P. Achar Lecture 2

G : conn reduc gp / $\mathbb{C}$, Gr : affine Grassmannian (most of today, top'l version)

$G(\mathbb{O})$-orbits in Gr $\xleftrightarrow{\text{bij}}$ X_*^+ = domin coweights

$\mathrm{Irr}(\check{G})$ $\xleftrightarrow{\text{bij}}$ $\|$ domin wts for $\check{G}$ (Langlands dual gp)

Goal Understand the relationship between topology of Gr & $\mathrm{Rep}(\check{G})$

I. Convolution

$\mathcal{F}_1, \mathcal{F}_2$: two sheaves or $\mathcal{D}$-modules or $\mathbb{C}$-valued functions on Gr

Assume : $G(\mathbb{O})$-equivariant (for fns : constant on orbits)

compactly supported (i.e. supp on a fin union of $G(\mathbb{O})$-orbits)

Convolution diagram.

$$\begin{tikzcd} \mathrm{Gr}\times\mathrm{Gr} & G(\mathbb{K})\times\mathrm{Gr} \arrow[l, "p"'] \arrow[r, "q"] & \boxed{G(\mathbb{K})\times^{G(\mathbb{O})}\mathrm{Gr}} \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"] & \mathrm{Gr} \\ & & \mathrm{Gr} & \end{tikzcd}$$

$\mathrm{Gr}\,\tilde{\times}\,\mathrm{Gr} = \mathrm{Conv}_2$

$\mathrm{Gr}\times\mathrm{Gr} = G(\mathbb{K})/G(\mathbb{O})$

quotient by $G(\mathbb{O})$ in 1st factor

"middle" action of $G(\mathbb{O})$

$g(h, x) = (hg^{-1}, gx)$

a free action

quotient by middle action

$(h, x) \mapsto hx$, m proper

makes $\mathrm{Gr}\,\tilde{\times}\,\mathrm{Gr}$ into a bdle over Gr w/ fibers Gr

Can make this picture manageable by looking at finite dim'l pieces $\overline{\mathrm{Gr}_\lambda}, \overline{\mathrm{Gr}_\mu} \subset \mathrm{Gr}$

$\overline{\mathrm{Gr}_\lambda}\,\tilde{\times}\,\overline{\mathrm{Gr}_\mu} \subset \mathrm{Gr}\,\tilde{\times}\,\mathrm{Gr}$

Goal define a new sheaf / $\mathcal{D}$-mod / fn " $\mathcal{F}_1 * \mathcal{F}_2$ "

3 steps : Step 0. Form $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ sheaf / fn on $\mathrm{Gr}\times\mathrm{Gr}$

Step 1. Pull back along p : $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ ← Automatically equivariant for the middle $G(\mathbb{O})$-action

Step 2. Descend along q

For fns : $\exists!$ fn on $\mathrm{Gr}\,\tilde{\times}\,\mathrm{Gr}$ called $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$ s.t. $q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2) = p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$

For sheaves : $q^* : \mathrm{Sh}(\mathrm{Gr}\,\tilde{\times}\,\mathrm{Gr}) \longrightarrow \mathrm{Sh}_{G(\mathbb{O})}(G(\mathbb{K})\times\mathrm{Gr})$ ← equivariance

is an equivalence of cat

Descend : Apply the inverse functor to $p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$, get $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$

Step 3. $\mathcal{F}_1 * \mathcal{F}_2 \overset{\text{def}}{=} m_*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$

fn version $(\mathcal{F}_1 * \mathcal{F}_2)(x) =$ "$\int_{m^{-1}(x)}$" $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$

Easiest way to make this pricise : Switch to an $\mathbb{F}_q$ -version

$$Gr = G(\mathbb{F}_q((t))) / G(\mathbb{F}_q[[t]])$$

In this setup, supp $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$ is a finite set of pts $\int_{m^{-1}(x)}$ means $\sum$

Conclusions

① $*$ makes the (derived) category of $G(\mathbb{O})$-eqvt sheaves / $\mathcal{D}$-modules into a monoidal category.

② $*$ makes the space of $G(\mathbb{O})$-eqvt fns into a ring.

$\mathcal{H}_{sph}$ = the spherical Hecke alg

= the ring of cpt supp $G(\mathbb{O})$-eqvt fns on Gr

(or bi-$G(\mathbb{O})$-equiv functions on $G(\mathbb{K})$)

Basis: $\chi_\lambda(x) = \begin{cases} 1 & \text{if } x \in Gr_\lambda \\ 0 & \text{otherwise} \end{cases}$ $\{\chi_\lambda \mid \lambda \in X_*^+\}$ basis for $\mathcal{H}_{sph}$.

II. Satake isomorphism

Thm (Satake 1963) $\mathcal{H}_{sph} \xrightarrow{\sim} \mathbb{C}[X_*]^W$ ← Weyl gp invariants

$\chi_\lambda \longmapsto \sum_{\mu \in W\cdot\lambda} e^\mu$

$\mathbb{C}[X_*]$: group ring of X_*

Elts : fin sums $\sum_{\mu \in X_*} c_\mu e^\mu$

In particular, $\mathcal{H}_{sph}$ is commutative ! □.

