The Linearization Problem, Old and New

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Algebraic Groups and Invariant Theory
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"I would like to thank the organizers . . .

. . . bla bla bla . . . "

(Just to spare my voice!)

Thanks a lot Karin, Donna, and Sasha!!!






The Basic Problem

One of the fundamental questions in affine algebraic geometry is the following. (For simplicity we will work over $\mathbb{C}$.)

Question

How can an algebraic group G act on affine n-space $\mathbb{A}^n$?

- Actions of the additive group $\mathbb{C}^+$ and of unipotent groups?
- Actions of $\mathbb{C}^*$ and of tori?
- Actions of finite groups?
- Actions of reductive groups?
- Fixed points?
- Invariants and quotient $\mathbb{A}^n/\!/G$?

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Recall that the group $G\mathbb{A}_n$ of *polynomial automorphisms* of $\mathbb{A}^n$ has the structure of an infinite dimensional algebraic group:

$$G\mathbb{A}_n = \bigcup_d G\mathbb{A}_n^{(d)}$$

where $G\mathbb{A}_n^{(d)}$ denotes the automorphisms $\varphi = (\varphi_1, \ldots, \varphi_n)$ of degree $\deg \varphi := \max(\deg \varphi_i) \leq d$ (SHAFAREVICH, 1966).

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What is the algebraic & geometric structure of the group $G\mathbb{A}_n$?

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Two Specific Questions

Linearization Problem

Is every action of a reductive algebraic group G on affine n-space $\mathbb{A}^n$ *linearizable*, i.e. is there a G-equivariant isomorphism $\mathbb{A}^n \xrightarrow{\sim} V$ where V is a representation of G?

Equivalently, is every reductive subgroup $G \subset G\mathbb{A}_n$ conjugate to a subgroup of GL_n?

Fixed Point Problem

Does every action of a reductive algebraic group G on affine n-space $\mathbb{A}^n$ have a fixed point?

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Early Results

1. $G\mathbb{A}_2$ is an amalgamated product $\mathrm{Aff}_2 \star_{B_2} \mathrm{Jonc}_2$ (VAN DER KULK 1953) and so every reductive group action on $\mathbb{A}^2$ is linearizable (KAMBAYASHI 1979).
2. A faithful action of a torus T on $\mathbb{A}^n$ is linearisable if $\dim T \geq n-1$ (BIALYNICKI-BIRULA, 1966/67).
3. If a reductive group action on $\mathbb{A}^n$ has no invariants (i.e. $\mathbb{A}^n /\!\!/ G = \{*\}$), then it is linearizable (LUNA 1973).
4. A semisimple group action on $\mathbb{A}^3$ or $\mathbb{A}^4$ is linearizable (K.-POPOV 1985, PANYUSHEV 1984).

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Remarks

1. The structure of $G\mathbb{A}_2$ as an amalgamated product holds for every ground field K.
2. It implies the non-existence of non-trivial forms of $\mathbb{A}^2$ (or of the polynomial ring $\mathbb{C}[x, y]$).
   (This question is completely open in higher dimensions!)
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Counterexamples

1. First examples of non-linearizable actions: $O_2(\mathbb{C})$ on $\mathbb{A}^4$ and $SL_2(\mathbb{C})$ on $\mathbb{A}^7$ (SCHWARZ 1989).
2. Counterexamples for all connected reductive group except tori (KNOP 1991).
3. Examples and counterexamples for reductiv group actions with one-dimensional quotient $\mathbb{A}^n /\!\!/ G$ (K.-SCHWARZ 1992).
4. Counterexamples for many non-commutative finite groups (MASUDA-MOSER-J.-PETRIE 1991).

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Remarks

1. All conterexamples so far come from non-trivial G-vector bundles (idea of BASS-HABOUSH 1987).
2. No such counterexamples for commutative reductive groups (work of MASUDA-MOSER-PETRIE, DE CONCINI-FAGNANI)
3. All conterexamples so far are holomorphically linearizable (*equivariant Oka-principle*, HEINZNER-KUTZSCHEBAUCH 1994).
4. Counterexamples in the holomorphic setting for every complex reductive group (DERKSEN-KUTZSCHEBAUCH 1998).
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1. $\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable (KORAS-RUSSELL 1997).
2. Free $\mathbb{C}^+$-actions on $\mathbb{A}^3$ are translations (KALIMAN 2004; not true in dimension ≥ 5.).
3. $G\mathbb{A}_3$ is not generated by Aff_3 and Jonc_3 (SHESTAKOV-UMIRBAEV 2004).

Theorem (K.-RUSSELL 2008)

Action of non-finite reductive groups on $\mathbb{A}^3$ are linearizable. (I.e. we know all non-finite reductive subgroups of $G\mathbb{A}_3$.)

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Recall

$G\mathbb{A}_n$ has the structure of an infinite dimensional algebraic group:

$$G\mathbb{A}_n = \bigcup_d G\mathbb{A}_n^{(d)}$$

$G\mathbb{A}_n^{(d)} := \{\varphi \mid \deg \varphi \leq d\}$
$\deg \varphi := \max(\deg \varphi_i)$

Questions

1. *Closed subgroups and closures of subgroups of $G\mathbb{A}_n$?*
2. *Locally finite automorphisms?*
3. *Structure of conjugacy classes in $G\mathbb{A}_n$?*
4. *"Discrete" subgroups? (I.e. $G \cap G\mathbb{A}_n^{(d)}$ finite for all d.)*

Caution!

