Semicentre of parabolic subalgebras
ASCONA CONFERENCE

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31/08/2009

References

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I. Polynomiality of the semicentre of parabolic subalgebras.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$: $\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$ which contains the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.
Denote by $\mathfrak{g}'$ the Levi of $\mathfrak{p}$.
Both algebras $\mathfrak{g}'$ and $\mathfrak{g}$ are semisimple Lie algebras.
Denote by π the set of simple roots of $\mathfrak{g}$ and $\pi' \subset \pi$ the set of simple roots of $\mathfrak{g}'$ (wrt the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$).

Definitions

Recall that the adjoint action of $\mathfrak{p}$ (coming from the Lie bracket) induces a derivation on the symmetric algebra $\mathrm{S}(\mathfrak{p})$ of $\mathfrak{p}$, which we will still denote by *ad*.

Definition

An element $s \in \mathrm{S}(\mathfrak{p})$ is said to be *semi-invariant* when for all $x \in \mathfrak{p}$ there exists $\lambda(x) \in \mathbb{C}$ such that $(ad\, x)(s) = \lambda(x)s$.
The *semicentre* $\mathrm{Sy}(\mathfrak{p})$ of $\mathrm{S}(\mathfrak{p})$ is the $\mathbb{C}$-vectorspace generated by the semi-invariants of $\mathrm{S}(\mathfrak{p})$.

Remarks

- Remark 1: we have a similar definition for the semicentre $\mathrm{Sz}(\mathfrak{p})$ of the enveloping algebra $\mathbf{U}(\mathfrak{p})$ of $\mathfrak{p}$.
- Remark 2: let $\mathfrak{p}' = [\mathfrak{p}, \mathfrak{p}]$ denote the derived algebra of $\mathfrak{p}$. Then $\mathrm{Sy}(\mathfrak{p}) = \mathrm{S}(\mathfrak{p})^{\mathfrak{p}'}$.
- Fact: The semicentres $\mathrm{Sy}(\mathfrak{p})$ and $\mathrm{Sz}(\mathfrak{p})$ are $\mathbb{C}$-algebras which are isomorphic, by an extension of the Duflo map (result of Rentschler and Vergne).

Main theorem

Theorem

(Joseph+FM - 2005) Suppose that $\mathfrak{g}$ is a product of simple Lie algebras of type A or C (shortly $\mathfrak{g}$ of type AC), and $\mathfrak{p}$ is any parabolic subalgebra of $\mathfrak{g}$.
Then the semicentre $\mathrm{Sy}(\mathfrak{p})$ of $\mathrm{S}(\mathfrak{p})$ is a polynomial $\mathbb{C}$-algebra in $\mathrm{Card}(\Pi)$ generators, each generator having a weight equal to δ_Γ, $\Gamma \in \Pi$ and a degree also given by some "receipt".

Remark 1: The polynomiality of $\mathrm{Sy}(\mathfrak{p})$ is still true for $\mathfrak{g}$ in other types and particular parabolic subalgebras $\mathfrak{p}$.
Remark 2: In type E_8, the polynomiality of $\mathrm{Sy}(\mathfrak{p})$ fails for some particular $\mathfrak{p}$ (result of Yakimova).

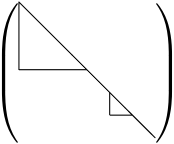
More about the Theorem (1)

- What is Π ?

Let w_0, resp. w_0', denote the longest element of the Weyl group of $\mathfrak{g}$, resp. of $\mathfrak{g}'$.
Define involutions i and j of π as follows.
For all $\alpha \in \pi$, $j(\alpha) = -w_0(\alpha)$,
for all $\alpha \in \pi'$, $i(\alpha) = -w_0'(\alpha)$
and for all $\alpha \in \pi \setminus \pi'$, $i(\alpha) = j(ij)^r(\alpha)$ where $r \in \mathbb{N}$ is the smallest nonnegative integer such that $j(ij)^r(\alpha) \notin \pi'$.
Then Π is the set of the $\langle ji \rangle$-orbits of π.

First example

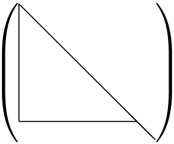
Suppose $\mathfrak{g}$ of type A_6, $\pi = \{\alpha_1, \ldots, \alpha_6\}$ and $\pi' = \pi \setminus \{\alpha_4, \alpha_6\}$



Then $i(\alpha_4) = \alpha_6$ because $j(\alpha_4) = \alpha_3 \in \pi'$ and $jij(\alpha_4) = \alpha_6 \notin \pi'$ and then $i(\alpha_6) = \alpha_4$.
There are three $\langle ji \rangle$-orbits of π which are $\Gamma_1 = \{\alpha_1, \alpha_4\}$, $\Gamma_2 = \{\alpha_2, \alpha_5\}$ and $\Gamma_3 = \{\alpha_3, \alpha_6\}$.
Thus the semicentre $\mathrm{Sy}(\mathfrak{p})$ is a polynomial algebra in three generators.

