DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 10

Problems due 13.01.2017

10.1 Identify $\bigwedge^2(\mathbb{R}^3)$ with $\mathbb{R}^3$ by identifying $e_1 \wedge e_2$ with e_3, $e_2 \wedge e_3$ with e_1 and $e_3 \wedge e_1$ with e_2. Show that under this identification, the exterior product $v \wedge w \in \bigwedge^2(\mathbb{R}^3) = \mathbb{R}^3$ is the same as the cross product $u \times w \in \mathbb{R}^3$.

10.2 (a) Let V have basis $\{e_1, e_2\}$ and let $T : V \to V$ be given by $T(e_1) = ae_1 + ce_2$ and $T(e_2) = be_1 + de_2$. Compute $\bigwedge^2 T : \bigwedge^2(V) \to \bigwedge^2(V)$ in terms of this basis. What is $\det(T)$?

(b) Let V be a finite dimensional vector space. Let $x \in \bigwedge^r(V)$, $y \in \bigwedge^s(V)$ and $z \in \bigwedge^t(V)$. Show that $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and that $x \wedge y = (-1)^{rs} y \wedge x$ for $x \wedge y$ as defined in Lemma 3.35.

10.3 Let A be a commutative ring with $1_A \neq 0_A$ and K a field. Show that A is an algebra over K if and only if A contains (an isomorphic copy of) K as a subring.

(See Remark 4.28 (1) for the notion of an algebra over a field).

10.4 (a) Let M be the additive group of rational numbers. Show that any two elements of M are linearly dependent (over $\mathbb{Z}$). Show then, that M cannot have a basis over $\mathbb{Z}$.

(b) Let $a, b \in \mathbb{Z}$, let $d = \gcd(a, b)$. Argue that $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$.