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Mathematics and Arts: Towards a balance between artistic intuition and mathematical complexity

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Website of the project: `http://thecollaborativemind.com/`

1 Introduction

An engineer, a physicist and a mathematician travel on a train through Scotland and watch the passing landscape. Suddenly, the engineer exclaims: "Hey, there are black sheep in Scotland!", upon which the physicist corrects: "Actually, we can only say that there is at least one black sheep in Scotland." The mathematician says: "Precisely speaking, there is at least one sheep with one black side in Scotland".

Recalling this first-year's joke, is it a funny thing to talk of one-sided sheep?

Let's ask a different question: What would happen if there were also an artist present? What would an artist associate with a one-sided sheep? And what might happen if the mathematician and the artist were to join teams?

The research project entitling this report brings together two mathematicians and two artists of separate fields: Karin Baur (Algebra), Gerhard Eckel (Computer-music, Sound-installations), Klemens Fellner (Partial Differential Equations and Applications), Tamara Friebel (Composition, Architecture).

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Mathematics and the arts have been linked by common roots since the dawn of human civilisation. They are spoken of as manifestations of inner truths of the world. It is easy to name a few highlights where the expressions of mathematics and the arts touched one another throughout history: The Vedic scriptures and their intrinsic realisations of music and architecture, the Greek philosophers and the harmonic world of Pythagoras, the Renaissance with da Vinci's universality and Palladio's architecture.

In the last decades, however, it seems that mathematical structures and dynamical systems have increasingly influenced the arts.

The architect *Le Corbusier* used proportional sequences constructed from the golden ratio and human body proportions to create his outstanding buildings. The composer *Xenakis* (in his early years a co-worker of Le Corbusier) transferred the interest in proportional systems and mathematical/graphical structures into the field of music.

The contemporary designs of architects like *Zaha Hadid* are based on curved geometric forms often discovered by time-costly experimenting with parametric design software like *GenerativeComponents*. In *aleatoric music*, composers turned to stochastic processes in an attempt to overcome the constraints of twelve-tone music and serialism.

The composer *Ligeti*, on the other hand, used a more metaphorical, transformative approach of scientific research in his concept of musical "permeability", which describes the freedom of choice of intervals of a musical structure.

To succeed artistically when utilising mathematical structures or dynamical systems, a crucial link seems to be the individual intuition of the artist.

Le Corbusier wrote eloquently about the construction of his *modulor* sequences, yet failed to convey how a proportional system yields great architecture. In fact, he remained the only leading architect using the modulor, by which means he intended to revolutionise architecture for ever.

Ligeti arrives at a new micropolyphonic method of composition where the polyphony cannot be actively heard by the listener, but remains underwater, hidden from the listener. He thus uncovered novel and fresh methods of listening and organising sound.

Tamara Friebel's work complex *Canto Morph* is a contemporary example of how mathematics can inspire intuitive artistic processes across the fields of architecture and composition. Based on proportional relations found within a strange attractor set, Friebel designed the Canto Morph pavilions, a performance space specifically designed to host the opera "Canto Morph" also composed by Friebel (see Fig. 1). The artistic and intuitive use of mathematical systems was researched by Friebel in the context of composition [10] as well as architecture [11, 12].

In the context of the artistic research project "The Choreography of Sound" [4]