Example of the "big" maximal parabolic subalgebra

Suppose $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and $\mathfrak{p}$ the maximal parabolic subalgebra where the last row is zero. Then $\pi = \{\alpha_1, \ldots, \alpha_n\}$ and $\pi' = \pi \setminus \{\alpha_n\}$.



Here $i(\alpha_n) = \alpha_n$ and then $(ji)(\alpha_n) = \alpha_1$, $(ji)^2(\alpha_n) = j(\alpha_{n-1}) = \alpha_2$,..., and finally $(ji)^k(\alpha_n) = \alpha_k$ for all $1 \leq k \leq n$.
Thus $\mathrm{Card}(\Pi) = 1$ and $\mathrm{Sy}(\mathfrak{p}) = \mathbb{C}[d]$: polynomial $\mathbb{C}$-algebra in one generator $d \in \mathrm{Sy}(\mathfrak{p})$: result already proved by Dixmier and by Joseph in 1976 and 1977.

More about the Theorem (2)

- What is the weight δ_Γ of each generator of the polynomial algebra $\mathrm{Sy}(\mathfrak{p})$?

Denote by $\{\varpi_\alpha\}_{\alpha\in\pi}$, resp. $\{\varpi'_\alpha\}_{\alpha\in\pi'}$, the set of fundamental weights associated to π, resp. to π'.
Assign to each $\Gamma\in\Pi$ the elements $d_\Gamma=\sum_{\gamma\in\Gamma}\varpi_\gamma$ and $d'_\Gamma=\sum_{\gamma\in\Gamma\cap\pi'}\varpi'_\gamma$.
Then the weight δ_Γ of each generator of the polynomial algebra $\mathrm{Sy}(\mathfrak{p})$ is
$\delta_\Gamma=w'_0(d_\Gamma)-w_0(d_\Gamma)=w'_0(d'_\Gamma)-d'_\Gamma+d_\Gamma-w_0(d_\Gamma)\in\sum_{\alpha\in\pi\setminus\pi'}\mathbb{N}\varpi_\alpha$.

Examples

First example: $\mathfrak{g}$ of type A_6, $\pi = \{\alpha_1, \ldots, \alpha_6\}$ and $\pi' = \pi \setminus \{\alpha_4, \alpha_6\}$.
Here we get $\varpi_i = \varpi'_i + \frac{i}{4}\varpi_4$ for all $1 \leq i \leq 3$ and $\varpi_5 = \varpi'_5 + \frac{1}{2}\varpi_4 + \frac{1}{2}\varpi_6$.
The weight of each generator of $\mathrm{Sy}(\mathfrak{p})$ is equal to $\delta_{\Gamma_i} = 2\varpi_4 + \varpi_6$ for all $1 \leq i \leq 3$.
Example of the "big" maximal parabolic: here we get $\varpi_i = \varpi'_i + \frac{i}{n}\varpi_n$ for all $1 \leq i \leq n-1$ and the weight of the unique generator d of $\mathrm{Sy}(\mathfrak{p})$ is equal to $(n+1)\varpi_n$.

More about the Theorem (3)

- What is the degree of each generator of $\mathrm{Sy}(\mathfrak{p})$?

Let $\Gamma \in \Pi/\langle -w_0 \rangle$. If $\Gamma = -w_0\Gamma$ then there is only one generator s_Γ in $\mathrm{Sy}(\mathfrak{p})$ corresponding to Γ.Otherwise there are two generators s_Γ and t_Γ in $\mathrm{Sy}(\mathfrak{p})$ corresponding to Γ and $\deg(t_\Gamma) = \deg(s_\Gamma) + 1$.
When $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, the degree of each s_Γ can be computed as follows.
Here $\pi = \{\alpha_1, \ldots, \alpha_n\}$. Assign to each simple root $\alpha \in \pi$ the integer $\partial_\alpha := \min\{i,\, n+1-i \mid \alpha = \alpha_i\}$.
Each connected component of π' is again of type A, so assign to each $\alpha \in \pi'$ an integer ∂'_α by the same procedure applied to each component.
Finally set $\partial'_\alpha = 0$ if $\alpha \in \pi \setminus \pi'$.
Then $\deg(s_\Gamma) = \sum_{\alpha \in \Gamma} (\partial_\alpha + \partial'_\alpha)$.

