Multiplicity spaces in classical symplectic branching

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- Irreducible (polynomial) representations of $GL(n, \mathbb{C})$:

$$\lambda \in \Lambda_n^+ \leftrightarrow V_\lambda$$

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&\cong \bigoplus_{\substack{\lambda^{(i)} \in \Lambda_i^+ \\ \lambda^{(i)} < \lambda^{(i+1)}}} V_{\lambda^{(1)}} \otimes \mathcal{N}_{\lambda^{(1)}}^{\lambda^{(2)}} \otimes \cdots \otimes \mathcal{N}_{\lambda^{(n-1)}}^{\lambda^{(n)}}
\end{aligned}
$$

where the sum is over all $\lambda^{(i)} \in \Lambda_i^+$ such that $\lambda^{(i)} < \lambda^{(i+1)}$ for $i = 1, \ldots, n-1$ and $\lambda^{(n)} = \lambda$.

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- This branching is **not** multiplicity-free.

A problem

Problem

Is it possible to resolve the multiplicities and construct a Gelfand-Zeitlin type basis for the symplectic group?

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- Our approach to this problem is based on classical invariant theory.

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- $\mathcal{M}_\mu^\lambda = Hom_{Sp(n-1,\mathbb{C})}(W_\mu, W_\lambda)$
- Generically $\dim \mathcal{M}_\mu^\lambda > 1$.

Our goal

We will show that there is a natural irreducible action of

$$L = \prod_{i=1}^{n} SL(2, \mathbb{C})$$

on $\mathcal{M}_{\mu}^{\lambda}$.

Starting point

- Let $\mu \in \Lambda_{n-1}^{+}$ and $\lambda \in \Lambda_{n}^{+}$. Then $\mathcal{M}_{\mu}^{\lambda}$ is an $SL(2,\mathbb{C})$-module:

$$SL(2,\mathbb{C}) \subset Z_{Sp(n,\mathbb{C})}(Sp(n-1,\mathbb{C})).$$

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- What's the $SL(2,\mathbb{C})$-module structure of $\mathcal{M}_\mu^\lambda$?

Double Interlacing

Theorem

Let $\mu \in \Lambda^+_{n-1}$ *and* $\lambda \in \Lambda^+_n$. *Then*

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$$\lambda_i \geq \mu_i \geq \lambda_{i+2}.$$
- If $\mu \ll \lambda$ then we call (μ, λ) a **double interlacing pair**. Let
$$\mathfrak{D} = \{(\mu, \lambda) | \mu \ll \lambda\}$$
be the set of all double interlacing pairs.

The module structure of symplectic multiplicity spaces

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Theorem (Molev '99, Wallach-Y '09)

Let $(\mu,\lambda) \in \mathfrak{D}$. *Then*

$$\mathcal{M}_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)}$$

as $SL(2,\mathbb{C})$*-modules.*

An example

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- $$\mathcal{M}_\mu^\lambda \cong F_2 \otimes F_1 \otimes F_0 \otimes F_0 \otimes F_1$$

Another question

- For $(\mu, \lambda) \in \mathfrak{D}$ set

$$\mathcal{A}_{\mu}^{\lambda} = \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)}$$

an irreducible $L = \prod_{i=1}^{n} SL(2, \mathbb{C})$-module.

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- The previous theorem can be reformulated as:

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- The previous theorem can be reformulated as:
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where $SL(2,\mathbb{C}) \subset L$ is the diagonal subgroup.
- Does there exist a natural action of L on $\mathcal{M}_\mu^\lambda$ such that $\mathcal{M}_\mu^\lambda \cong \mathcal{A}_\mu^\lambda$?

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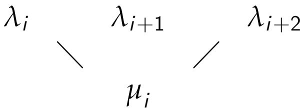
- 

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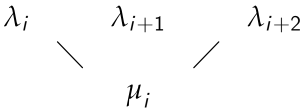

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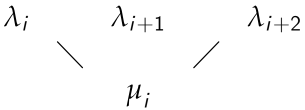

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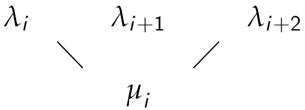
- 

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- E.g. $\lambda = (3, 2, 1)$ and $\mu = (3, 0)$. Then (μ, λ) is of order type $(\geq\leq)$.

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- $\mathcal{M}_\sigma$ is an $SL(2, \mathbb{C})$-subalgebra of $\mathcal{M}$.

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Theorem

Let $\sigma \in \Sigma$. Then $\mathcal{M}_\sigma$ and $\mathcal{O}(V)$ are canonically isomorphic as $SL(2,\mathbb{C})$-algebras. In particular, $\mathcal{M}_\sigma$ is a polynomial algebra.

Glueing the actions

- The above theorem allows us to canonically transfer the L-action from $\mathcal{O}(V)$ to $\mathcal{M}_\sigma$. We have a family of L-algebras

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- The action of L is well-defined on the intersections of these subalgebras, allowing us to glue them together obtaining a representation of L on $\mathcal{M}$.

Main result

Theorem

There is a **unique** *representation* $(\Phi, \mathcal{M})$ *of* $L = \prod_{i=1}^{n} SL(2,\mathbb{C})$ *such that,*

1. *for all* $(\mu, \lambda) \in \mathfrak{D}$, $\mathcal{M}_\mu^\lambda \cong \mathcal{A}_\mu^\lambda = \bigotimes_{i=1}^n F_{r_i(\mu,\lambda)}$, *and*
2. *for all* $\sigma \in \Sigma$, *L acts as algebra automorphisms on* $\mathcal{M}_\sigma$.

Moreover, $\Phi|_{SL(2,\mathbb{C})}$ *is the natural action of* $SL(2,\mathbb{C})$ *on* $\mathcal{M}$.

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$$\{\gamma \in \Lambda_n^+ : \mu < \gamma < \lambda^+\}.$$
- Let $\mathcal{M}_\mu^\lambda(\gamma)$ be the T_L-weight space indexed by γ.

An application continued...

- $\lambda \in \Lambda_n^+$.

$$W_\lambda \quad \cong \quad \bigoplus_{\substack{\mu \in \Lambda_{n-1}^+ \\ \mu \ll \lambda}} W_\mu \otimes \mathcal{M}_\mu^\lambda$$

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