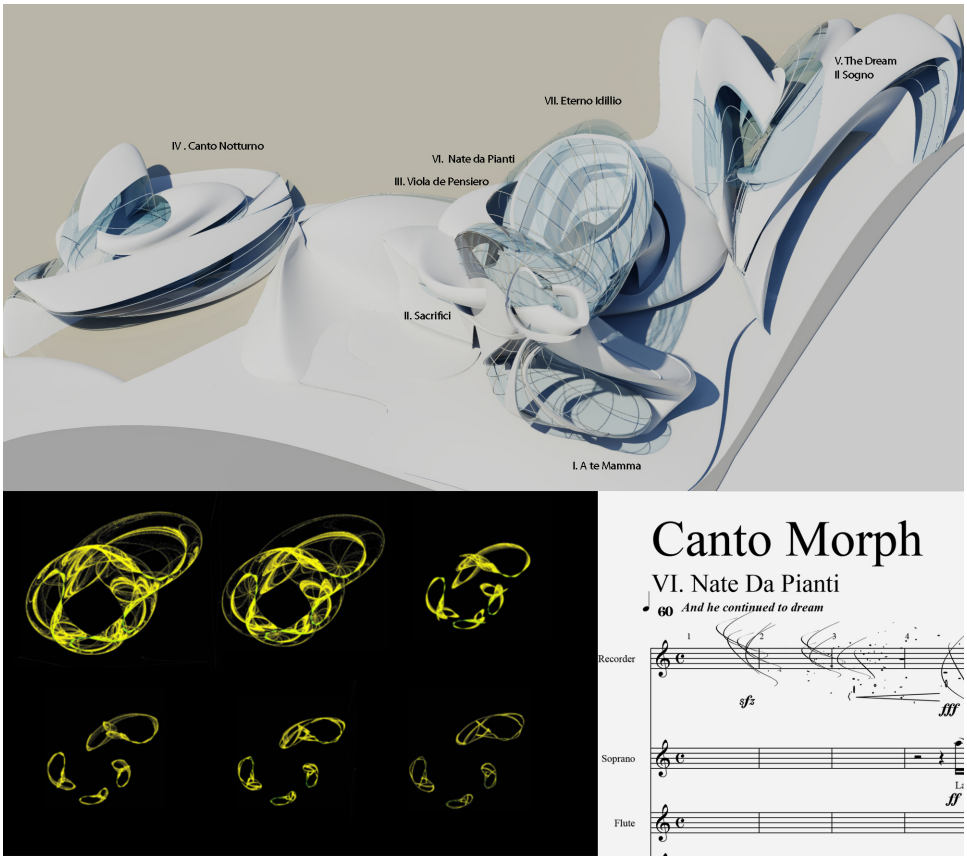


Figure 1: A set of strange attractors and its musical and architectonic derivatives.

Gerhard Eckel used dynamical systems to create movement paths for the spatialisation of sound objects in realtime.

A *major disadvantage in using strange attractor sets* in any creative process is the lack of control: The proportions of a strange attractor are as uncontrollable as the evolution process of a chaotic system is unpredictable.

2 Aims and communication aspects of the project

Our project aims to *research and explore mathematical structures and dynamical systems*, which offer *control points* for non-mathematicians (like artists, composers or architects) *and inspire an intuitive dialogue between artistic creation and mathematical complexity*.

To illustrate an example of such a mathematical system (which is already well used in electroacoustic music) one can imagine a single leader surrounded by a

self-organising swarm of individuals, whose behaviour is the result from following the leader and from interacting with every other individual of the swarm: While the motion of the leader can be easily controlled by a sound artist, the collective flocking of the swarm around the leader constitutes a highly complex dynamical behaviour, which has been used to spatialise music in multi-speaker systems in a controllable (via the leader) yet complexly self-organising way.

Note that the above paragraph does not only describe an example of a suitable mathematical system, it also does so by using *mathematical diction*. The above paragraph describes the leader-swarm behaviour from the viewpoint of the mathematician observing sheep in Scotland (even without mentioning gradient flows on metric spaces, which form the mathematical basis of so many swarming models). The viewpoint of any artist will certainly be quite different.

Therefore, an *intrinsic challenge* of the project is to open communication pathways between mathematical ways of thinking and talking and artistic ways of thinking and speaking. The communication between the project team is of crucial importance; in particular because it is explicitly not a goal of the project to build "design machines" or to derive "composition algorithms"!

On the contrary, our project aims to cultivate a dialogue between mathematicians and artists on equal terms, as a collaborative model: A dialogue, which allows artists to become inspired by mathematics, which they would not have otherwise encountered. A dialogue, which allows mathematicians to approach mathematics, for instance, with the naivety of somebody from outside the field or with a certain kind of artistic liberty.

Referring back to the first paragraph of this section, this implies that the question of control points to a complex mathematical system is first and foremost an artistic decision, not a mathematical one. Also, this artistic decision can only be made on the basis of communication between mathematicians and artists.

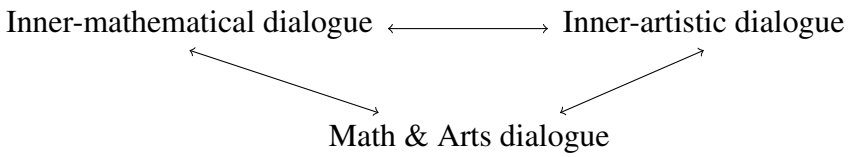
As a consequence, the meetings of the project team are very rich. Communication and cross-wise inspiration occur on many different levels and these experiences have turned out to be hugely rewarding and exciting for all the participants.

