# MV-polytopes/cycles and affine buildings

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- $\mathcal{G}$ the affine Grassmannian of $G$.

# Geometric Framework

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**Definition**

Let $\lambda \in X_+^\vee$ and $\nu \in X^\vee$. If $\mathcal{G}_\lambda \cap S_\nu \neq \emptyset$, then the irreducible components $\mathrm{Irr}(\overline{\mathcal{G}_\lambda \cap S_\nu})$ are called **Mirković-Vilonen cycles** (MV-cycles for short) of coweight $(\lambda, \nu)$.

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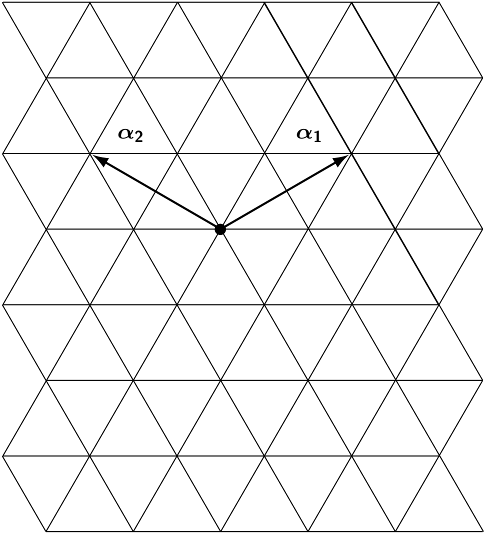
**Definition**

A convex polytope $P$ in $X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ is called **MV-polytope** of coweight $(\lambda, \nu)$ if there exists an MV-cycle $Z$ of coweight $(\lambda, \nu)$, such that $\boldsymbol{\mu}(Z) = P$.

# Example: MV-polytopes for $A_2$



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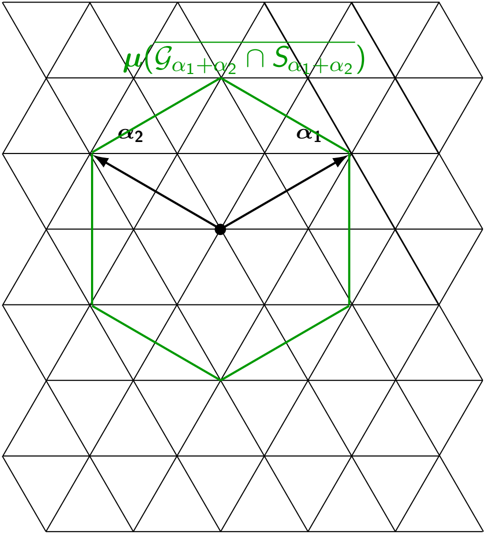

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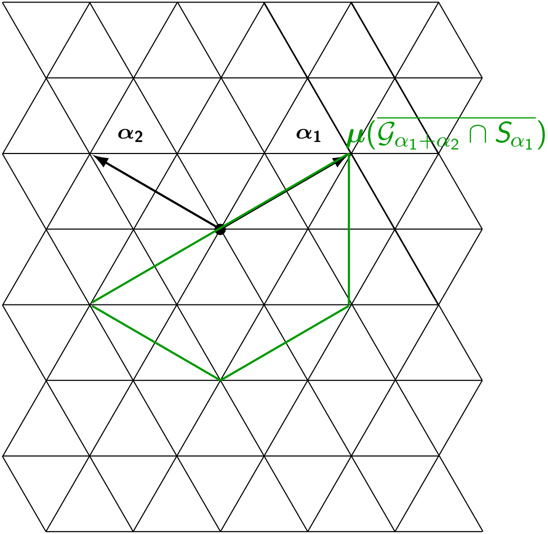

