DISCRETE AND ALGEBRAIC STRUCTURES, WS 2016/17

Exercise Sheet 1

Problems due 14.10.2016

(1) Proof the multinomial theorem: For any $n \in \mathbb{N}$,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

where the sum is over all $k_1, k_2, \ldots, k_m \in \mathbb{N} \cup \{0\}$ satisfying $k_1 + k_2 + \cdots + k_m = n$.

(2) (a) Compute the generalized binomial coefficient $\binom{\frac{1}{2}}{k}$ for $k \in \mathbb{N}$.

(b) Let k, n be nonnegative integers. Show that the number of 0-1 sequences consisting of precisely k ones and n zeros is

$$\binom{n+k}{k}.$$

How many of these sequences have no two adjacent ones?

(3) Suppose that $f, g, h \colon \mathbb{N} \to \mathbb{R}$ satisfy

$$f(n) = 1 + O(h(n)), \quad g(n) = 1 + O(h(n)), \quad \text{and} \quad h(n) = o(1).$$

Show that

$$f(n) \cdot g(n) = 1 + O(h(n)) \quad \text{and} \quad \frac{1}{f(n)} = 1 + O(h(n)).$$

Deduce that

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}} \cdot \left(1 + O\left(\frac{1}{n}\right)\right).$$

(4) For which initial conditions a_0, a_1, a_2 does the solution of the recurrence

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \quad n \geq 3$$

satisfy

a) $\lim_{n\to\infty} a_n = \infty$, b) $\lim_{n\to\infty} a_n = -\infty$, c) $\lim_{n\to\infty} a_n = c \in \mathbb{R}$,

d) none of the above?