DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 5

Problems due 18.11.2016

(5.1) Let $G = (V, E)$ be a graph with $n \geq 2$ vertices.
  (a) Prove that if G is a tree, then it contains at least two leaves.
  (b) Suppose there is a vertex $v \in V$ with $d(v) = 1$. Prove that G is a tree if and only if $G[V \setminus \{v\}]$ is a tree.
  (c) Prove that G is a tree if and only if G is connected and $|E| = |V| - 1$.

(5.2) (a) Let $G = (V, E)$ be a connected plane graph and suppose that every face has at least g edges on its boundary. Show that
$$[E] \leq \frac{g}{g-2}(|V| - 2)$$
  (b) Prove that K_5 and $K_{3,3}$ are not planar.
  (c) For non-connected graphs, embeddings and plane graphs are defined the same way as for connected graphs. Find and prove an equation for the number of faces of a (not necessarily connected) plane graph G.

(5.3) Let T be a tree on the vertex set $\{1, 2, \dots, n\}$ and let $s = (s_1, \dots, s_{n-2})$ its Prüfer code.
  (a) Determine all trees T for which $s_1, \dots, s_{n-2}$ are pairwise distinct.
  (b) Determine all trees for which $s_1 = \dots = s_{n-2}$.
  (c) What bijection similar to the Prüfer code one could be used to count *rooted* trees on $\{1, 2, \dots, n\}$? (It is not necessary to re-prove statements that are already known for Prüfer codes; just point out the differences to your construction.)

(5.4) Prove that the following statements about a graph $G = (V, E)$ are equivalent.
  (i) G is a tree;
  (ii) G contains a unique path between any two of its vertices;
  (iii) G is minimally connected, i.e. G is connected, but deleting any edge from G disconnects the graph;
  (iv) G is maximally acyclic, i.e. G does not contain cycles, but for any two non-adjacent vertices x, y adding the edge $\{x, y\}$ produces a cycle.

(5.5) Let P be the Petersen graph (it is a 3-regular graph with 10 vertices - look it up). Show that P is non-planar:
  (a) using Euler's formula;
  (b) by showing that P contains $K_{3,3}$ as a toplogical minor;
  (c) by showing that P contains K_5 as a minor.