Exercise Sheet 7

Problems due 2.12.2016 (you might need the 3rd lecture for 7.3 and 7.4)

(7.1) (1) Let $V_1, \dots, V_m$, W and W' be $\mathbb{C}$-vector spaces.
Prove that if $\varphi : V_1 \times \dots \times V_m \to W$ is multilinear and $T : W \to W'$ is linear, then $T \circ \varphi$ is multilinear.

(2) Let $V_1, \dots, V_m$, $W_1, \dots, W_m$ and W be $\mathbb{C}$-vector spaces.
Let $\varphi : W_1 \times \dots \times W_m \to W$ be multilinear and $T_i : V_i \to W_i$ linear for all i.
Define $\psi : V_1 \times \dots \times V_m \to W$ by $\psi(v_1, \dots, v_m) = \varphi(T_1 v_1, \dots, T_m v_m)$.
Show that ψ is multilinear.

(7.2) (1) Assume that the determinant function $\det : \mathbb{C}^n \times \dots \times \mathbb{C}^n \to \mathbb{C}$ is defined via the Leibniz formula, for $A = (a_{ij})_{1 \le i,j \le n}$ and S_n the permutations of n elements and sgn the signum of a permutation

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

Show that det is multilinear but that for $n > 1$, it is not a tensor map, i.e. that
$\dim\langle \operatorname{im}(\det) \rangle < \prod_{i=1}^n \dim(\mathbb{C}^n)$.

(2) Define $\varphi : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}_{m \times n}$ by $\varphi(x, y) = xy^T$. Show that φ is multilinear and that it is a tensor map.

(7.3) Let V be a vector space over $\mathbb{C}$. An inner product on V is a function $(\cdot,\cdot)$: $V \times V \to \mathbb{C}$ such that

1. $(u, v) = \overline{(v, u)}$ for all $u, v \in V$.
2. $(\alpha_1 v_1 + \alpha_2 v_2, u) = \alpha_1 (v_1, u) + \alpha_2 (v_2, u)$ for all $v_i, u \in V$, $\alpha_i \in \mathbb{C}$.
3. $(v, v) \ge 0$ for all $v \in V$ and $(v, v) = 0$ if and only if $v = 0$.

(1) Let $F := \{f_1, \dots, f_n\}$ be a basis of the $\mathbb{C}$-vector space V. Then there exists a unique inner product $(\cdot,\cdot)$ on V such that F is orthonormal.

(2) Let $E = \{e_1, \dots, e_n\}$ be a basis of V. For any $u = \sum_{i=1}^n a_i e_i$ and $v = \sum_{i=1}^n b_i e_i$, show that $(u, v) := \sum_{i=1}^n a_i \overline{b_i}$ is the unique inner product on V so that E is an orthonormal basis.

(7.4) Let V be a $\mathbb{C}$-vector space and $V^* = \operatorname{Hom}(V, \mathbb{C})$.

(1) Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of V with inner product $(\cdot,\cdot)$. Show that $\{f_1, \dots, f_n\}$ is a dual basis to it (this means that $f_i(e_j) = \delta_{ij}$ for all i, j) if and only if $f_j(v) = (v, e_j)$ for all $v \in V$ and $j = 1, \dots, n$.

(2) Let $\{f_1, \dots, f_n\}$ be a basis of V^*. Show that if $v \in V$ is a vector with $f_j(v) = 0$ for all j, then $v = 0$.