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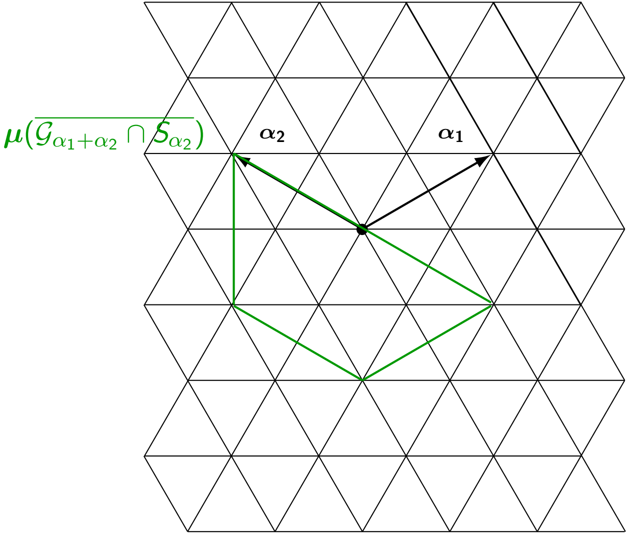

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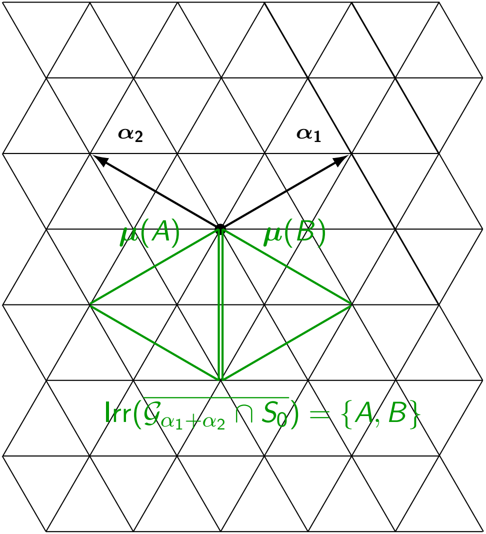

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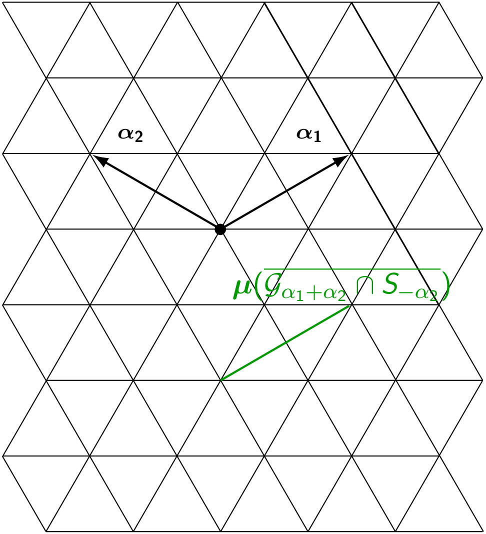

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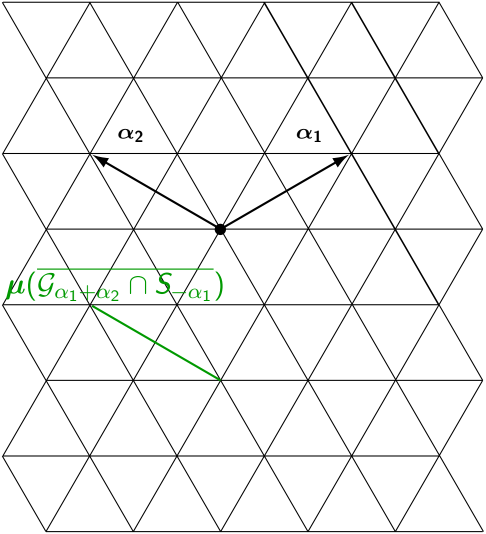

