Superbosonization of invariant matrix ensembles

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joint work with
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The problem

$G = GL_n(\mathbb{C})$ (or one of the classical groups $O_n(\mathbb{C}), Sp_{2m}(\mathbb{C})$)

$K = U_n$ (more generally, a maximal compact subgroup of G)

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$$V_1 = Hom(\mathbb{C}^n, \mathbb{C}^q) \oplus Hom(\mathbb{C}^q, \mathbb{C}^n)$$

with action $g \cdot ((A, B), (C, D)) = ((gA, Bg^{-1}), (gC, Dg^{-1}))$.

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with action $g \cdot ((A, B), (C, D)) = ((gA, Bg^{-1}), (gC, Dg^{-1}))$.

We are interested in the following problems:

The problem

1) $\mathcal{A}_V^G$ = algebra of G-equivariant holomorphic maps

$$f : V_0 \longrightarrow \Lambda^\bullet V_1^*$$

Try to mimic categorical quotient in the sense that G-equivariant holomorphic maps will come from holomorphic maps

$$F : W_0 \longrightarrow \Lambda^\bullet W_1^*$$

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for some (to be specified) $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$.

2) given $f \in \mathcal{A}_V^G$, consider integral in the sense of Berezin

$$\Omega_V(f) := \int_{V_{0,\mathbb{R}}} f^{\text{top degree part}} d\mu$$

Aim is to simplify the integral by reduction of the number of variables: replace integral over $f^{\text{top degree part}}$ by integral over $F^{\text{top degree part}}$.

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Classical invariant theory: (recall $n \geq p$)
GL_n-invariant polynomials: $\mathbb{C}[V_0]^G \simeq \mathbb{C}[M_{p,p}]$, the isomorphism being induced by the quotient map:

$$\pi : V_0 = M_{n,p} \oplus M_{p,n} \longrightarrow M_{p,p}, \quad (A, B) \mapsto BA.$$

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Same holds for holomorphic invariant functions (Luna):

Theorem

Every holomorphic invariant function on V_0 is the pull back of the form $f(A, B) = F(BA)$, F a holomorphic function on $W_0 = M_{p,p}$. $\diamond$

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Step 2: Given $f \in \mathcal{A}_V^G$, simplify the integral

$$\Omega_V(f) := \int_{V_{0,\mathbb{R}}} f \, \mathrm{d}\mu \qquad (\text{here } f = f^{\text{top}})$$

by reducing the number of variables.

$V_{0,\mathbb{R}} = M_{n,p} \hookrightarrow M_{n,p} \oplus M_{p,n}$, $A \mapsto (A, \overline{A}^t)$
Assumption: f analytic on $V_{0,\mathbb{R}}$, rapid decay

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Aim: reduction to an integral over a function on the quotient.

Recall:
quotient map $\pi : V_0 \to M_{p,p}$, $(A, B) \mapsto BA$,

$$\pi(V_{0,\mathbb{R}}) = \text{non-neg. hermitian } p \times p\text{-matrices}$$

The simplest case: $q = 0$, $G = GL_n$, $n \geq p$

Let $V'_{0,\mathbb{R}} \subset V_{0,\mathbb{R}} \simeq M_{n,p}$ be the matrices of maximal rank.

$V_{0,\mathbb{R}}$ and $V'_{0,\mathbb{R}}$ are stable under the induced U_n-action

An element $L = (A, \overline{A}^t) \in V'_{0,\mathbb{R}}$ gives a decomposition $\mathbb{C}^n = ker(A) \oplus im(\overline{A}^t)$ and hence an element in the Grassmann variety $(U_p \times U_{n-p})\backslash U_n$ of p-planes in $\mathbb{C}^n$.

Fixing a unitary basis of $im(\overline{A}^t)$ one can identify the restriction of A to $im(\overline{A}^t)$ with a matrix in $GL_p(\mathbb{C})$.

