DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 2

**Problems** due 21.10.2016

(1) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be combinatorial classes, with OGFs $A(z) = \sum_{n\geq 0} a_n z^n$, $B(z) = \sum_{n\geq 0} b_n z^n$ and $C(z) = \sum_{n\geq 0} c_n z^n$. Prove the following (from the lecture):
  (a) If $\mathcal{C} = \mathcal{A} + \mathcal{B}$, then $C(z) = A(z) + B(z)$;
  (b) If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then $C(z) = A(z) \cdot B(z)$;
  (c) If $a_0 = 0$ and $\mathcal{C} = \textsc{Seq}(\mathcal{A})$, then $C(z) = \frac{1}{1-A(z)}$
  (d) If $a_0 = 0$ and $\mathcal{C} = \textsc{Mset}(\mathcal{A})$, then
  $C(z) = \prod_{n\geq 1}(1-z^n)^{a_n} = \exp(A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \dots)$.

Note (for c)): The formal power series $\exp(z)$ is defined as usual; also $\log(z)$. First show that $\exp(A(z))$ is well defined if $a_0 = 0$ and that $\log(A(z))$ is well defined if $a_0 = 1$. The property $\exp(\log(z)) = z$ and the rules for exp and log of sums and products can be used without proof (even if the sums and products in question are infinite).

(2) Let $A = \sum_{ne\geq 0} a_n z^n$ be a formal power series.
  (a) Prove that $A(z)$ has a reciprocal if and only if $a_0 \neq 0$ and that the reciprocal is unique.
  (b) Show that the infinite sum

$$1 + A(z) + A(z)^2 + \dots$$

  is well defined (i.e. for every $n$, the series $\sum_k [z]^n A(z)^k$ converges absolutely) if and only if $|a_0| < 1$ and that it is the reciprocal of $1 - A(z)$.

Hint (part b)): For the "if" direction, prove first that for $n > 0$

$$[z^n](1+A(z)+A(z)^2+\dots) = \sum_{k=1}^{n}\left[\left(\sum_{\substack{j_1,\dots,j_k\geq 1\\ j_1+\dots+j_k=n}} \prod_{i=1}^{k} a_{j_k}\right)\cdot\left(\sum_{l=0}^{\infty}\binom{k+l}{k}a_0^l\right)\right]$$

and show that the infinite sums on the right hand converge absolutely for every fixed $k$.

(3) Find the generating function for the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$.

(4) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be combinatorial classes, with EGFs $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$, $B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ and $C(z) = \sum_{n\geq 0} c_n \frac{z^n}{n!}$. Prove the following (from the lecture):
  (a) If $A(z) \cdot B(z) = \sum_{n\geq 0}(\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k})\frac{z^n}{n!}$;
  (b) If $\mathcal{C} = \mathcal{A} * \mathcal{B}$, then $\mathcal{C}(z) = \mathcal{A}(z) \cdot \mathcal{B}(z)$;
  (c) If $a_0 = 0$ and $\mathcal{C} = \textsc{Set}(\mathcal{A})$, then $C(z) = \exp(A(z))$
  (d) If $a_0 = 0$ and $\mathcal{C} = \textsc{Cyc}(\mathcal{A})$, then $C(z) = \log(\frac{1}{1-A(z)})$.