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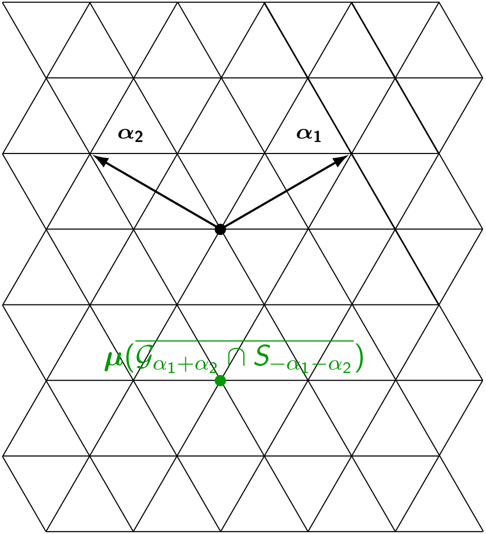

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- Kamnitzer:
  - Description of MV-polytopes as solutions of tropical equations.
  - Gave a construction of MV-cycles starting with MV-polytopes: Let $P$ be an MV-polytope, such that $P = \mathrm{conv}(\mu_w \mid w \in W)$, and

$$M := \underbrace{\overline{\bigcap_{w \in W} U^-(\mathcal{K})^w \underline{t}_{\mu_w}}}_{\text{GGMS stratum}},$$

then M is an MV-cycle with $\boldsymbol{\mu}(M) = P$.

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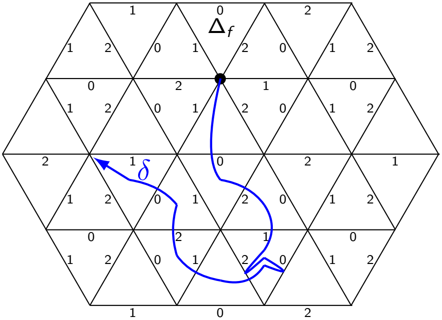
## Idea

We want to modify the approach of Gaussent and Littelmann to obtain more fixed points.

# Example: LS-gallery for $SL_3(\mathbb{C})$




LS-gallery $\delta$ of type $[s_0, s_2, s_1, s_2, s_0, s_2, s_1, s_2, s_0]$.

## Remark

We have the following diagram:

$$\begin{array}{ccccc} \Gamma(\gamma_\lambda) & \xrightarrow[i]{} & \Sigma(\gamma_\lambda) & \xrightarrow[r_w]{} & \Gamma(\gamma_\lambda) \\ & & \big\downarrow \pi & & \\ & & \overline{\mathcal{G}_\lambda} & & \end{array}$$

$\Sigma(\gamma_\lambda)$ denotes the Bott-Samelson resolution of $\overline{\mathcal{G}_\lambda}$ of type $\gamma_\lambda$.
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**Remark (Bijection of Gaussent/Littelmann)**

Let $\delta$ be an LS-gallery of type $\gamma_\lambda$. Then

$$M_\delta = \overline{\pi(C_e(\delta))}, \text{ for } C_e(\delta) = r_e^{-1}(\delta),$$

is an MV-cycle. Here $e$ denotes the unit of $W$.

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Let $\Gamma_{LS}(\gamma_\lambda, \nu)$ be the set of LS-galleries of type $\gamma_\lambda$ ending in $\nu$, then

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**Lemma**

*This observation holds for all Weyl group elements,*

$$\pi(r_w^{-1}(\Gamma_{LS}^w(\gamma_\lambda, \nu))) = \left(U^-(\mathcal{K})^w \underline{t}_\nu\right) \cap \mathcal{G}_\lambda.$$

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**Remark**

- The gallery $\delta_e^M$, where $e$ is the unit of $W$, is the LS-gallery $\delta$ such that $M_\delta = M$.
- For all $w \in W$ it holds that $\delta_w^M = r_w(x)$ for a generic point $x \in r_e^{-1}(\delta_e^M)$.