Particular key functions of the team meetings are not only to actually work on the pending projects. On a secondary level the team enjoys a continuing dialogue between mathematical creativity and intuition and artistic creativity and intuition: A dialogue of "How to take steps towards the unknown?" and of "How are these steps taken in different fields?"

On a tertiary level the team also experiences elements of current art-philosophy, as discussed, for instance, in Epistemologies of Aesthetics by Dieter Mersch, [14], who deconstructs and displaces the terminology that typically accompanies the question of the relationship between art and scientific truth. Identifying artistic practices as modes of thought that do not make use of language in a way that can easily be translated into scientific discourse, Mersch advocates for an aesthetic

mode of thought beyond the linguistic turn, a way of thinking that cannot be substituted by any other disciplinary system.

Besides all these aspects of the communication within the team members, the discourse of the team meetings is also fluently shifting within the following triangle of dialogues:



Moments of inner-mathematical discussions morph into phases of inner-artistic discourses and both feed into the work on the tasks of the Mathematics and Arts project, which currently comprise the following three topics:

- Cluster algebras, triangulations and frieze pattern, see Section 3 below,
- Models of collective behaviour,
- Models with "wavy" entropy decay.

So far, the biggest part of the project focussed on algebra and combinatorics of triangulations, which is the subject of the following Section 3.

Concerning models of collective behaviour, we are in particular interested in models, where the swarming behaviour can be controlled by the energy of the swarm. Since serving artistic inspiration is the only purpose for these models, the mathematicians are here given the liberty of studying objects just for the sake of them "looking good".

Models with "wave" entropy are models, where the interaction between (a complex) microscopic and (a controllable) macroscopic scale leads to a dynamical system, which allows evaluation in terms of suitable entropy functionals.

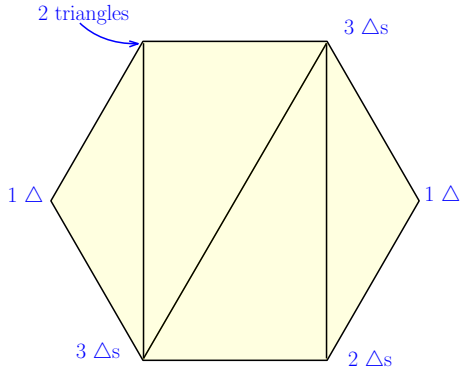
3 Cluster algebras, triangulations and frieze patterns

Around 2000, Fomin and Zelevinsky ([8]) invented cluster algebras - certain rings of variables, generated by possibly infinitely many overlapping sets of generators. The authors were motivated by phenomena observed in the study of dual canonical bases of enveloping algebras and of total positivity of matrices/of algebraic groups. Zelevinsky's beautiful description in [15] is recommended as a

first introduction to the topic. Fomin and Zelevinsky showed in [9] how cluster algebras arise from triangulations of polygons. This approach has been developed further since then, notably by Fomin, Shapiro and Thurston in [7]. Under this correspondence, cluster algebras give rise to frieze patterns of positive numbers, if one specialises the cluster variables in a cluster to 1. Frieze patterns have first been studied by Coxeter [5] and then Conway-Coxeter [3] in the 70's. A frieze pattern consists of sequences of rows of integers, forming a lattice as in Figure 2, starting with a row of zero's, then a row of one's, and then several rows of positive integers, such that in any diamond formed by four entries, the product of the horizontal neighbours is one plus the product of the vertical neighbours:

$$\left.\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}\right\} \quad \Longrightarrow d := \frac{bc-1}{a} \in \mathbb{N}.$$

The frieze is finite if it ends after finitely many rows with a row of one's followed by a row of zero's. It is infinite otherwise. Conway and Coxeter showed in [3] that every finite frieze is invariant under a glide reflection and that it arises through the triangulation of a (regular) polygon: the first non-trivial row is given as the so-called matching numbers for the triangulation, i.e. the number of triangles incident with the vertices of the polygon. The glide symmetry implies in particular that the frieze is periodic, with period a divisor of the size of the polygon. The frieze of Figure 2 arises from a snake triangulation of a hexagon as in the figure here. The matching numbers are 2,1,3,2,1,3 (going counterclockwise around the polygon).



