P. Achar — Lecture 3.

Today : Application to Rep Theory in pos. char.

Thm (Mirković - Vilonen " Geometric Satake")

Let $\Bbbk$ be a comm Noetherian ring of fin global dim

$$\begin{tikzcd} \mathrm{Perv}_{G(\mathbb{O})}(\mathrm{Gr}, \Bbbk) \arrow[rr, dashed, "\exists! \ \sim"] \arrow[dr, "\mathbb{H}^\bullet"'] & & \mathrm{Rep}(\check{G}_\Bbbk) \arrow[dl, "\text{forget}"] \\ & \Bbbk\text{-mod} & \end{tikzcd}$$

coeff's of sheaves

split gp scheme over $\Bbbk$

Important case $\Bbbk = \mathbb{Z}$

$\Bbbk =$ alg closed field of pos. char.

I. Weyl modules $X_*^+ =$ domin cowts for G $=$ domin wts for $\check{G}$

$\lambda \in X_*^+ \rightsquigarrow L_{\mathbb{C}}(\lambda) =$ irred $\check{G}_{\mathbb{C}}$-rep of h. wt. λ

Fact. $\exists!$ (up to isom) minimal $\underbrace{\check{G}_{\mathbb{Z}}\text{-stable free ab subgp of rank} = \dim_{\mathbb{C}} L_{\mathbb{C}}(\lambda)}_{\text{"lattice"}}$

of $L_{\mathbb{C}}(\lambda)$. Call it $\Delta_{\mathbb{Z}}(\lambda)$

$\forall$ field $\Bbbk$. let $\Delta_\Bbbk(\lambda) = \Bbbk \otimes_{\mathbb{Z}} \Delta_{\mathbb{Z}}(\lambda)$: a $\check{G}_\Bbbk$-module Weyl module.

In general, $\Delta_\Bbbk(\lambda)$ is NOT irred

Has unique irred quot $\Delta_\Bbbk(\lambda) \twoheadrightarrow L_\Bbbk(\lambda)$ irred $\check{G}_\Bbbk$-rep of h. wt. λ

$\mathrm{ch}\, \Delta_\Bbbk(\lambda) = \mathrm{ch}\, L_{\mathbb{C}}(\lambda) =$ Weyl character formula

II. MV cycles. Let $j_\lambda : \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}$ be the inclusion map.

Def-n $\mathcal{I}_!(\lambda, \Bbbk) = {}^p\mathcal{H}^0\big(j_{\lambda !}(\underline{\Bbbk}[\dim \mathrm{Gr}_\lambda])\big)$

Thm Under geom Satake, $\mathcal{I}_!(\lambda, \Bbbk) \longmapsto \Delta_\Bbbk(\lambda)$

Use $\mathrm{Gr}_T = X_* \hookrightarrow \mathrm{Gr}_G$

$\lambda \longmapsto \underline{t}_\lambda$

Choose $B = TU$ Borel, torus, unipotent

Def-n $S_\lambda = U(\mathcal{K})\, \underline{t}_\lambda \subset \mathrm{Gr}_G$

Thm. (Mirković - Vilonen)

1) $S_\mu \cap Gr_\lambda \neq \phi \iff \mu$ is a weight of $\Delta_{\Bbbk}(\lambda)$

If non-empty, equi-dim of dim $\rho(\mu+\lambda)$

2) The μ-wt space of $\mathbb{H}^\cdot(I_!(\lambda, \Bbbk)) = \Delta_{\Bbbk}(\lambda)$ is

canonically isom to $H_c^{2\rho(\mu+\lambda)}(S_\mu \cap Gr_\lambda, \Bbbk)$

singular cohom w/ compact supp.

Proof of 2 uses $(j_{\lambda!}, j_\lambda^!)$ adjunction.

Conseq : μ-wt space of $\Delta_{\Bbbk}(\lambda)$ has a basis indexed by irred comp of $S_\mu \cap Gr_\lambda$

For Baumann's talk : use $I_!(\lambda, \mathbb{C}) = IC(\overline{Gr_\lambda}, \mathbb{C})$

Important observation : Works for $\Bbbk = \mathbb{Z}$.

Exercise $G = GL_3$ Compute MV cycles for $S_{(0,0,0)} \cap Gr_{(1,0,-1)}$ (should find 2 components)

Related : Find the space of upper triangular nilp matrices with 2 Jordan blocks

III. Principal block. From now on, $\Bbbk$: a field of characteristic $p > h$ = Coxeter # for G

Basic problem : Compute $\mathrm{ch}\, L_{\Bbbk}(\lambda)$

Related : Compute $\mathrm{ch}\, T_{\Bbbk}(\lambda)$ indecomposable tilting module of h. wt. λ

has a Δ-filtration and (dual Δ)-filtration

1^{st} reduction : can reduce to solving this problem for $L_{\Bbbk}(\lambda)$, $T_{\Bbbk}(\lambda) \in \mathrm{Rep}_0(\check{G}_{\Bbbk})$

// principal block.

Def-n $\mathrm{Rep}_0(\check{G}_{\Bbbk}) = \langle L_{\Bbbk}(\quad) \rangle$

should be of the form $w\rho - \rho + pw\lambda$ for some $w \in W$, $\lambda \in X_*$ (not nec dom.)

