DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 11

Problems due 20.01.2017

11.1 Let M be an abelian group and let $\operatorname{End} M$ be the set of all endomorphisms on M, i.e. the set of all group homomorphisms $f : M \to M$.

(a) Show that $\operatorname{End} M$ is an abelian group under the operation $(f, g) \mapsto f + g$ where $(f + g)(x) := f(x) + g(x)$. Show also that

1) $(\operatorname{End} M, +, \circ)$ is a ring with unit;

2) M is an $\operatorname{End} M$-module under the action $\operatorname{End} M \times M \to M$ given by

$$(f, m) \mapsto f \cdot m = f(m);$$

(b) 3) if R is a ring (with unit) and $\mu : R \to \operatorname{End} M$ a ring morphism with $\mu(1_R) = \mathrm{id}_M$, then M is an R-module under the action $R \times M \to M$ given by $(\lambda, m) \mapsto \lambda m = (\mu(\lambda))(m)$;

4) Describe the kernel of the ring morphism μ of part (b) 3).

11.2 (a) Let G be a finite abelian group with $|G| = m$. Show that if $n, t \in \mathbb{Z}$ then

$$n \equiv t \mod m \Longrightarrow ng = tg \quad \forall\, g \in G$$

Deduce that G is a $\mathbb{Z}/m\mathbb{Z}$-module under the action $\mathbb{Z}/m\mathbb{Z} \times G \to G$ given by $(n + m\mathbb{Z}, g) \mapsto ng$. Conclude that every finite abelian group whose order is a prime p can be regarded as a vector space over a field of p elements.

(b) Prove that the ring of endomorphisms of the abelian group $\mathbb{Z}$ is isomorphic to the ring $\mathbb{Z}$, and that the ring of endomorphisms of the abelian group $\mathbb{Q}$ is isomorphic to the field $\mathbb{Q}$.

(Hint: use problem 11.1; note that if $f \in \operatorname{End} \mathbb{Z}$ then $f = \mu(f(1))$.)

11.3 If A is a ring (with unit) define its *centre* to be

$$Z(A) := \{x \in A : xy = yx \ \forall\, y \in A\}$$

Show that $Z(A)$ is a ring (with unit). If R is a commutative ring (with unit) prove that A is a unitary associative R-algebra if and only if there is a 1-preserving (sending 1_R to $1_{Z(A)}$) ring morphism $\varphi : R \to Z(A)$.

11.4 (a) Let M be an R-module. If S is a non-empty subset of M, define the *annihilator of S in R* to be

$$\mathrm{Ann}_R S = \{r \in R : rx = 0_M \ \forall\, x \in S\}.$$

Show that $\mathrm{Ann}_R S$ is a left ideal of R and that it is a two-sided ideal whenever S is a submodule of M.

(b) Let M be an R-module. If $r, s \in R$ show that

$$r - s \in \mathrm{Ann}_R M \Longrightarrow rx = sx \ \forall\, x \in M$$

Deduce that M can be considered as an $R/\mathrm{Ann}_R M$-module. Show that the annihilator of M in $R/\mathrm{Ann}_R M$ is zero.