DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 8

Problems due 9.12.2016

(8.1) (1) Set $\Gamma_{n:m} := \{\alpha : \alpha = (\alpha(1), \dots, \alpha(m)), 1 \le \alpha(i) \le n, i = 1, \dots, m\}$ (this is equal to $\Gamma(n, \dots, n)$ with m factors in the lecture notes).
Prove that $\prod_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{\gamma \in \Gamma_{m,n}} \prod_{i=1}^{m} a_{i\gamma(i)}$

(2) Let $v_1, \dots, v_k \in V$ and $A \in \mathbb{C}_{k\times k}$. Suppose $AA^T = I_k$ (identity matrix) and $u_j = \sum_{i=1}^{k} a_{ij} v_i$, $j = 1, \dots, k$.
Prove that $\sum_{i=1}^{k} u_i \otimes u_i = \sum_{i=1}^{k} v_i \otimes v_i$.

(8.2) Suppose that the multilinear map $\varphi : V_1 \times \cdots \times V_m \to P$ has the universal factorization property. Let $T = T(\psi)$ in the diagram of the definition of the universal factorization property (Definition 3.7).
Show that the linear map T (in the universal factorization property) is unique if and only if $\langle \mathrm{Im}\varphi \rangle = P$.

(8.3) Let $z \in U \otimes V$ so that z can be represented as $z = \sum_{i=1}^{k} u_i \otimes v_i$. Prove that if k is the smallest number among all such representations, then both $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ are linearly independent sets.

(8.4) (1) Let $E_i = \{e_{i1}, \dots, e_{in_i}\}$ be an ONB of V_i, $i = 1, \dots, m$. Let $E := \{e_\gamma^{\otimes} := e_{1\gamma(1)} \otimes \cdots \otimes e_{m\gamma(m)} : \gamma \in \Gamma\}$ be the corresponding basis of $V_1 \otimes \cdots \otimes V_m$. For $u := \sum_{\gamma\in\Gamma} a_\gamma e_\gamma^{\otimes}$ and $v := \sum_{\gamma\in\Gamma} b_\gamma e_\gamma^{\otimes}$ define

$$(u, v) := \sum_{\gamma\in\Gamma} a_\gamma \overline{b_\gamma}.$$

Check that E is an ONB w.r.t. this inner product.

(2) Define $\otimes : \mathbb{C}^k \times \mathbb{C}^n \to \mathbb{C}_{k\times n}$ by $x \otimes y := xy^T$. Let $\mathbb{C}^k$ and $\mathbb{C}^n$ be equipped with the standard inner products.
Prove that for any $A, B \in \mathbb{C}_{k\times n} = \mathbb{C}^k \otimes \mathbb{C}^n$, the induced inner product is given by

$$(A, B) = \mathrm{tr}(B^* A).$$

(Notation: $\mathrm{tr}(B^*A)$ is the trace of the square matrix B^*A)