DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 3

Problems due 4.11.2016

Every permutation σ of $\{1, 2, \ldots, n\}$ can be written as a product of disjoint cycles, $\sigma = (a_{1,1}a_{1,2} \ldots a_{1,r_1})(a_{2,1} \ldots a_{2,r_2}) \cdots (a_{s,1} \ldots a_{s,r_2})$ with $a_{ij} \in \{1, 2, \ldots, n\}$ and $a_{ij} = a_{i'j'}$ if and only if $(i, j) = (i', j')$. The cycle notation $(a_{1,1}a_{1,2} \ldots a_{1,r_1})$ means that $a_{1,m}$ is sent to $a_{1,m+1}$ for $1 \leq m < r_1$ and that a_{1,r_1} is sent to a_{11}.

As an example: the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ can be written as (132). The permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ can be written as (13)(24): $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$.

(3.1) A *derangement of length n* is a permutation of $\{1, 2, \ldots, n\}$ without fixed points, an *r-derangement* is a permutation all of whose cycles have length larger than r. Let $\mathcal{D}$ be the class of all derangements and $\mathcal{D}^{(r)}$ the class of all r-derangements. Determine the exponential generating function $D(z)^{(r)}$ and calculate the number of derangements (i.e.$r = 1$) of length n by extracting the coefficients $[z^n]D(z)$.
Recall that for an EGF $A(z) = \sum_n a_n \frac{z^n}{n!}$ we write $[z^n]A(z) := \frac{a_n}{n!}$.

(3.2) Let $T(z) = \sum_n t_n \frac{z^n}{n!}$ be the EGF of the class $\mathcal{T}$ of rooted labelled trees, putting $t_0 = 0$.
A graph is a *forest* if it has no cycles. Let $\mathcal{F}$ be the class of labelled forests where each component is rooted. For $k > 0$ let $\mathcal{F}_k$ be the class of $\mathcal{F}$ consisting of the forests with exactly k components. Find expressions for the EGFs $F(z)$ and $F_k(z)$ in terms of $T(z)$ and determine the number of forests in $\mathcal{F}$ and in $\mathcal{F}_k$ on n vertices by applying Lagrange Inversion.

(3.3) Involutions and permutations without long cycles. A permutation σ is an *involution* if σ^2 =Id with Id the identity permutation. Clearly, an involution can have only cycles of sizes 1 and 2. A *pairing* is an involution without fixed points (i.e. all cycles have length 2). Use the combinatorial method to describe the combinatorial classes $\mathcal{I}$ of all involutions and $\mathcal{J}$ of pairings and determine their EGFs. Determine I_n and J_n (in case of J_n for n even).

(3.4) Consider the function

$$f(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$$

Use singularity analysis to determine the coefficient $[z^n]f(z)$ up to a multiplicative error of $1 + O(n^{-1})$.