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In other words:

$$V'_{0,\mathbb{R}} \simeq GL_p \times_{U_p} U_{n-p} \backslash U_n.$$

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Theorem. f invariant holomorphic function on V_0, F the corresponding holomorphic function on $M_{p,p}$, then the Berezin integral

$$\Omega_V(f) = 2^{-pn} \frac{vol(U_n)}{vol(U_{n-p})} \int_{D_p = p\times p \ \ pos.herm.matrices} F(x) \det{}^n(x) d\mu_{D_p}.$$

where we identify GL_p/U_p with the pos. herm. matrices by $gU_p \mapsto g\overline{g}^t$. ◇

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Remark: In this case the formula is not new for the physicists. Advantage: generalizes directly to the other classical groups.

The case: $q = 0$, $n \geq p$ respectively $n \geq 2p$.

Theorem

$G = GL_n(\mathbb{C}), O_n(\mathbb{C}), Sp_n(\mathbb{C})$: f a G-invariant holomorphic function on V_0, F the corresponding holomorphic function on the (algebraic) quotient $W_0 = V_0 /\!/ G$, then the Berezin integral

$$\Omega_V(f) = 2^{-p(n+m)} \frac{vol(K_n)}{vol(K_{n,p})} \int_{D_p = G_p/K_p} F(x) \det{}^{n'}(x) d\mu_{D_p}.$$

where

G	K_n	G_p	K_p	$K_{n,p}$	m	n'
$GL_n(\mathbb{C})$	U_n	$GL_p(\mathbb{C})$	U_p	U_{n-p}	0	n
$O_n(\mathbb{C})$	O_n	$GL_{2p}(\mathbb{R})$	$O_{2p}(\mathbb{R})$	$O_{n-2p}(\mathbb{R})$	1	$\frac{n}{2}$
$Sp_n(\mathbb{C})$	USp_n	$GL_p(\mathbb{H})$	USp_{2p}	USp_{n-2p}	-1	$\frac{n}{2}$

Remark: view G_p/K_p embedded in G_p via Cartan embedding $g \mapsto g\Theta(g^{-1})$, so $\det^{\frac{n}{2}}$ makes sense.

Berezin integral

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Given: $f : V_0 \longrightarrow \Lambda^\bullet V_1^*$

$$\Omega_V(f) := \int_{V_{0,\mathbb{R}}} D_{Z,\tilde{Z},\xi,\tilde{\xi}} f(Z, \tilde{Z}, \xi, \tilde{\xi})$$

- $Z, \tilde{Z}$ – commuting variables $z_{i,j}, \tilde{z}_{j,i}$ on $V_0 = M_{n,p} \oplus M_{p,n}$

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So $f(Z, \tilde{Z}, \xi, \tilde{\xi})$ = short way of writing for $f : V_0 \to \Lambda^\bullet V_1^*$

$$\sum_{\text{ordered exterior products}} f_{i_1,j_1,\ldots,k_1,l_1,\ldots}(z_{i,j}, \tilde{z}_{i,j})\, \xi_{i_1,j_1} \wedge \ldots \wedge \xi_{i_t,j_t} \wedge \tilde{\xi}_{k_1,l_1} \wedge \ldots \wedge \tilde{\xi}_{k_s,l_s}$$

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- Berezin form: $D_{Z,\tilde{Z},\xi,\tilde{\xi}} =$

$$2^{2pn} \prod_{c=1}^{p} \prod_{i=1}^{n} |d\Re e(z_{i,c})\ d\Im m(z_{i,c})| \otimes (2\pi)^{qn} \prod_{e=1}^{q} \prod_{i=1}^{n} \frac{\partial^2}{\partial \xi_{j,e} \partial \tilde{\xi}_{e,j}}$$

- Convention: $\frac{\partial^2}{\partial\xi\partial\tilde{\xi}} \tilde{\xi}\xi = 1$.

Up to constants: $D_{Z,\tilde{Z},\xi,\tilde{\xi}}$ projects f on the component of maximum degree f^{top} in the anti-commuting variables, Lebesgue measure on $V_{0,\mathbb{R}}$.

