Exercise Sheet 12

Problems due 27.01.2017

12.1 If R is a commutative ring (with unit) show that the set $P_n(R)$ of all polynomials over R of degree less than or equal to n is an R-module. Show also that the set $P(R)$ of all polynomials over R is a unitary associative R-algebra.

12.2 Let R be a commutative ring (with unit). Prove that a map $f : R \times R \to R$ is an R-morphism if and only if there exist $\alpha, \beta \in R$ such that

$$f(x, y) = \alpha x + \beta y \quad \text{for all } x, y \in R$$

12.3 If A and B are submodules of an R-module M, establish a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{\gamma} A \times B \xrightarrow{\pi} A + B \longrightarrow 0$$

(Hint: Observe that the 'obvious' definitions of γ and π, namely $\gamma(x) = (x, x)$ and $\pi(x, y) = x + y$ do not work. Try $\pi(x, y) = x - y$)

12.4 A short exact sequence of the form

$$(f, E, g) : \quad 0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} B \longrightarrow 0$$

is called an *extension of A by B*.

(a) Given any R-modules A and B show that at least one extension of A by B exists.

(b) (i) Two extensions (f_1, E_1, g_1) and (f_2, E_2, g_2) of A and B are said to be *equivalent* if there exists an R-morphism $h : E_1 \to E_2$ such that $h \circ f_1 = f_2$ and $g_2 \circ h = g_1$. Prove that such an R-morphism is an isomorphism.

(ii) Show that the following two extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_4$ are not equivalent:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$