Infinite friezes arise from triangulations of annuli, where the arcs have endpoints on both boundaries of the figure, [2]: the first non-trivial row is given by the matching numbers for a triangulation of an annulus (with marked points on both boundaries).

Through discussions in the group, we have started various projects related with triangulations of surfaces. On one hand, the observation of the fast growing entries in infinite friezes led to a collaboration on the growth of such patterns, [1].

0	0	0	0	0	0	0
	1	1	1	1	1	1
2	**1**	**3**	**2**	**1**	**3**	**2**
	1	2	5	1	2	5
2	1	3	2	1	3	2
	1	1	1	1	1	1
0	0	0	0	0	0	0

Figure 2: A six- (and three-) periodic finite frieze pattern arising from a hexagon

On the other hand, some conceptual similarities between calculating the matching numbers forming an n-periodic frieze and models of n interacting individuals led the team also to look for possible entropy functionals, which might be useful to describe the growth behaviour of infinite frieze patterns. And indeed, by reinterpreting formulas of matching numbers of n-periodic friezes in terms of probabilities of observing first- and higher order relationships between vertices, we were able to identify a class of *Shannon-type entropy functionals*, which seem to characterise the growth behaviour of infinite friezes. In fact, these entropy functionals evaluate line-wise n-periodic frieze pattern, by taking (roughly speaking) the expected value of the information content associated to the relationship probabilities and average these over n neighbouring entries of the frieze.

Interestingly, while the entries of frieze pattern are not necessarily line-wise monotone, the decay of these Shannon-type entropy functional appears to be strictly line-wise monotone and approximately exponentially decaying towards an entropy minimum as depicted in Figure 3.

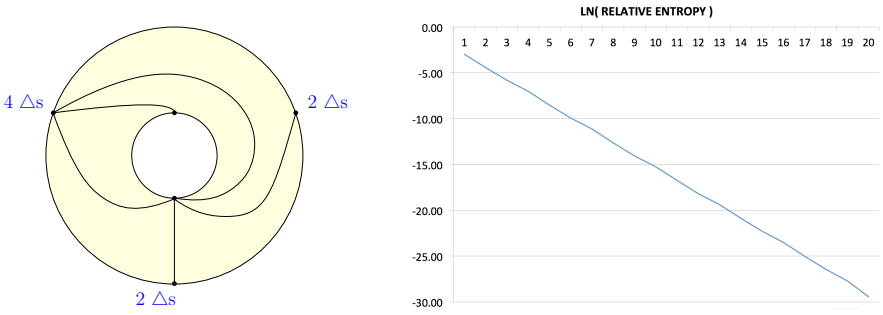


Figure 3: Decay of relative a Shannon-type entropy functional for an infinite frieze pattern derived from a topological triangulation of an annulus.

The detailed properties of these entropy functionals and their possible use to char-