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- $V_{0,\mathbb{R}} = \{(A, {}^t\overline{A}) | A \in M_{n,p}(\mathbb{C})\} \subset V_0$, $f|_{V_{0,\mathbb{R}}}$ analytic, rapid decay.

Superbosonization - A brief characterization

Given ensemble of disordered Hamiltonians (for example hermitian $n \times n$ matrices, real symmetric matrices), probabilty distribution rapid decay at infinity, or bounded support (for example Gaussian distribution).
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Supersymmetry method:

Starting point: characteristic function of the probabilty measure of a given ensemble of disordered Hamiltonians.

$$\mathcal{F}(K) = \int e^{-i\mathrm{Tr}(KH)} d\mu(H)$$

What is the Fourier variable K?

The exact meaning of the Fourier variable K depends on what observable is to be computed.

In our case one should think of a lattice with $(p+q)$-sites, associated to each site a vector space of dimension n, n is the number of orbitals / states per site (*granular materials*).

$$V = V_0 \oplus V_1, \begin{cases} V_0 = M_{n,p} \oplus M_{p,n} \text{ commuting variables } z_{i,j}, \tilde{z}_{j,i} \\ V_1 = M_{n,q} \oplus M_{q,n} \text{ anti-commuting variables } \zeta_{k,l}, \tilde{\zeta}_{l,k} \end{cases}$$

p-bosonic and q-fermionic copies of the vector space $\mathbb{C}^n$.

Interest: study behavior for $n \to \infty$

The ˜-variables come into the picture due to the complexification of the picture (analytic $\to$ holomorphic).

For such a situation, in general the matrix entries of K will be of the form

$$K_{i,j} = \sum_{l=1}^{p} z_{i,l}\tilde{z}_{l,j} + \sum_{m=1}^{q} \zeta_{i,m}\tilde{\zeta}_{m,j}$$

where $z_{i,j}, \tilde{z}_{j,i}$ commuting variables, $1 \le i \le n, 1 \le j \le p$
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To calculate the spectral correlation function for example, one has to calculate the Berezin integral for

$$f = exp\left(i \sum_{l,l'} z_{l,l'} E_{l'} \tilde{z}_{l',l} + i \sum_{k,k'} \zeta_{k,k'} F_{k'} \tilde{\zeta}_{k',k} \right) \mathcal{F}(K)$$

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Note: f is a map $f : V_0 \rightarrow \Lambda^\bullet V_1^*$, a so-called *superfunction*.
parameters E_l, F_l - physical meaning of energy

Leaving out many details, to calculate for example the spectral correlation function one ends up with the following problem: have a holomorphic map or superfunction

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$$\Omega_V(f) = \int_{M_{n,p}} D_{z,\tilde{z},\zeta,\tilde{\zeta}} f(z, \tilde{z}, \zeta, \tilde{\zeta})$$

where $M_{n,p}$ is embedded in $V_0 = M_{n,p} \oplus M_{p,n}$ as a real subspace via

$$A \mapsto (A, \overline{A}^t).$$

Assume: f analytic on the *diagonal* $M_{n,p}$, rapid decay (Schwartz function, functions that decrease faster than any power).
Remark: doubling of the variables, complexification of a real situation

Summarizing:

$V = V_0 \oplus V_1$ where

- $V_0 = M_{n,p} \oplus M_{p,n}$ commuting variables $z_{i,j}, \tilde{z}_{i,j}$
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- $G = GL_n(\mathbb{C})$ (or a classical group: $Sp_n(\mathbb{C}), O_n(\mathbb{C})$)
- probability measure $d\mu(H)$ on the the ensemble is K-invariant
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Superbosonization (Efetov): use the additional symmetry to simplify the Berezin integral
- assume $n \geq p$ for $G = GL_n(\mathbb{C})$, $n \geq 2p$ for $Sp_n(\mathbb{C}), O_n(\mathbb{C})$

Example

$$\Omega_V(f) = 2^{-pn} \frac{vol(U_n)}{vol(U_{n-p})} \int_{D_p = p\times p \;\; pos.herm.matrices} F(x)\det{}^n(x) d\mu_{D_p}.$$

The other extreme case: $p = 0, q > 0$.