For $\check{G} = PGL_2$ $(X_* = 2\mathbb{Z})$

Dom wts allowed in $\mathrm{Rep}_0(\check{G}_{\Bbbk})$ are : $0, 2p-2, 2p, 4p-2, 4p, \ldots$

Fact $\exists$ bij $X_* \longleftrightarrow$ dom wts allowed in $\mathrm{Rep}_0(\check{G}_{\Bbbk})$

$\lambda \longmapsto w\rho - \rho + pw\lambda$ where $w \in W$ min'l s.t. $w\lambda \in X_*^+$

call this "λ"

Irred in $\mathrm{Rep}_0(\check{G}_{\Bbbk})$: $L_{\Bbbk}(``\lambda")$ $\leftarrow$ λ any wt

Problem : compute $\mathrm{ch}\, L_{\Bbbk}(``\lambda")$, $T_{\Bbbk}(``\lambda")$

IV. Iwahori orbits

$$\begin{array}{ccc} G(\mathbb{O}) & \xrightarrow[e]{t \mapsto 0} & G(\mathbb{C}) \\ \cup & & \cup \\ I = e^{-1}(B) & & B \end{array}$$

↖ an Iwahori subgroup.

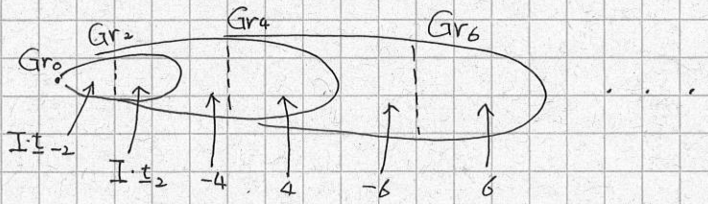
e.g. for GL_2 $\quad I = \begin{bmatrix} \mathbb{O}^\times & \mathbb{O} \\ t\mathbb{O} & \mathbb{O}^\times \end{bmatrix}$

Thm 1) $\{I\text{-orbits on } Gr\} \longleftrightarrow X_*$

$$I \cdot t_\lambda \longleftarrow\!\!\shortmid \lambda$$

2) If $\lambda \in X_*^+$, $Gr_\lambda = \bigsqcup_{\mu \in W\lambda} I \cdot t_\mu$

Example $G = SL_2$, $X_* = 2\mathbb{Z}$



□.

Idea / Hope : $Perv_{(I)}(Gr, \Bbbk)$ ← a refinement of $Perv_{G(\mathbb{O})}(Gr, \Bbbk)$

should be related to $Rep_0(\check{G}_\Bbbk)$.

Plan

graded Finkelberg - Mirković conjecture

$$D^b Rep_0(\check{G}_\Bbbk) \xleftrightarrow[\sim]{③} D^b_{triv}(\check{B}) \xleftarrow[F]{②} D^b Coh^{\check{G} \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \xleftrightarrow[P]{\sim\ ①} D^{mix}_{(I)}(Gr, \Bbbk)$$

$V \mapsto V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}$

$$Rep(\check{G}_\Bbbk) \xleftrightarrow[\sim]{g.\ Satake} Perv_{G(\mathbb{O})}(Gr, \Bbbk)$$

$\cup$ $\quad Tilt\ Perv(Gr, \Bbbk) \overset{\sim}{\longleftrightarrow}$ ④ A_{sph}

Riche - Williamson

"graded"

① $D^{mix}_{(I)}(Gr, \Bbbk)$: mixed derived var of $Perv_{(I)}(Gr, \Bbbk)$

$\tilde{\mathcal{N}} = \check{G} \times^{\check{B}} \check{\mathfrak{u}}$ = Springer resolution for $\check{G}$

↖ Lie (unip radical of $\check{B}$)

$\check{G} \times \mathbb{G}_m \circlearrowright \tilde{\mathcal{N}}$ $\quad \mathbb{G}_m$ acts by scaling

Thm (A - Rider 2014, (also Mautner - Riche 2015))

$\exists$ an equiv of cat ① that's compat w/ geom Satake.

Precursor to ①

Thm (Brylinski 1989) $\lambda \in X_*^+$, $\mu \in X_*$

$$M_\lambda^\mu(q) = \sum_{n \geq 0} \Big(\sum_i (-1)^i \dim Ext^i \big(L_G(\lambda) \otimes \mathcal{O}, \mathcal{O}_{\tilde{\mathcal{N}}}(\mu)\langle n \rangle \big) \Big) q^n$$

Exercise Check this for SL_2

②,③ A-Riche

$D^b_{\mathrm{triv}}(\check{B}) \subset D^b\mathrm{Rep}(\check{B})$ generated by $\Bbbk(p\lambda)$

F is a "degrading functor"

$$\bigoplus_n \mathrm{Hom}(\mathcal{F}, \mathcal{G}\langle -n\rangle[n]) \xrightarrow{\sim} \mathrm{Hom}(F(\mathcal{F}), F(\mathcal{G}))$$

④ A-Makisumi-Riche-Williamson

Thm (A-Riche) $\mathrm{ch}\, L_\Bbbk(``\lambda") = \sum_i (-1)^i \,\mathrm{rank}\, {}^pH^i(IC(I\cdot t_\lambda)|_{I\cdot t_\mu})\cdot \mathrm{ch}\,\Delta_\Bbbk(``\mu")$

Thm (AMRW) anti-spherical p-KL-polynomials

$$\mathrm{ch}\, T_\Bbbk(``\lambda") = \sum {}^p n_{\lambda,\mu}(1)\cdot \mathrm{ch}\,\Delta_\Bbbk(``\mu").$$