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**Remark**

For each simple root $\alpha$, one can define **Kashiwara operators**

$$e_\alpha : \Gamma_{LS}(\gamma_\lambda) \to \Gamma_{LS}(\gamma_\lambda) \cup \{0\} \text{ and}$$

$$f_\alpha : \Gamma_{LS}(\gamma_\lambda) \to \Gamma_{LS}(\gamma_\lambda) \cup \{0\},$$

where 0 is an element that is not included in the set $\Gamma_{LS}(\gamma_\lambda)$.

# Combinatorial construction of $\Xi_w(\delta)$

## Definition

Let $\delta \in \Gamma_{LS}(\gamma_\lambda)$ and $\alpha$ a simple root, then we define

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## Definition

Let $w \in W$ and $w = w's_\alpha$ with $l(w') < l(w)$, then we define $\Xi_w(\delta)$ recursively by

$$\Xi_w(\delta) = \Xi_{w's_\alpha w'^{-1}}\left(\Xi_{w'}(\delta)\right).$$

# Properties of the $\Xi_w(\delta)$

**Theorem (E.)**

*Let $\delta \in \Gamma_{LS}(\gamma_\lambda)$. There exists a dense open subset $O \subset C_e(\delta)$ such that for all $x \in O$ and $w \in W$ it holds:*

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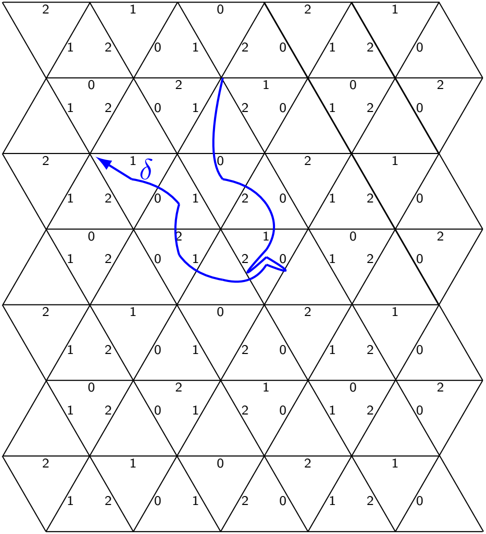
*Hence $\delta_w = \Xi_w(\delta)$ for all $w \in W$.*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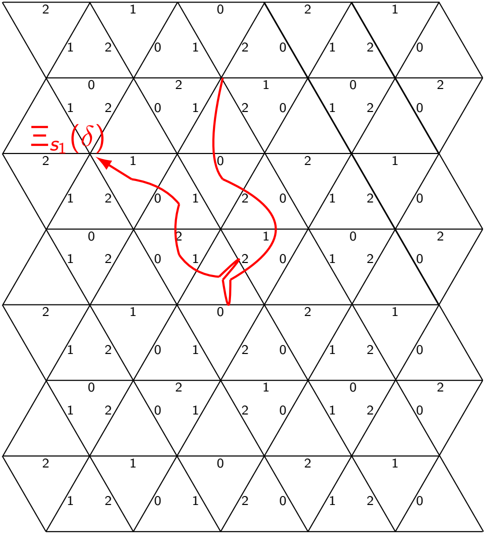

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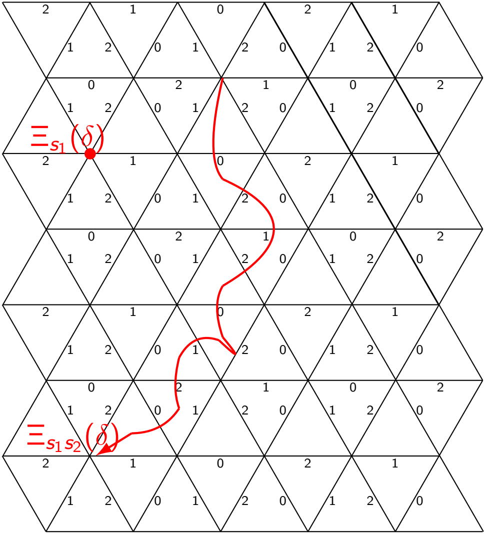