In general, for a subset $C \subset G\mathbb{A}_n$,

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A Strange Example

Example (HILLE-K.-KRAMMER 2008)

There is an action of the braid group B_3 on $\mathbb{A}^3$ as a discrete subgroup with one invariant $f = xyz - x^2 - y^2 - z^2$ and two fixed points, the singular points of $f = 0$.

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Some Properties of $G\mathbb{A}_n$

1. $G\mathbb{A}_n$ is "rationally" connected.
2. $G\mathbb{A}_n$ is N-transitive (on $\mathbb{A}^n$) for every $N \in \mathbb{N}$.
3. If $F \subset \mathbb{A}^n$ is a finite subset, then $F = (\mathbb{A}^n)^G$ for some (closed) subgroup $G \subset G\mathbb{A}_n$.
4. Usual Galois correspondence:

$$\{\text{closed subsets } X \subset \mathbb{A}^n\} \leftrightarrow \{\text{closed subgroups } G \subset G\mathbb{A}_n\}$$

5. φ locally finite $\Leftrightarrow$ $\overline{\langle\varphi\rangle}$ is a commutative algebraic group, and $\overline{\langle\varphi\rangle}$ is isomorphic to

$$\mathbb{C}^+ \times {\mathbb{C}^*}^r \times F \ \text{ or } \ {\mathbb{C}^*}^r \times F \quad \text{with } F \text{ finite cyclic.}$$

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Well-known Facts about Algebraic Groups

Let G be an algebraic group.

1. The unipotent elements form a closed subset.
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What about $G\mathbb{A}_n$?

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Semisimple Conjugacy Classes

Proposition (FURTER-K. 2008)

1. *If the conjugacy class of a semisimple element $s \in G\mathbb{A}_n$ is closed then s is diagonalizable.*
2. *If the conjugacy class of a G-action with fixed points is closed, then the action is linearizable.*

Lemma

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Let X be a variety. A *family of automorphisms* of $\mathbb{A}^n$ is an automorphism $\rho = (\rho_x)_{x \in X}$ of $X \times \mathbb{A}^n$ over X.

Similarly one defines *a family of actions* of an algebraic group G on $\mathbb{A}^n$.

Proposition (K. 1989)

A family of linear actions of a reductive group G is locally trivial in the Zariski-topology. It is given by a vector bundle $\mathcal{V} \to X$ of the form

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Corollary

Every G-action by Jonquière automorphisms is linearizable.

Caution!

Lifts from Z to $Z \times \mathbb{A}^n$ for $n > 1$ are in general not trivial!
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Generic Triviality

Theorem (K.-Russell 2005)

Let A, B be two G-varieties and $\varphi\colon A \to X$ and $\psi\colon B \to X$ two affine G-invariant morphisms. Assume that the fibers A_x and B_x are G-isomorphic for all $x \in X$. Then there is an étale dominant morphism $U \to X$ such that the pull-backs $U \times_X A$ and $\mu\colon U \times_X B$ are G-isomorphic.

$$\begin{array}{ccccccc}
A & \longleftarrow & U \times_X A & \xrightarrow{\sim} & U \times_X B & \longrightarrow & B \\
\varphi\downarrow & & \downarrow & & \downarrow & & \downarrow\psi \\
X & \xleftarrow{\mu} & U & = & U & \xrightarrow{\mu} & X
\end{array}$$

Local Triviality

Lemma

A family $\rho = (\rho_x)_{x \in X}$ which is locally finite on an open dense set $U \subset X$ is locally finite on X.

Does this hold if $U \subset X$ is only dense?

Proposition

Assume that there is a dense set $X' \subset X$ such that all ρ_x, $x \in X'$, are conjugate to a fixed locally finite automorphism ρ_0. Then ρ is locally finite. Moreover, if ρ_0 is semisimple or unipotent, then so are all ρ_x.

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Here a representation V is called "nice" if every G-equivariant automorphism of V is linear.
E.g. the *adjoint representation* of a simple group is nice, but there exist non-nice representations (A. KURTH, 1997).

Corollary

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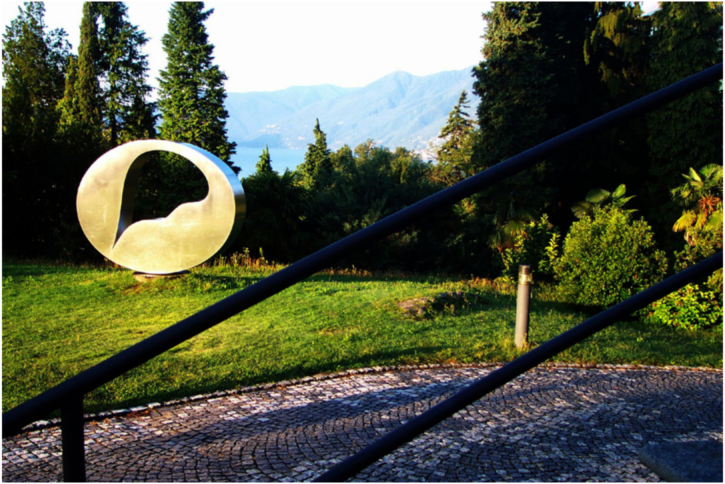

Thank you for your attention!