DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 4

Problems due 11.11.2016

(4.1) (a) Prove Halls theorem that a bipartite graph with $V = A \dot{\cup} B$ has a matching covering if and only if

$$|N(S)| \geq |S| \quad \text{for every } S \subseteq A$$

(b) Give an example that shows that Hall's theorem fails for graphs with infinitely many vertices.

(4.2) (a) Show that every connected graph G on n vertices contains a path of length $\min\{2\delta(G), n-1\}$.

(b) Let G be a connected graph with at least 3 vertices and let e be an edge of G. Prove that G^3 has a Hamiltonian cycle that uses e.
Hint: Induction on the number of vertices.

(4.3) Prove the following direction of a theorem by Tutte from the lecture.

Let G be a graph and suppose that there exists a sequence $G_0 = K_4, G_1, \ldots, G_n = G$ with the following property: For every $i \in \{0, 1, \ldots, n-1\}$ there is an edge e in G_{i+1} between vertices v, w both of degree ≥ 3 such that G_i is obtained from G_{i+1} by contracting e. Then G is 3-connected.

(4.4) Let k, n be positive integers and let X be a set of size kn. Prove that for any two partitions

$$X = \biguplus U_i \quad \text{and} \quad X = \biguplus V_i \quad \text{with } |U_i| = |V_i| = k \text{ for all } i$$

there exists a common set of representatives $Y \subseteq X$ (i.e. $|U_i \cap Y| = |V_i \cap Y| = 1$ for all i). Show that this is not true if we start with three partitions.