Geometry and Representation Theory, ESI Vienna, Jan. 2017

P. Achar, Introduction to affine Grassmannians and the geometric Satake equivalence.

Lecture I.

Affine Grassmannians

1. Today : Definitions (4 or 5 versions)
2. Wednesday : Convolution, Satake isomorphism
3. Friday : Other topics in rep theory

Exercises on the conference webpage (go to mini-course page)

I. Topological version for GL_n

$\mathbb{O} = \mathbb{C}[[t]]$ = formal power series $\sum_{i=0}^{\infty} a_i t^i$
t : indeterminate

$\cap$

$\mathbb{K} = \mathbb{C}((t))$ = formal Laurent series $\sum_{i \geq -r}^{\infty} a_i t^i$

Rmk $\mathbb{O}^\times$ = power series w/ non-zero constant term

$\mathbb{K}$: a field

Def-n A lattice in $\mathbb{K}^n$ is a free $\mathbb{O}$-submodule of rank n

Analogues - free abelian subgroups of $\mathbb{R}^n$

- free $\mathbb{Z}_p$ submodules of $\mathbb{Q}_p$

The standard lattice $\mathcal{L}^0 = \mathbb{O}^n \subset \mathbb{K}^n$

Another example. $n = 2$ $\mathcal{L} = \mathrm{Span}_{\mathbb{O}} \left\{ \begin{bmatrix} t^{-5} + e^t \\ t \sin t \end{bmatrix}, \begin{bmatrix} 1 \\ t^3 \end{bmatrix} \right\}$ ... (*)

Def-n 1a The affine Grassmannian for GL_n, $Gr = Gr_{GL_n}$ is the set of lattices in $\mathbb{K}^n$

1st goal. Give this set a topology

Tools. 1) Valuation

Say $\mathcal{L} = \mathrm{Span}_{\mathbb{O}} \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$

$\det \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \in \mathbb{K}^\times$ ← not well-defined
(i.e. not indep of basis)
but smallest power t is well-defined

Valuation of $\mathcal{L} = v(\mathcal{L}) = \min \{ n \mid t^n \text{ occurs in } \det(\text{basis}) \}$

Example (*) $v(\mathcal{L}) = -2$

2) Comparison with std lattice

Example (*) $\quad \mathcal{L} \subset t^{-5}\mathcal{L}^0$, Easy $\quad t^5\mathcal{L}^0 \subset t^3\mathcal{L}^0 \subset \mathcal{L}$

More generally $\quad \forall \mathcal{L} \ \exists a \geq 0$ s.t. $\quad t^a\mathcal{L} \subset \mathcal{L} \subset t^{-a}\mathcal{L}$

Let (temporary) $\quad Gr^{k,[-a,a]} = \{ \mathcal{L} \mid v(\mathcal{L}) = k,\ t^a\mathcal{L}^0 \subset \mathcal{L} \subset t^{-a}\mathcal{L}^0 \}$

So $\quad Gr = \bigcup Gr^{k,[-a,a]}$

$$Gr^{k,[-a,a]} \hookrightarrow \left\{ \begin{array}{c} \text{subspaces of } t^{-a}\mathcal{L}^0 / t^a\mathcal{L}^0 \simeq \mathbb{C}^{2na} \\ \text{of dim } na-k \end{array} \right\}$$

$$\mathcal{L} \longmapsto \mathcal{L}/t^a\mathcal{L}^0$$

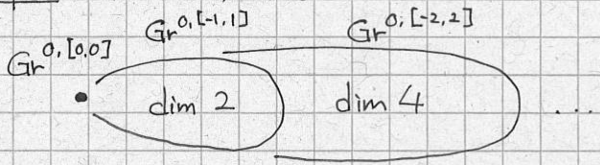
becomes a proj variety

$\mathbb{C}$-vec space of dim $na-k$

an ordinary Grassmannian $\hookrightarrow \mathbb{P}^\bullet$ Plücker embedding

Thm This data equips Gr with the structure of an ind-variety

Example $\quad n = 2$, val 0



$\dim Gr^{0,[-a,a]} = 2a$

Observations

$GL_n(\mathbb{K}) \circlearrowright \mathbb{K}^n$ sends lattices to lattices, acts transitively on Gr

Stabilizer of $\mathcal{L}^0 = GL_n(\mathbb{O})$

$\exists_{\text{bij}} \quad Gr \xleftarrow{\sim} GL_n(\mathbb{K})/GL_n(\mathbb{O})$