First consider again $G = GL_n(\mathbb{C})$.
Since $p = 0$ one has $V_0 = 0$ and

$$V = V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) = M_{n,q}(\mathbb{C}) \oplus M_{q,n}(\mathbb{C}).$$

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The action of G on $V = V_1$: $g \cdot (A, B) = (gA, Bg^{-1})$, induces an action on $\mathcal{A}_V$, and the algebra $\mathcal{A}_V^G$ of G-equivariant holomorphic maps $V_0 \to \Lambda^\bullet V_1^*$ are just the G-fixed points in $\Lambda^\bullet(V_1^*)$.

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Since $\mathcal{A}_V = \Lambda^\bullet V_1^*$, the algebras $\mathcal{A}_V$ and $\mathcal{A}_V^G$ come equipped with natural grading.
Berezin integral $=$ projection of $f \in \mathcal{A}_V^G$ onto its top-degree component:

$$\Omega_V : \mathcal{A}_V^G \to \Lambda^{2qn} V^*, \quad f \mapsto f^{top}$$

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To get a uniform formula also for cases $p, q \neq 0$, we would like to get a formula for $\Omega_V(f)$ in this case similar to the formula we had in the case $q = 0$:

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Old case $q = 0$: replace $f =$ invariant function on V_0 by $F =$ function on quotient space W, replace Berezin integral $\Omega(f)$ by: *factor* $\int_{symm.\ space\ D_p} F det^n d\mu_{D_p}$.

New case $p = 0$: replace $f \in (\Lambda^\bullet V_1^*)^G$ by an H-invariant function on W (but what is H, what space W??), replace projection by integration over a ?symmetric space?.

The other extreme case: $p = 0, q > 0$.

1st step Howe duality: For simplicity n even, $G = GL_n$.

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Set $N = qn$, let $\mathcal{C}(V_1 \oplus V_1^*)$ be the Clifford algebra, i.e.:

$\mathcal{C}(V_1 \oplus V_1^*)$ is the $\mathbb{C}$ algebra generated by $V_1 \oplus V_1^*$ subject to the condition

$$ww' + w'w = s(w, w') \cdot 1,$$

where for

$$w = v+\phi, w' = v'+\phi' \in V_1 \oplus V_1^* : \; s(v+\phi, v'+\phi') = \phi'(v)+\phi(v').$$

The other extreme case: $p = 0, q > 0$.

Recall: linear span of $ww' - w'w$ stable under $[\cdot, \cdot]$

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get *Spin*-representation of $Spin_{4N}$ on $\Lambda^\bullet V_1^*$ (recall $N = qn$)

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Seems to be leading away from the problem...but

The other extreme case: $p = 0, q > 0$.

By Howe duality:

$\Lambda^\bullet V_1^*$ ist a direct sum

$$\Lambda^\bullet V_1^* \simeq \bigoplus U_i \otimes N_i$$

where U_i, N_i irreducible G resp. G'-modules,
and $\forall i \neq j$: $U_i \not\simeq U_j, N_i \not\simeq N_j$.

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In particular: $\mathcal{A}_V^G = \Lambda^\bullet (V_1^*)^G$ is an irreducible G'-module:

$$\mathcal{A}_V^G \simeq V(\frac{n}{2}, \dots, \frac{n}{2}, -\frac{n}{2}, \dots, -\frac{n}{2}).$$

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Note: $\Lambda^0(V_1^*) \subset \Lambda^\bullet(V_1^*)^G$ and $\Lambda^{2N}(V_1^*) \subset \Lambda^\bullet(V_1^*)^G$ are highest (vacuum) / lowest weight vectors. So Berezin integral becomes projection onto the lowest weight vector.