Another basis for $\mathbb{C}[X_*]^W$ (different from χ_λ's)

$$ch\, L(\lambda) = \sum (\dim L(\lambda)_\mu)\, e^\mu$$

$L(\lambda)$: irr $\check{G}$-rep of h. wt λ

Question Can you see this basis in the topology of Gr ?

Answer. (Lusztig 1983) Yes, sort of : Let q vary

Combinatorial interlude. $\lambda \in X_*^+$, $\mu \in X_*$

$\mathcal{M}_\lambda^\mu(q)$: Lusztig's q-analogue of the wt multiplicity (defn in exercise)

a q-deformation of Kostant formula

$$\mathcal{M}_\lambda^\mu(1) = \dim(L(\lambda)_\mu).$$

Thm (Lusztig) If $\lambda, \mu \in X_*^+$

$$\mathcal{M}_\lambda^\mu(q) = \sum_i \operatorname{rank} H^{\bullet}\left(IC(\overline{Gr_\lambda})|_{Gr_\mu}\right) q^{i/2}$$

$\bullet = \dim Gr_\mu - \dim Gr_\lambda + i$

Consequences

1) $\mathcal{H}^\cdot(IC(\overline{Gr_\lambda}))$ obeys a parity vanishing condition

2) $\mathcal{M}^\mu_\lambda(q)$ has coeffs ≥ 0 for $\lambda, \mu \in X^+_*$

3) $\dim \mathbb{H}^\cdot(IC(\overline{Gr_\lambda})) = \dim L(\lambda)$

4) Suppose $L(\lambda) \otimes L(\mu) = \bigoplus L(\nu_i)$

Then $IC(\overline{Gr_\lambda}) * IC(\overline{Gr_\mu}) = \bigoplus IC(\overline{Gr_{\nu_i}})$

combinatorial, not (yet) functorial

5) $*$ is exact for perv sh / $\mathcal{D}$-modules !

Goal : make this functorial

Main missing ingredient : commutativity of $*$

IV. Fusion product

Recall Def-n 2b Gr represents $R \mapsto (\mathcal{L}, \beta)$

$\left(\begin{array}{l} \mathcal{L} : \text{a } G\text{-bundle on } \mathcal{D} \\ \beta : \mathcal{L}|_{\mathcal{D}^\times} \xrightarrow{\sim} \mathcal{L}^0|_{\mathcal{D}^\times} \end{array} \right.$

Thm ("def-n 2c") Let $x \in \mathbb{A}^1$

Gr represents $R \mapsto (\mathcal{L}, \beta)$ $\quad \mathcal{L}$: a G-bundle on $\mathbb{A}^1$

$\beta : \mathcal{L}|_{\mathbb{A}^1 - x} \xrightarrow{\sim} \mathcal{L}^0|_{\mathbb{A}^1 - x}$

Idea (Drinfeld, Mirković-Vilonen)

Take two points in $\mathbb{A}^1$, Let them vary

The fusion space Fus = the ind-scheme that represents

$R \longmapsto (x_1, x_2, \beta, \mathcal{L})$

$\left(\begin{array}{l} x_1, x_2 \in \mathbb{A}^1, \ \mathcal{L} : \text{a } G\text{-bundle on } \mathbb{A}^1 \\ \beta : \mathcal{L}|_{\mathbb{A}^1 - \{x_1, x_2\}} \xrightarrow{\sim} \mathcal{L}^0|_{\mathbb{A}^1 - \{x_1, x_2\}} \end{array} \right.$

$$\begin{tikzcd} Gr \times \mathbb{A}^1 \arrow[r, hook, "i"] \arrow[d] & \text{Fus} \arrow[d] & Gr \times Gr \times \mathcal{U} \arrow[l, hook', "j"'] \arrow[d] \\ \mathbb{A}^1 \arrow[r, hook, "\text{diag}"] & \mathbb{A}^2 & \mathcal{U} = \{(x_1, x_2) \mid x_1 \neq x_2\} \arrow[l, hook'] \end{tikzcd}$$

Thm (Mirković-Vilonen) $\mathcal{F}_1 * \mathcal{F}_2 = i^*(j_{!*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \underline{\mathbb{C}}_{\mathcal{U}}))|_{Gr \times \{x\}}$

your favorite point

constant sheaf on $\mathcal{U}$

There's an autom of <u>Fus</u> : Swap x_1 & x_2

On $Gr \times \underset{\uparrow \text{diagonal}}{\mathbb{A}^1}$: identity

On $Gr \times Gr \times \mathcal{U}$: Swaps the two Gr's

<u>Cor.</u> $\mathcal{F}_1 * \mathcal{F}_2 \simeq \mathcal{F}_2 * \mathcal{F}_1$

<u>Thm</u> (Mirković - Vilonen)

$$\begin{tikzcd} \mathrm{Perv}_{G(\mathcal{O})}(Gr) \arrow[rr, "\exists(!)\text{ equiv.}", "\sim"'] \arrow[dr, "H"'] & & \mathrm{Rep}(\check{G}) \arrow[dl, "\text{forget.}"] \\ & \mathrm{Vect} & \end{tikzcd}$$

or $\mathcal{D}$-modules