$GL_n(\mathbb{O}) \circlearrowright Gr$ preserves valuation & comparisons w/ $\mathcal{L}^0$

So each $Gr^{k,[-a,a]}$ = union of $GL_n(\mathbb{O})$-orbits

Thm (Exercise)

$$GL_n(\mathbb{O})\text{-orbits on } Gr \longleftrightarrow \{ (a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \cdots \geq a_n \}$$

$$\mathrm{Span}\left\{ \begin{bmatrix} t^{a_1} \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ t^{a_2} \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \\ t^{a_n} \end{bmatrix} \right\} \longleftarrow (a_1, \dots, a_n)$$

II. Topological version for general G

G = conn. reductive gp / $\mathbb{C}$

Def-n (1b) The affine Grassmannian for G is $Gr_G = G(\mathbb{K})/G(\mathbb{O})$

Thm Left $G(\mathbb{O})$-orbits in $Gr_G \longleftrightarrow X_*^+$ = dominant coweights for G

$$Gr_\lambda \longleftarrow \lambda \in X_*^+$$

Def-n $\overline{Gr_\lambda} = \bigcup_{\mu \le \lambda} Gr_\mu$, $\mu \le \lambda$ means $\lambda - \mu = \Sigma$ of pos coroots.

Thm Each $\overline{Gr_\lambda}$ admits the structure of a projective variety making Gr into an ind-variety.

Proof sketch. Embed each $\overline{Gr_\lambda} \hookrightarrow \mathbb{P}^{\text{something}}$ (use rep theory of Kac-Moody groups)

Examples.

1) T : a torus, $Gr_T = X_*(T)$, $Gr_{GL_1} = Gr_{\mathbb{G}} \simeq \mathbb{Z}$.

2) GL_{SL_n} = lattices of val. 0

3) Other classical groups : exercises

4) Note : $SL_2(\mathbb{C}) \twoheadrightarrow PGL_2(\mathbb{C})$ surj

But $\underset{\substack{\uparrow \\ \text{conn.}}}{Gr_{SL_2}} \longrightarrow \underset{\substack{\uparrow \\ \text{2 comp.}}}{Gr_{PGL_2}}$ NOT surj

Facts 1) Each $Gr_\lambda \simeq$ affine space bdle over some G/P (depends on λ)

So simply-conn

2) $\dim Gr_\lambda = \langle \lambda, 2\rho \rangle$. $2\rho = \Sigma$ pos roots

3) $\pi_0(Gr) = \pi_1(G) = X_* /$ coroot lattice

Gr connected $\Longleftrightarrow$ G semisimple, simply-conn

III. Scheme version for GL_n

Def-n (2a) The affine Grassmannian for GL_n is the ind-scheme that represents the functor $\underset{\substack{\uparrow \\ \text{a } \mathbb{C}\text{-algebra}}}{R} \longmapsto$ set of $\underset{\substack{\uparrow \\ \text{fin gen projective } R[[t]]\text{-submodules of } R((t))^n \text{ that generates } R((t))^n \text{ after inverting } t}}{R[[t]]\text{-lattices}}$ in $R((t))^n$

Thm This functor is rep by an ind-scheme.

Obvious : $\mathbb{C}$-points of def-n (2a) = def-n (1a)

~~Example~~ Exercise Compute $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-points of ~~GL~~ $Gr_{\mathbb{G}_m}$

Show that $Gr_{\mathbb{G}_m}$ is not reduced.

IV. Scheme version for general G.

Def-n R : a $\mathbb{C}$-algebra

An R-family of G-bundles on the formal disc $\mathcal{D} = \operatorname{Spec} \mathbb{C}[[t]]$

is a $\otimes$-functor $\mathcal{L} : \mathrm{Rep}(G) \longrightarrow \{\text{fin gen projective } R[[t]]\text{-modules}\}$

Standard family. $\mathcal{L}^0 : V \longmapsto V \otimes_{\mathbb{C}} R[[t]]$

An R-family of G-bundles on the punctured disc $\mathcal{D}^\times$

same def-n ... $R((t))$-modules

Def-n 2b Gr_G is the ind-scheme that represents ~~set of isom classes of pairs~~

$R \longmapsto$ set of isom classes of pairs $(\mathcal{L}, \beta)$

$\mathcal{L} =$ an R-family of G-bdles on $\mathcal{D}$

$\beta : \mathcal{L}|_{\mathcal{D}^\times} \xrightarrow{\sim} \mathcal{L}^0|_{\mathcal{D}^\times}$

Exercise 2a & 2b agree for GL_n