Parity sheaves

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(joint with Daniel Juteau and Carl Mautner)

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Throughout:

- X will denote a complex algebraic variety equipped with the classical topology;
- X will be equipped with a Whitney stratification into smooth, connected, locally closed strata:

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

- k will denote a field (usually of characteristic > 0).

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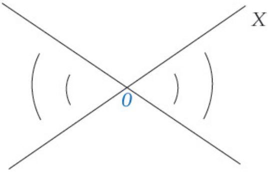
The categorical structure of $\mathbf{P}_\Lambda(X, k)$ reflects in subtle ways the topology of X and its subvarieties $\overline{X_\lambda}$.

Our favourite example



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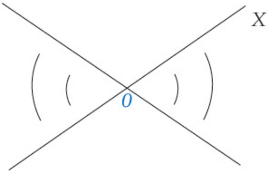

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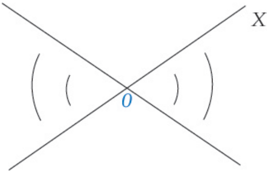

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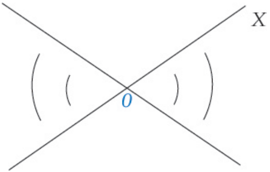

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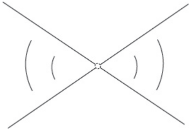
We stratify X into two pieces:

$X = X_{reg} \sqcup \{0\} =$  $\sqcup \{0\}$

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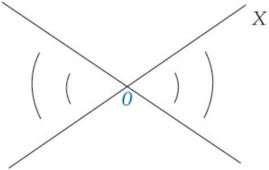
With this stratification:
$\mathbf{P}_\Lambda(X, k)$ is semi-simple
$\iff$
k is not of characteristic 2



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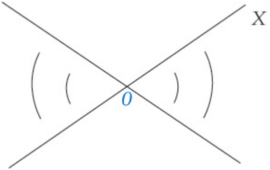

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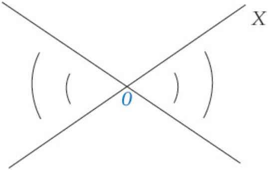

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which has 2-torsion in its cohomology.

Characteristic zero

Over the last thirty years many relations between perverse sheaves with characteristic zero coefficients (that is $k = \mathbb{Q}, \mathbb{C}$ etc.) and representation theory have been discovered. Examples include the (proof of the) Kazhdan-Lusztig conjecture, the Springer correspondence, the construction of canonical bases for quantum groups, and the geometric Satake isomorphism.

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This allows inductive calculations of the stalks of simple perverse sheaves. (E.g. Kazhdan-Lusztig polynomials etc.)

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- Juteau: "modular Springer correspondence"

$$\begin{matrix}\text{decomposition matrix} \\ \text{for } W \text{ over } k\end{matrix} \subset \begin{matrix}\text{decomposition matrix} \\ \text{for } \mathbf{P}_G(\mathcal{N}, \mathbb{Z}) \text{ and } \mathbf{P}_G(\mathcal{N}, k)\end{matrix}$$

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- understand the failure of the Decomposition Theorem;
- find general techniques for working with modular perverse sheaves on varieties arising in representation theory.

Parity sheaves

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Definition

A complex $F \in D^b_\Lambda(X,k)$ is **parity** *if the stalks of F and $\mathbb{D}F$ both vanish in even degree, or both vanish in odd degree.*

Assume, for all strata X_λ and all (G-equivariant) local systems $\mathcal{L}$ on X_λ

$$H^i_G(X_\lambda, \mathcal{L}) = 0 \quad \text{for odd } i.$$

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- *Given an irreducible (equivariant) local system $\mathcal{L}$ on some stratum X_λ there is, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\lambda, \mathcal{L})$ extending $\mathcal{L}[\dim X_\lambda]$.*
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We call such an $\mathcal{E}(\lambda, \mathcal{L})$ (if it exists) a **parity sheaf**.

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- ?? symplectic singularities.

Parity sheaves

- In all examples above, if k is of characteristic zero

$$\mathcal{E}(\lambda, \mathcal{L}) \cong \mathbf{IC}(\overline{X_\lambda}, \mathcal{L}).$$

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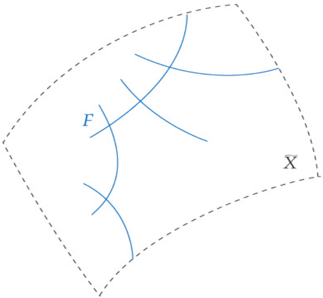
There are a number of reasons to think of parity sheaves as a "replacement" for simple perverse sheaves in positive characteristic.

Parity sheaves allow one to understand the failure of the decomposition theorem for certain semi-small maps arising in representation theory.