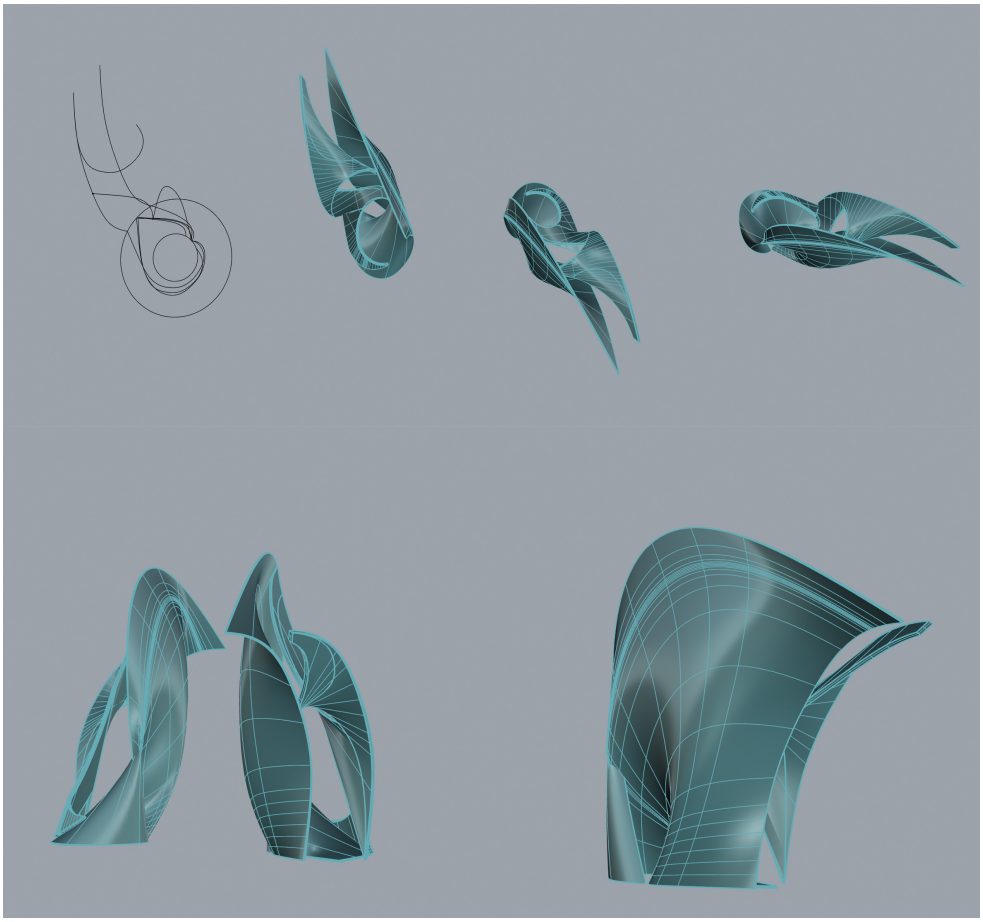


Figure 4: Exploring proportions and relations of triangulations.

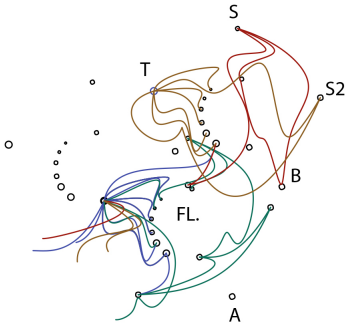
acterise frieze pattern are currently under investigation. The discovery of the existence of such functionals was only made possible by cross-referencing mathematical intuitions from algebra and partial differential equations and constitutes an inner-mathematical example of the creation of new ideas within our project.

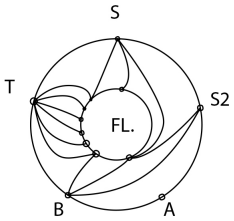
4 Attractive Privacies of Breathing Borders

Alongside the inner-mathematical discussions on frieze patterns, Tamara Friebel became also artistically interested in properties of triangulations and has developed her personal interpretation into two musical compositions. As a preliminary study, she derived 3d objects, which served as test cases to illustrate possible proportions and relations, see e.g. Figure 4.

The composition “Attractive Privacies of Breathing Borders” by Tamara Friebel

Attractive Privacies of Breathing Borders,
in one solitary universe.





A

Improvisation Schematic Diagram:
SSATB & FL

Improvise freely using this diagram to define the points of relation between the players.
Improvisation Duration: 1'00
Score Duration: 6'24

Performance Technique:
Each voice - Soprano, Soprano 2, Alto, Tenor, Baritone is related in different ways to each other. The singers stand in the following configuration, with the Flute in the centre as above:

