\rho_j}\right)\right)\right)$$
$$= \sum_{j=1}^{m} \rho_j^{-n}[z^n]g_j(z) + O\left(\rho_j^{-n} n^{\alpha_0 - 1}\right).$$

Remark 7.

$$[z^n]g_j(z) = [z^n]\sum_{k=1}^{\ell}(1-z)^{-\alpha_{j,k}} = \sum_{k=1}^{\ell}\frac{n^{\alpha_{j,k}-1}}{\Gamma(\alpha_{j,k})}.$$

Remark 8 (Recipe for singularity analysis)**.**

- *Find the grammar that determines the class and the corresponding generating function*
- *Locate the singularities*

- *Check the* Δ*-analyticity of the generating function*
- *Singular expansion at each singularity separately*
- *Extract and sum the asymptotics using the Transfer Theorem and basic scaling.*

Example 18. *Let* $\mathscr{P}$ *be the class of permutations with cycles of odd length. Then*

$$\begin{aligned}\mathscr{P} &= \textsc{Set}(\mathscr{A}(\mathscr{Z})),\\ \mathscr{A}(\mathscr{Z}) &= \textsc{Cyc}_{odd}(\mathscr{Z}).\end{aligned}$$

For the generating functions, this means

$$\begin{aligned}A(z) &= \sum_{k=odd}\frac{z^k}{k} = \frac{1}{2}\log\frac{1+z}{1-z},\\ P(z) &= e^{A(z)} = \exp\left(\frac{1}{2}\log\frac{1+z}{1-z}\right) = \sqrt{\frac{1+z}{1-z}}.\end{aligned}$$

The dominating singularities of this function are $\{\pm 1\}$.

$P(z)$ is Δ-analytic in the region $D = \Delta_0 \cap (-1)\Delta_0$ since $\sqrt{1-z}$ is analytic in $\mathbb{C}\setminus[1,\infty)$ and $\sqrt{1+z}$ is analytic in $\mathbb{C}\setminus(-\infty,-1]$.

Singular expansions:

$$\begin{aligned}P^{+}(z) &= \frac{g(1)}{\sqrt{1-z}} + \frac{g'(1)(z-1)}{\sqrt{1-z}} + \dots && (z \to +1 \text{ in } D)\\ &= \frac{\sqrt{2}}{\sqrt{1-z}} - 2^{-\frac{3}{2}}\sqrt{1-z} + O\left((1-z)^{\frac{3}{2}}\right)\\ P^{-}(z) &= \frac{1}{\sqrt{2}}\sqrt{1+z} + O\left((1+z)^{\frac{3}{2}}\right). && (z \to -1 \text{ in } D)\end{aligned}$$

Extract asymptotics:

$$\begin{aligned}[z^n]P(z) = \frac{2^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}n^{-\frac{1}{2}} - \frac{2^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)}n^{-\frac{3}{2}} + O\left(n^{-\frac{5}{2}}\right)\\ + \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}n^{-\frac{3}{2}} + O\left(n^{-\frac{5}{2}}\right)\end{aligned}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

Chapter 2

Graph Theory

(hand-written notes to be typed in LaTex)

2.1 Matchings, Eulerian Tour and Hamiltorian Cycles

2.1.1 Basic Terminologies

2.1.2 Matchings

2.1.3 Eulerian Tours

2.1.4 Hamiltorian Cycles

2.2 Graph Decompositions

2.2.1 Connected Graphs

2.2.2 2-Connected Graphs

2.2.3 3-Connected Graphs

2.3 Planar Graphs and Trees

2.3.1 Embedding of Graphs in the Plane

2.3.2 Euler's Formula and its Applications

2.3.3 Properties of Trees

2.3.4 Prüfer Code

2.4 Stochastic Aspects – Random Graphs

2.4.1 Binomial Random Graphs

2.4.2 Useful Inequalities

2.4.3 Ramsey Number and First Moment Method

2.4.4 Independence Number and Deletion Method

2.4.5 Subgraphs and Second Moment Method

Part II

Lectures by Karin Baur

Chapter 3

Multilinear Algebra

Chapter 4

Rings and Moduls