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Consider the "Levi decomposition":

$$\mathfrak{g}' = \text{Lie}\, GL_{2q}, \quad \mathfrak{g}' = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}^+$$

where $\mathfrak{h} = \text{Lie}\, GL_q \oplus \text{Lie}\, GL_q$, $\mathfrak{u}^-$ and $\mathfrak{u}^+$ are isomorphic to $M_q(\mathbb{C})$, so

$$\mathfrak{u}^- = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \mathfrak{h} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \mathfrak{u}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

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Set $\lambda = (\frac{n}{2}, \ldots, \frac{n}{2}, -\frac{n}{2}, \ldots, -\frac{n}{2})$, so $\mathcal{A}_V^G \simeq V(\lambda)$ as G'-module.

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λ extends to $\mathfrak{h} \oplus \mathfrak{u}^+$, consider the parabolic Verma module

$$M(\lambda) = U(\mathfrak{g}') \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$$

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It follows

$$\dim \mathrm{Hom}_H(M(\lambda), V(\lambda)_{-\lambda}) = \dim \mathrm{Hom}_H(\mathbb{C}[M_q] \otimes \mathbb{C}_\lambda, V(\lambda)_{-\lambda}) \leq 1.$$

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$$\mathrm{Hom}_H(V(\lambda), V(\lambda)_{-\lambda}) = \mathrm{Hom}_H(M(\lambda), V(\lambda)_{-\lambda})$$

is one dimensional and spanned by the Berezin integral

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So we can identify the Berezin integral in the following way with an element in $\mathrm{Hom}_H(M(\lambda), V(\lambda)_{-\lambda})$ and hence:

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and, as H-modules ($H = GL_q \times GL_q$):

$$\mathbb{C}[GL_q] = \mathbb{C}[M_q]_{\det} \supset \mathbb{C}[M_q] \otimes \mathbb{C}_{2\lambda} = \frac{1}{\det^n}\mathbb{C}[M_q]$$

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Here one has an obvious projector onto the invariants

$$\mathbb{C}[GL_q] \to \mathbb{C}, \quad F \mapsto \int_{U_q} F dk$$

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It follows:

$f \in (\Lambda^\bullet V_1^*)^G \quad \rightarrow \quad f \in V(\lambda)$ identification with an element in the $G' = GL_{2q}(\mathbb{C})$-module

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& \rightarrow & F(x)\det^{-n}(x) \in \mathbb{C}[M_q] \otimes \mathbb{C}_{2\lambda} \subset \mathbb{C}[GL_q(\mathbb{C})] \\
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 & = & \int_{U_q} F(k)\det{}^{-n}(k)\, dk.
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The other extreme case: $p = 0, q > 0$, $G = GL_n(\mathbb{C}), O_n(\mathbb{C}), Sp_n(\mathbb{C})$

Theorem

For $f \in \Lambda^\bullet(V_1^*)^G \simeq V(\lambda)$, let $F \in \mathbb{C}[W] \simeq U(\mathfrak{u}^-)$ be a lift. The Berezin integral $f \mapsto \Omega_V(f)$ can be computed as an integral over the compact symmetric space K/K_0:

$$\Omega_V(f) = (2\pi)^{qn} 2^{qm} \frac{vol(K_n)}{vol(K_{n,q})} \int_{D_q = K/K_0} F(y) \det^{-n'}(y) d\mu_{D_q}$$

where

G	K_n	$K_{n,q}$	K	K_0	m	n'
$GL_n(\mathbb{C})$	U_n	U_{n+q}	$U_q \times U_q$	$\Delta(U_q)$	0	n
$O_n(\mathbb{C})$	$O_n(\mathbb{R})$	$O_{n+2q}(\mathbb{R})$	U_{2q}	USp_{2q}	1	$n/2$
$Sp_n(\mathbb{C})$	USp_n	USp_{n+2q}	U_{2q}	$O_{2q}(\mathbb{R})$	-1	$n/2$

Note: in the case O_n one has $K/K_0 \subset$ skew symmetric matrices, so $\det^{\frac{1}{2}}$ is defined. ◇

The Berezinian or superdeterminant

In the formulas before ($G = GL_n(\mathbb{C})$):

$$\Omega_V(f) = constant \int_{D_p = p\times p \ \ pos.herm.matrices} F(x)\det{}^n(x) d\mu_{D_p}.$$

for $p = 0$

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For the general case we have to combine the positive powers of the determinant $\det^n(x)$ and the negative powers $\det^{-n}(y)$, this is done by the Berezinian.