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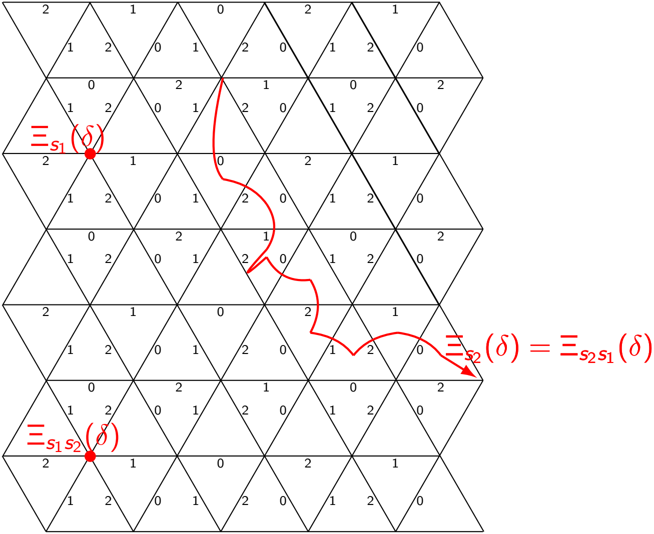

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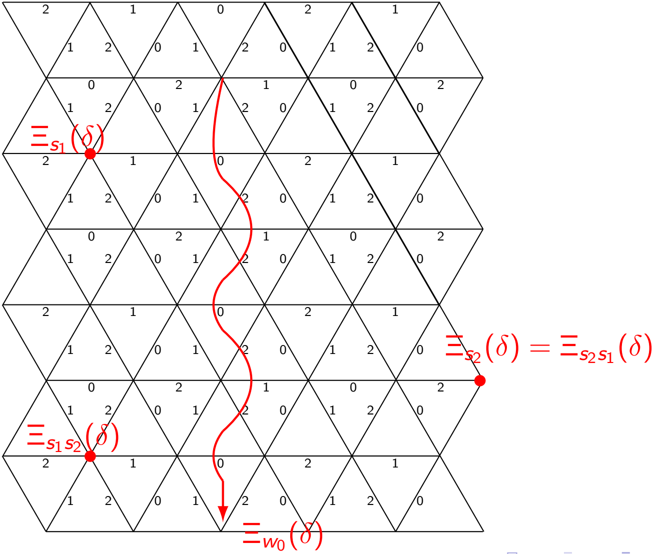

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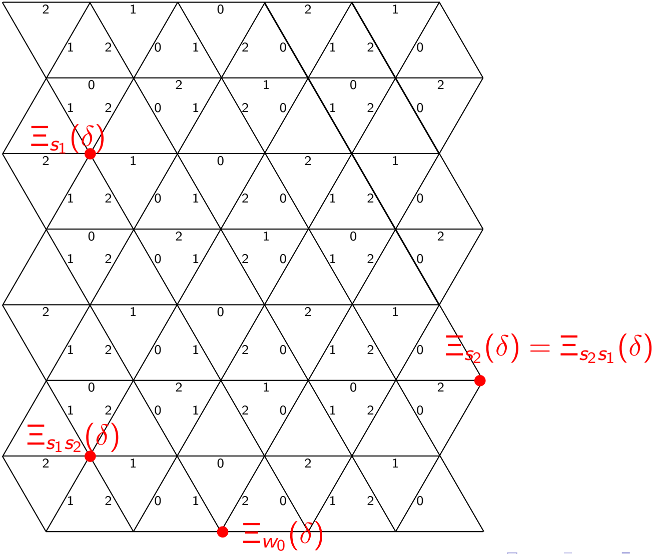

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