DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 6

Problems due 25.11.2016

(6.1) Let G be a graph on n vertices. Prove that it contains an independent set of size $\frac{n}{2d(G)}$.
Hint: choose a random subset S of the vertices of G by putting vertices into S independently with some suitable probability. Determine the expected number of vertices in S and of edges in $G[S]$. If $G[S]$ has much more vertices than edges, how do we obtain an independent set from S?

(6.2) Let X be a random variable and let $t > 0$. Suppose that X only takes non-negative values. Prove Markovs inequality

$$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

Show that the inequality is false for general X by constructing a random variable with non-negative expectation for which the inequality fails for *all* values of t.

(6.3) Let X be a random variable and let $t > 0$. Prove Chebyshev's inequality

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var[X]}}{t^2}$$

where $\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X]^2 - \mathbb{E}[X^2]$.

(6.4) Denote by $G(n, n, p)$ the random *bipartite* graph defined as follows. The vertex set is $[2n] = A \dot{\cup} B$ for $A = [n]$ and $B = \{n+1, \dots, 2n\}$. Independently for each $a \in A$ and $b \in B$, the edge $\{a, b\}$ is in $G(n, n, p)$ with probability p. Prove that

$$[\mathbb{P}(n, n, p) \text{ contains a cycle of length } 4] \overset{n \to \infty}{\longrightarrow} \begin{cases} 0 & \text{if } p = o(\frac{1}{n}) \\ 1 & \text{if } p = \omega(\frac{1}{n}) \end{cases}$$