Background: in the framework of supermanifolds the Berezinian plays the same role as the determinant when considering coordinate changes on a supermanifold.

The Berezinian or superdeterminant

Consider the $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = W_{0,0} \oplus W_{1,1} = M_{p,p} \oplus M_{q,q}$$

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An element $P \in W$ can be considered as a "block matrix" or "*super matrix*" for the "super space" $\mathbb{C}^{p|q}$:

$$P = \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} \qquad \begin{array}{ll} x \in W_{0,0} & \sigma \in W_{1,0} \\ \tau \in W_{0,1} & y \in W_{1,1} \end{array}$$

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Let $W'' \subset W$ be the subset of supermatrices such that x, y are invertible, then the superdeterminant:

$$SDet \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{\det(x)}{\det(y - \tau x^{-1}\sigma)} = \frac{\det(x - \sigma y^{-1}\tau)}{\det(y)}$$

is well defined.

The Berezinian or superdeterminant

$$SDet \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{\det(x - \sigma y^{-1}\tau)}{\det(y)}$$

For $q = 0$ set $SDet(x) = \det(x)$ and for $p = 0$ we set $SDet(y) = \det^{-1}(y)$.
A similar function which we will need is

$$J(x, y) = \frac{\det^q(x)\det^q(y)}{\det^{q-p}(y - \tau x^{-1}\sigma)}.$$

Superbosonization formula:

Consider now the general case for $G = GL_n(\mathbb{C})$:
assume $n \geq p \geq 0, q \geq 0$.

$$V_0 = M_{n,p} \oplus M_{p,n}, \quad V_1 = M_{n,q} \oplus M_{q,n}$$

$$W_0 = M_{p,p} \oplus M_{q,q}, \quad W_1 = M_{p,q} \oplus M_{q,p}.$$

Proposition: If $n \geq p$, then $\exists$ surjective homomorphism between
- holomorphic maps $F : W_0 \to \Lambda^\bullet(W_1^*)$

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Proof.
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$(S^\bullet V_0^* \otimes \Lambda^\bullet V_1^*)^G = (\textit{graded}\text{ symmetric algebra})^G = S^\bullet(V_0^* \oplus V_1^*)^G$

$$= \big(T(V_0^* \oplus V_1^*)/ < x \otimes x' - (-1)^{|x||x'|} x' \otimes x > \big)^G$$

Superbosonization formula:

Now the degree 2 part is

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invariants come from pairings of column and row vectors

$$W_0 = \text{even part} = \begin{array}{ccc} S^2(V_0^*)^G = S^2(M_{n,p}^* \oplus M_{p,n}^*)^G & = & M_{p,p}^* \\ \Lambda^2(V_1^*)^G = \Lambda^2(M_{n,q}^* \oplus M_{q,n}^*)^G & = & M_{q,q}^* \end{array}$$

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$$W_0 = \text{even part} = \begin{array}{lll} S^2(V_0^*)^G = S^2(M_{n,p}^* \oplus M_{p,n}^*)^G & = & M_{p,p}^* \\ \Lambda^2(V_1^*)^G = \Lambda^2(M_{n,q}^* \oplus M_{q,n}^*)^G & = & M_{q,q}^* \end{array}$$

and for the odd part we get $W_1 =$

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Superbosonization formula:

Now the degree 2 part is

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Isomorphism of vector spaces induces a surjective map (Howe)

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G. Schwarz: holomorphic equiv. maps are of the form $f = f_1 f_2$, where f_2 is an algebraic equiv. map and f_1 is an hol. invariant function on V_0.