Examples

First example Here we get three generators s_{Γ_1}, t_{Γ_1} and s_{Γ_2} which degrees are: $\deg(s_{\Gamma_1}) = 1+3+1 = 5$, $\deg(t_{\Gamma_1}) = \deg(s_{\Gamma_1}) + 1 = 6$ and $\deg(s_{\Gamma_2}) = 2+2+2+1 = 7$.

Example of the "big" maximal parabolic: If n is even,
$\deg(d) = 2(1+2+\ldots+\frac{n}{2}) + 2(1+2+\ldots+\frac{n-2}{2}) + \frac{n}{2} = n(n+1)/2$.
If n is odd,
$\deg(d) = 2(1+2+\ldots+\frac{n-1}{2}) + \frac{n+1}{2} + 2(1+2+\ldots+\frac{n-1}{2}) = n(n+1)/2$.

Sketch of proof

The proof is based on two results, one coming from the quantum case and the other from the classical case of the Borel.More precisely, we use the following theorems

Theorem 1

(Joseph+FM -2001) (coming from quantum case) Let $\mathbf{U}(\mathfrak{g})^\star$ be the Hopf dual of the enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$, $\mathfrak{m}$ the nilradical of the parabolic $\mathfrak{p}$ and ${}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star := \{f \in \mathbf{U}(\mathfrak{g})^\star ;\ f(\mathbf{U}(\mathfrak{g})\mathfrak{m}) = 0\}$. Let $({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star)^{\mathfrak{p}'}$ be the algebra of the invariants of ${}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star$ under the coadjoint action of $\mathfrak{p}'$.
Then $({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star)^{\mathfrak{p}'}$ is a $\mathbb{C}$-polynomial algebra in $\mathrm{Card}(\Pi)$ generators, each generator having a weight (for the coadjoint action of $\mathfrak{h}$) equal to δ_Γ, $\Gamma \in \Pi$.

Theorem 2

(Joseph - 1977) (Borel) Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra, $\mathfrak{b}$ a Borel of $\mathfrak{g}$ and $\mathfrak{n}$ the maximal nilpotent subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$ wrt a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ ($\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$).
Then the Poisson centre $\mathrm{Y}(\mathfrak{n}) = \mathrm{S}(\mathfrak{n})^{\mathfrak{n}}$ of $\mathrm{S}(\mathfrak{n})$ and the semicentre $\mathrm{Sy}(\mathfrak{b}) = \mathrm{S}(\mathfrak{b})^{\mathfrak{n}}$ of $\mathrm{S}(\mathfrak{b})$ are polynomial $\mathbb{C}$-algebras, each generator having a degree and a weight well defined.
Moreover $\mathrm{Y}(\mathfrak{n})$ and $\mathrm{Sy}(\mathfrak{b})$ have the same set $\mathcal{B}$ of weights and, for each weight $B \in \mathcal{B}$, the weight subspace $\mathrm{Y}(\mathfrak{n})_B$ is one-dimensional.

The lower bound

We can define on $\mathbf{U}(\mathfrak{g})^\star$ a decreasing filtration $\mathcal{F}_K$, called Kostant filtration, such that we get the following isomorphism of algebras and $\mathbf{U}(\mathfrak{p})$-modules

$$\mathrm{gr}_{\mathcal{F}_K}({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star) \simeq \mathrm{S}(\mathfrak{p})$$

and then

$$\mathrm{gr}_{\mathcal{F}_K}(({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star)^{\mathfrak{p}'}) \subset (\mathrm{gr}_{\mathcal{F}_K}({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^\star))^{\mathfrak{p}'} \simeq \mathrm{S}(\mathfrak{p})^{\mathfrak{p}'} = \mathrm{Sy}(\mathfrak{p})$$

The upper bound

There exist suitable graduations gr' and gr'' on $\mathrm{S}(\mathfrak{p})$ and a polynomial subalgebra S of $\mathrm{S}(\mathfrak{p})$ such that

$$\mathrm{gr}'(\mathrm{gr}''(\mathrm{Sy}(\mathfrak{p}))) \subset \mathrm{S}$$

- Remark 1: the algebra S is essentially equal to $\mathrm{Y}(\mathfrak{n})\mathrm{Y}(\mathfrak{n}'^{-})$ where $\mathfrak{n}'^{-}$ is the subalgebra of $\mathfrak{n}^{-}$ such that $\mathfrak{p} = \mathfrak{n}'^{-} \oplus \mathfrak{b}$.
- Remark 2: these graduations gr' and gr'' respect the natural degree on $\mathrm{S}(\mathfrak{p})$ and are invariant under the adjoint action of $\mathfrak{h}$.
- Remark 3: the polynomial algebra S has $\mathrm{Card}(\Pi)$ generators, each of them having a weight equal to δ_{Γ} or to $\delta_{\Gamma}/2$ (always δ_{Γ} when $\mathfrak{g}$ is of type AC).