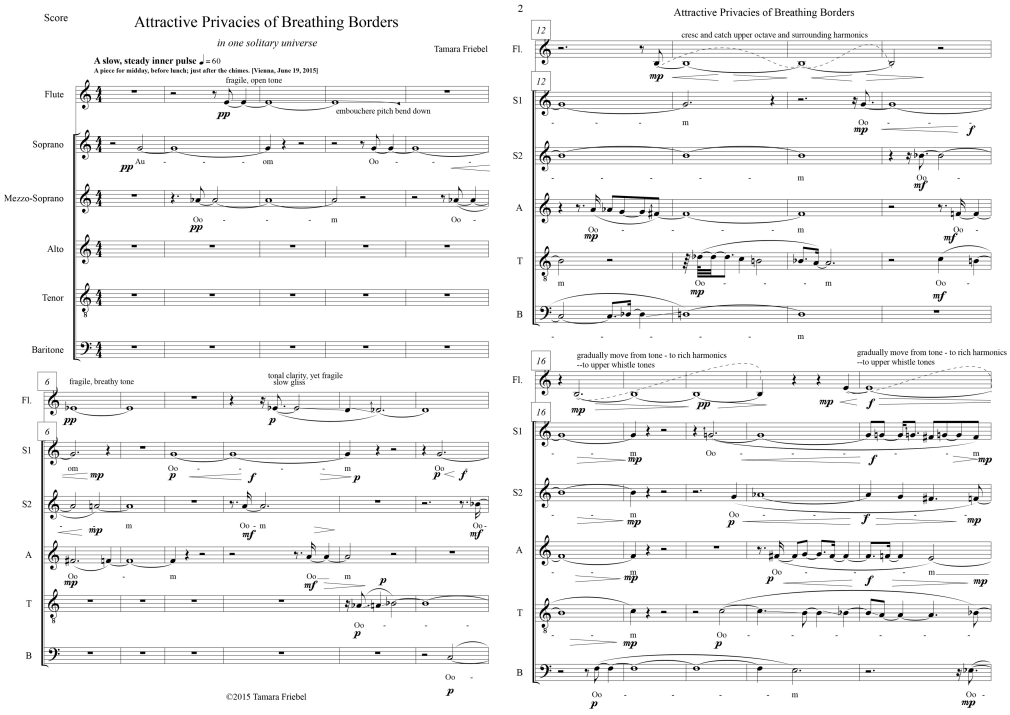


Figure 5: "Attractive Privacies of Breathing Borders" (SSATB + flute, 2015), excerpt of the concept sheet and beginning of the score.

for flute and five vocal soloists (SSATB) was presented at the SALT festival at the University of Victoria, Canada. The piece is drawn from the mathematical research of Section 3. Mathematical elements like triangulations became hereby artistic elements like connectivity and transparency of borders: Constraints, which Friebel uses in the composition process.

Figure 5 provides excerpts of the cover sheet and the score of Attractive Privacies of Breathing Borders. A triangulation of an annulus is not only taken for the staging of the vocalists around the central flute, but served also as a starting point to derive communication and separation patterns between the six parts.

In the first performance of the piece, the musicians were asked as a first part to improvise several constellations represented by graphical scores. The musicians were so given the opportunity to explore individual associations. Then, as a second part, the performance of the composer's score followed.

A second composition "Zwielichter, herumtanzend" (Twilights, dancing around us) for baroque flute and harpsichord was first performed at the Essl Museum Klosterneuberg in December 2015 (Ensemble Klingekunst, see [13] for a preview on youtube). "Zwielichter, herumtanzend" continued to explore schematic diagrams based on topological triangulations, which were then intuitively read and transcribed into individual "strings" of music. In a second step these strings were then further layered and juxtaposed.

5 Outlook

The current work within the project continues to explore the three main topics, i) triangulations and friezes ii) models of collective behaviour and iii) models with "wavy" entropies.

The project team is moreover currently contemplating formats of a workshop event to be organised in fall, which shall be able to reflect and stimulate the many different aspects of the project: mathematical and artistic research, artistic practice and performances, educational aspects, ...

A further project concerns variants of the sound installation *Zeitraum* by Gerhard Eckel, [6]. Zeitraum is a sound environment exposing the interrelation of time and space in acoustic communication, composed of many identical sound sources dispersed irregularly in a large space. When listened to from a particular location, the so called sweet spot, the pattern is perceived as an isochronous pulse. When distancing oneself from the sweet spot, then physical as well as psycho-acoustic effects compel the listener's brain to "compose" a personal sound experience.

And finally, we shall always wholeheartedly recommend any mathematician traveling Scotland, to never do so without an artist. Because it is really exciting to explore the full range of possibilities of one-sided sheep.

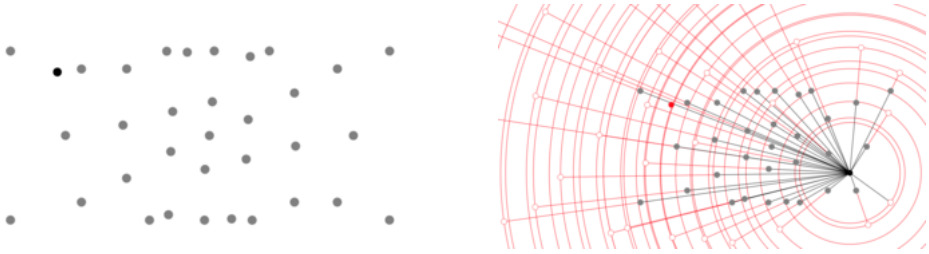


Figure 6: Speaker arrangement and sound wave alignment of “Zeitraum”.

6 Acknowledgments

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