Intersection forms

F	$\subset$	$\tilde{X}$
equidimensional		smooth
$\dim_{\mathbb{C}}(F) = n$		$\dim_{\mathbb{C}}(\tilde{X}) = 2n$



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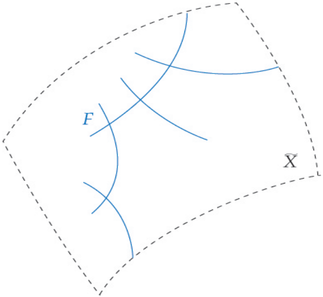

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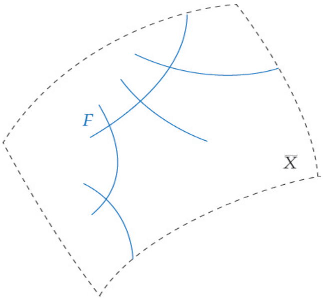

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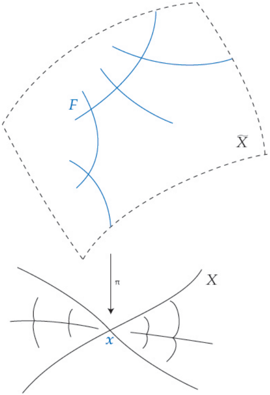
The inclusion $F \hookrightarrow \tilde{X}$ yields an **intersection form**:

$$H^{BM}_{top}(F) \times H^{BM}_{top}(F) \to \mathbb{Z}$$

$$\begin{array}{ccc} F & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{x\} & \xrightarrow{\;i\;} & X \end{array}$$

π: proper, semi-small.
($\widetilde{X}$, F as before)



The Decomposition Theorem at the "most singular point"

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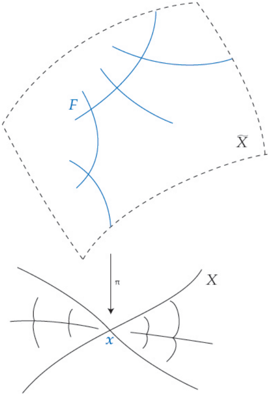
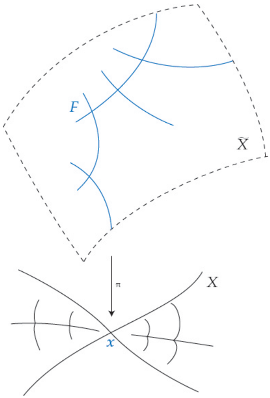
The multiplicity of $i_* \underline{k}_x$
as a summand in $\pi_* \underline{k}_{\tilde{X}}[2n]$
is given by the rank of the form:



$$\mathrm{Hom}(i_* \underline{k}_x, \pi_* \underline{k}_{\tilde{X}}[2n]) \times \mathrm{Hom}(\pi_* \underline{k}_{\tilde{X}}[2n], i_* \underline{k}_x) \to k$$

In fact, both homomorphism spaces may be canonically identified with $H^{BM}_{top}(F)$ and the pairing is the intersection form.



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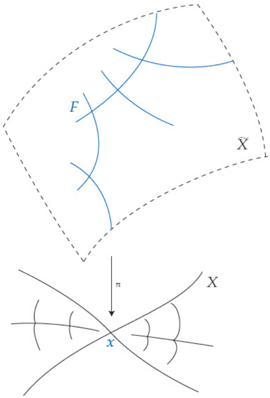
Conclusion: The Decomposition Theorem is true at x if and only if the intersection form on the fibre is non-degenerate.



The general case

In the general case, suppose we have:

$$\pi : \tilde{X} \to X$$

proper and semi-small.

To each stratum one may associate a local system:

$$\mathcal{L}_\lambda = \text{“local system of top Borel-Moore homology”}$$

Moreover, $\mathcal{L}_\lambda$ is equipped with an intersection form B_λ.

The general case

If the Decomposition Theorem is true one has:

$$\pi_* \underline{k}_{\widetilde{X}}[\dim \widetilde{X}] \cong \bigoplus \mathbf{IC}(\overline{X_\lambda}, \mathcal{L}_\lambda) \tag{1}$$

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With characteristic zero coefficients this fact was observed and used by de Cataldo and Migliorini to give Hodge theoretic proofs of the decomposition theorem for semi-small maps.

(See also "The Contravariant Form" in Chriss-Ginzburg.)

However, if $\pi_* \underline{k}_{\widetilde{X}}[\dim \widetilde{X}]$ is parity one can be more precise:

Decomposing Parity Sheaves

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Theorem

Suppose that $\pi_* \underline{k}_{\widetilde{X}}[\dim \widetilde{X}]$ *is parity. Then one has:*

$$\pi_* \underline{k}_{\widetilde{X}}[\dim_{\mathbb{C}} \widetilde{X}] \cong \bigoplus \mathcal{E}(\lambda, \mathcal{L}_\lambda / \operatorname{rad} B_\lambda).$$

Hence the multiplicity of $\mathcal{E}(\lambda, \mathcal{L})$ *as a summand of* $\pi_* \underline{k}_{\widetilde{X}}[\dim \widetilde{X}]$ *is equal to the multiplicity of* $\mathcal{L}$ *in* $\mathcal{L}_\lambda / \operatorname{rad} B_\lambda$.

Parity sheaves on the affine Grassmannian

Recall the geometric Satake isomorphism:

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This gives a local characterisation of tilting sheaves on $\mathcal{G}r$ (compare *Tilting Sheaves*, Beilinson–Bezrukavnikov–Mirkovic).

Literature

These slides are available at:

http://people.maths.ox.ac.uk/~williamsong/parity.pdf

For more details, see:

JMW, *Perverse sheaves and modular representation theory*, http://arxiv.org/abs/0901.3322

(Survey + lots of examples and references).

JMW, *Parity sheaves*, http://arxiv.org/abs/0906.2994

(General theory + large classes of examples)

Fiebig, W., *The p-smooth locus of Schubert varieties*, in prep.

(Relation between parity sheaves and Fiebig's work)