End of proof

If $\mathfrak{g}$ is of type AC, then we can show that the lower and the upper bounds coincide.
Indeed the formal character of $({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^{\star})^{\mathfrak{p}'}$ is less than the formal character of $\mathrm{gr}'(\mathrm{gr}''(\mathrm{Sy}(\mathfrak{p})))$, which is less than the formal character of S.
But, in case AC, the polynomial algebras $({}^{\mathfrak{m}}\mathbf{U}(\mathfrak{g})^{\star})^{\mathfrak{p}'}$ and S have exactly the same number of generators with the same weight so they have the same formal character.
Thus we have the equality $\mathrm{gr}'(\mathrm{gr}''(\mathrm{Sy}(\mathfrak{p}))) = \mathrm{S}$ in this case and we easily deduce that $\mathrm{Sy}(\mathfrak{p})$ is also a polynomial algebra with the same number of generators with the same weight and degree as for S. □

II. A conjectural construction of the invariants

Since $\mathfrak{g}$ is semisimple, we have

$$\mathrm{S}(\mathfrak{g}) = \mathrm{Y}(\mathfrak{g}) \oplus (ad\,\mathbf{U}(\mathfrak{g})_+)(\mathrm{S}(\mathfrak{g}))$$

where $\mathbf{U}(\mathfrak{g})_+$ is the kernel of the augmentation of $\mathbf{U}(\mathfrak{g})$.
Denote by q the projection on the first factor.
Choose, for each $B \in \mathcal{B}$, an element a_B in $\mathrm{Y}(\mathfrak{n}^-)_{-B}$.
To each $b \in \mathrm{Sy}(\mathfrak{b})_B$, associate the element $\varphi(b) = q(a_B\, b) \in \mathrm{Y}(\mathfrak{g})$ and extend this map linearly to the whole $\mathrm{Sy}(\mathfrak{b}) = \oplus_{B\in\mathcal{B}}\mathrm{Sy}(\mathfrak{b})_B$.

- Remark: for each $b \in \mathrm{Sy}(\mathfrak{b})_B$, $\varphi(b) \in (ad\mathbf{U}(\mathfrak{g})(a_B\, b))^{\mathfrak{g}}$ and $\dim(ad\mathbf{U}(\mathfrak{g})(a_B\, b))^{\mathfrak{g}} \leq 1$.
  Indeed $(ad\mathbf{U}(\mathfrak{g})(a_B\, b)) = (ad\mathbf{U}(\mathfrak{g})(a_B))(ad\mathbf{U}(\mathfrak{g})(b))$.
  Moreover $B = -w_0 B$ (property of weights in $\mathcal{B}$) and then $(ad\mathbf{U}(\mathfrak{g})(a_B))$ and $(ad\mathbf{U}(\mathfrak{g})(b))$ are simple $\mathbf{U}(\mathfrak{g})$-modules dual of each other.

Properties and conjecture about φ

Proposition

(Joseph+FM - 2008) The linear morphism $\varphi : \mathrm{Sy}(\mathfrak{b}) \longrightarrow \mathrm{Y}(\mathfrak{g})$ is injective iff $\mathfrak{g}$ is of type AC.
If $\mathfrak{g}$ is of type AC, then φ is also surjective.

Conjecture 1

The linear morphism φ is always surjective.

Conjecture 2

There exist a subset S of $\mathcal{B} \times \mathcal{B}$ and elements $a_\lambda \in \mathrm{Y}(\mathfrak{n}^-)_{-\lambda}$ and $b_\mu \in \mathrm{Sy}(\mathfrak{b})_\mu$ for all $(\lambda,\, \mu) \in S$ st

$$\mathrm{S}(\mathfrak{g}) = \sum_{(\lambda,\mu)\in S} (ad\, \mathbf{U}(\mathfrak{g})(a_\lambda\, b_\mu))$$

III. Towards a more precise description of the semi-invariants

Suppose $\mathfrak{g}$ of type A_n and $\mathfrak{g}'$ consisting of two blocks $(k,\ n+1-k)$ st k and $n+1-k$ are coprime. Then $\mathrm{Sy}(\mathfrak{p}) = \mathbb{C}[d] = \mathrm{Y}(\mathfrak{p}')$.
Let V be a finite dimensional simple $\mathbf{U}(\mathfrak{g}')$-module and, for a $\mathbf{U}(\mathfrak{g}')$-module M, denote by M^V the isotypical component of type V in M.
Let $\mathrm{H}(\mathfrak{g}')$ be the space of harmonic elements of $\mathrm{S}(\mathfrak{g}')$ and $\mathrm{H}_s(\mathfrak{g}')$ its subspace of homogeneous polynomials of degree s.