Superbosonization formula:

$G = GL_n(\mathbb{C})$

Theorem:(Superbosonization formula) Let $F : W_0 \to \Lambda^\bullet(W_1^*)$ be a lift for $f : V_0 \to \Lambda^\bullet(V_1^*)$, then the Berezin integral:

$$\Omega_V(f) = \text{factor} \int_D proj_{high.\ deg.}\, J(x,y) SDet^n(x,y) F(x,y) d\mu_{Dp} dk$$

where

$$D = (\textit{space of positive hermitian } p \times p \textit{ matrices}) \times U_q$$

and $SDet$ = super-determinant / Berezinian

$$SDet \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{\det(x)}{\det(y - \tau x^{-1}\sigma)}$$

and $J(x,y) = \det^q(x) \det^q(y) / \det^{q-p}(y - \tau x^{-1}\sigma)$.

Superbosonization formula:

The general case: $G = O_n, Sp_n$

$V_0 = M_{n,p} \oplus M_{p,n}, \quad V_1 = M_{n,q} \oplus M_{q,n}$

$W_0 = Sym_{2p,2p} \oplus Alt_{2q,2q}, \quad W_1 = M_{2p,2q}$ for $G = O_n$.

$W_0 = Alt_{2p,2p} \oplus Sym_{2q,2q}, \quad W_1 = M_{2p,2q}$ for $G = Sp_n$.

Lemma: If $n \geq 2p$, then $\exists$ surjective homomorphism between
- holomorphic maps $F : W_0 \to \Lambda^\bullet(W_1^*)$

and

- holomorphic G-equivariant map $f : V_0 \to \Lambda^\bullet(V_1^*)$.

Set $J(x, y) = \det^q(x) \det^{q-m/2}(y) / \det^{q-m/2-p}(y - \tau x^{-1}\sigma)$.

Here $m = 1, -1$ for $G = O_n, Sp_n$.

Superbosonization formula:

The general case: $G = GL_n$, $n \geq p$ resp. $G = O_n, Sp_n$, $n \geq 2p$

Theorem

(Superbosonization formula) Let $F : W_0 \to \Lambda^\bullet(W_1^*)$ be a lift for the G-equivariant holomorphic map $f : V_0 \to \Lambda^\bullet(V_1^*)$, then the Berezin integral:

$$\Omega_V(f) = factor \int_D proj_{high.\ deg}\, J(x,y) SDet^{n'}(x,y) F(x,y) d\mu_{Dp} dk$$

where $n' = n/(1+|m|) \geq p$, $m = 0, 1, -1$ for $G = GL_n, O_n, Sp_n$, and the domain for the integration is:

$$D = GL_p(\mathbb{C})/U_p \times U_q \text{ for } G = GL_n$$

$$D = GL_{2p}(\mathbb{R})/O_{2p} \times U_{2q}/USp_{2q} \text{ for } G = O_n$$

$$D = GL_p(\mathbb{H})/USp_{2p} \times U_{2q}/O_{2q} \text{ for } G = Sp_n$$

Two remarks:

- factors can be made precise, so formulas can be used for calculations

- method extends to case where one uses products of these groups.

Summarizing:

- Advantage of the new method: by conversion from its original role as the number of integrations to do, the (usually) big integer n has been turned into an exponent

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- Advantage of the new method: by conversion from its original role as the number of integrations to do, the (usually) big integer n has been turned into an exponent
- applicable in cases where other methods (non-Gaussian distribution) did not work so far
- even in cases where other methods work get interesting equalities. As an example, Martin Zirnbauer applied the new method Wegners n-orbital model with n orbitals per site and unitary symmetry. Be warned, however, that this equivalence of the formulas obtained by the new method and Hubbard-Stratonovich is by no means easy to see directly.

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- the restriction $n \geq p$ has been removed in a paper by Bunder, Efetov, Kravtsov, Yevtushenko, and Zirnbauer (but formula gets more complicated...naturally)

Thank you very much!