Hypotheses

Suppose that there are a finite dimensional simple $\mathbf{U}(\mathfrak{g}')$-module V and a couple of nonnegative integers $(s,\ t)$ st
i) $\mathrm{gr}'(\mathrm{gr}''(d)) \in \mathrm{S}_s(\mathfrak{g}')\mathrm{S}_t(\mathfrak{m})^V$,
ii) $[\mathrm{S}_t(\mathfrak{m}) : V] = 1$,
iii)$[\mathrm{H}_s(\mathfrak{g}') : V^*] = 1$.

$\mathfrak{g}'$-invariant coming from principal term

Definition

Under the hypotheses above, the unique - up to scalars - $\mathfrak{g}'$-invariant d_P coming from the "principal term" of d is the element in $(\mathrm{H}_s(\mathfrak{g}')^{V^*}\mathrm{S}_t(\mathfrak{m})^V)^{\mathfrak{g}'}$

- Remark: this element d_P exists and is unique - up to scalars - by Schur's lemma.

Towards a more precise description of the semi-invariants

- Example of the "big" maximal parabolic.

Suppose $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and $\pi' = \pi \setminus \{\alpha_n\}$. Then $\mathrm{Sy}(\mathfrak{p}) = \mathrm{Y}(\mathfrak{p}') = \mathbb{C}[d]$ and a nice description of d was given by Joseph (1977).
Observe that the "principal term" of d satisfies

$$\mathrm{gr}'(\mathrm{gr}''(d)) \in \mathrm{S}_{n(n-1)/2}(\mathfrak{g}')\mathrm{S}_n(\mathfrak{m}).$$

It was shown by Joseph that

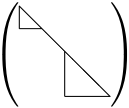
$$d = d_P$$

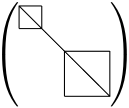
up to scalars.
Indeed $\mathfrak{m}$ is isomorphic, as a $\mathbf{U}(\mathfrak{g}')$-module, to the standard representation of $\mathfrak{g}'$. Moreover its k-fold symmetric power is still irreducible and its dual occurs in $\mathrm{H}(\mathfrak{g}')$ iff n divides k.

Another example

- What happens for another example of the same type?

For example, for $\mathfrak{g}$ of type A_4, and $\pi' = \pi \setminus \{\alpha_2\}$, that is $\mathfrak{g}'$ consisting of blocks (2, 3) of $\mathfrak{sl}_5(\mathbb{C})$.

$\mathfrak{p} =$ 

$\mathfrak{g}' =$ 

Case (2, 3) in $\mathfrak{sl}_5(\mathbb{C})$

In this case, we still have $\mathrm{Sy}(\mathfrak{p}) = \mathbb{C}[d] = \mathrm{Y}(\mathfrak{p}')$ for some $d \in \mathrm{Sy}(\mathfrak{p})$, $\deg(d) = 9$ and d is of weight $5\varpi_2$.
To obtain a similar description of d as for the previous case, the difficulty is that $\mathfrak{m} \simeq L(\varpi'_1 + \varpi'_4)$ as $\mathbf{U}(\mathfrak{g}')$-module but the k-fold symmetric power of this irreducible module is no more irreducible.
Set $V = L(2(\varpi'_1 + \varpi'_3 + \varpi'_4))$ and $W = L(3\varpi'_3)$: irreducible $\mathfrak{g}'$-modules resp. of highest weight $2(\varpi'_1 + \varpi'_3 + \varpi'_4)$ and $3\varpi'_3$.The principal term of d satisfies

$$\mathrm{gr}'(\mathrm{gr}''(d)) \in \mathrm{S}_3(\mathfrak{g}')\mathrm{S}_6(\mathfrak{m})^V.$$

We can show that

$$d \in (\mathrm{H}_3(\mathfrak{g}')^{V^*}\mathrm{S}_6(\mathfrak{m})^V)^{\mathfrak{g}'} \oplus (\mathrm{H}_3(\mathfrak{g}')^{W^*}\mathrm{S}_6(\mathfrak{m})^W)^{\mathfrak{g}'}$$

but, for the moment, we are not able to claim that both contributions are non zero.