The Solution of a Differential System Given by Lie Algebras

GHERSTEGA NATALIA,POPA MIHAIL

Institute of Mathematics and Computer Science
Academy of Science of Moldova

E-mail: ghersteja@gmail.com,popam@math.md

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Linear differential operators

Let us consider linear differential operators [1] of the following variables

$$X_1 = x^1 \frac{\partial}{\partial x^1} - D_1,$$

where

$$D_1 = \sum_{i=1}^{s} \sum_{k=0}^{m_i} \left[(m_i - k - 1) \overset{i}{a}{}^1_k \frac{\partial}{\partial \overset{i}{a}{}^1_k} + (m_i - k) \overset{i}{a}{}^2_k \frac{\partial}{\partial \overset{i}{a}{}^2_k} \right],$$

$$X_2 = x^2 \frac{\partial}{\partial x^1} - D_2,$$

where

$$D_2 = \sum_{i=1}^{s} \sum_{k=0}^{m_i} \left[k \left(\overset{i}{a}{}^1_{k-1} \frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k-1} \frac{\partial}{\partial \overset{i}{a}{}^2_k} \right) - \overset{i}{a}{}^2_k \frac{\partial}{\partial \overset{i}{a}{}^1_k} \right],$$

Linear differential operators

$$X_3 = x^1 \frac{\partial}{\partial x^2} - D_3,$$

where

$$D_3 = \sum_{i=1}^{s} \sum_{k=0}^{m_i} \left[(m_i - k) \left(\overset{i}{a}{}^1_{k+1} \frac{\partial}{\partial \overset{i}{a}{}^1_k} + \overset{i}{a}{}^2_{k+1} \frac{\partial}{\partial \overset{i}{a}{}^2_k} \right) - \overset{i}{a}{}^1_k \frac{\partial}{\partial \overset{i}{a}{}^2_k} \right],$$

$$X_4 = x^2 \frac{\partial}{\partial x^2} - D_4,$$

where

$$D_4 = \sum_{i=1}^{s} \sum_{k=0}^{m_i} \left[k \overset{i}{a}{}^1_k \frac{\partial}{\partial \overset{i}{a}{}^1_k} + (k-1) \overset{i}{a}{}^2_k \frac{\partial}{\partial \overset{i}{a}{}^2_k} \right],$$

which form the reductive Lie algebra L_4. Denote by $\Gamma = \{m_i\}_{i=1}^{s}$ a finite set of different non-negative integers, and $x = (x^1, x^2)$, $a = (\overset{i}{a}{}^1_l, \overset{i}{a}{}^2_l)$ $(i = \overline{1, s}; l = \overline{0, m_i})$.

The system of differential equations

Let us consider the following system of equations

$$\left\{ \begin{array}{lllll} X_1(F) & = & X_4(F) & = & -gF, \ g \in \mathbb{Z}, \\ X_2(F) & = & X_3(F) & = & 0. \end{array} \right. \qquad (1)$$

The solutions of the system (1) are of great importance in the study of the invariants of the polynomial dynamic systems [1-2].

The system of differential equations

Adducing the system (1) to the form

$$\begin{cases} (X_1 - X_4)(F) & = & X_2(F) & = & X_3(F) = 0, \\ X_4(F) & = & -gF, & & g \in \mathbb{Z}. \end{cases} \tag{2}$$

we suppose the operators $X_1 - X_4,\ X_2,\ X_3$ form a semi-simple Lie algebra L_3, which is an ideal in L_4.

The solutions $F(x, a)$ of the system (1) or (2)

In the following let us examine the solutions $F(x, a)$ of the system (1) or (2),when they are homogeneous polynomials of the type $(d) = (\delta, d_1, d_2, \ldots, d_s)$, where δ and d_i are degrees of homogeneity related to variables x^1, x^2, and $\overset{i}{a}{}^{j}_{l}$ $(i = \overline{1, s};\ j = 1, 2; l = \overline{0, m_i})$, respectively.

The space of solutions of the system (1) by $V_\Gamma^{(d)}$

One can observe that if g is fixed then the set of solutions of the system (1) or (2) form a linear space.
In case of homogeneous polynomials of the type (d) we denote the space of solutions of the system (1) by $V_\Gamma^{(d)}$ and the space of solutions of the first three equations of the system (2) by $S_\Gamma^{(d)}$.

The space of solutions of the system (1) by $V_\Gamma^{(d)}$

Following [1], when $V_\Gamma^{(d)} \cong S_\Gamma^{(d)}$ and $dim_{\mathbb{R}} S_\Gamma^{(d)} < \infty$, the sum $S_\Gamma = \sum_{(d)} S_\Gamma^{(d)}$ forms finitely defined graded algebra, i.e.

$$S_\Gamma^{(d)} = \langle F_1, F_2, \dots F_{\sigma_1} | f_1^1, f_2^1, \dots, f_{\sigma_2}^1; f_1^2, f_2^2, \dots, f_{\sigma_3}^2; \dots \rangle,$$

where $\sigma_i (i = 1, 2, 3, \dots)$ are Betti numbers in series of Poincare of the algebra S_Γ.

Hilbert series

The arising here sequence is $\{dim_{\mathbb{R}} S_\Gamma^{(d)}\}_{(d)}$, and the corresponding generalized Hilbert series [3] is

$$H(S_\Gamma, u, z_1, z_2, \ldots, z_s) = \sum_{(d)} dim_{\mathbb{R}} S_\Gamma^{(d)} u^\delta z_1^{d_1} z_2^{d_2} \ldots z_s^{d_s}, \qquad (3)$$

where $dim_R S_\Gamma^{(0)} = 1$.

The common Hilbert series is obtain from the generalized one as follows:

$$H_{S_\Gamma}(u) = H(S_\Gamma, u, u, u, \ldots, u). \qquad (4)$$

Hilbert series

Note 1. We remark that the transcendence degree over R of the field of quotients of algebra S_Γ is named its Krull dimension; this dimension is equal to the maximum number of algebraically independent homogeneous elements in S_Γ, and also to the order of the pole of common Hilbert series at the unit.

The generating function

It is known [1] that $dim_{\mathbb{R}} S_\Gamma^{(d)}$ is equal to the coefficient of $u^\delta z_1^{d_1} z_2^{d_2} \dots z_s^{d_s}$ in the expansion of the initial generating function

$$\varphi_\Gamma^{(0)}(u) = (1 - u^{-2})\psi_{m_1}^{(0)}(u)\psi_{m_2}^{(0)}(u)\dots\psi_{m_s}^{(0)}(u), \tag{5}$$

in non-negative powers of variables $u, z_1, z_2, \dots, z_s$, where

$$\psi_{m_i}^{(0)}(u) = \begin{cases} \frac{1}{(1-uz_i)(1-u^{-1}z_i)} & \text{for } m_i = 0, \\ \frac{1}{(1-u^{m_i+1}z_i)(1-u^{-m_i-1}z_i)\prod_{k=1}^{m_i}(1-u^{m_i-2k+1}z_i)^2} & \text{for } m_i \neq 0, \end{cases} \tag{6}$$

for each $\Gamma = \{m_i\}_{i=1}^s$.
We remark for
$\Gamma = \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}$ with the help of Hilbert series the homogeneous polynomial solutions for system (1)-(2) were found [1].

Hilbert series for algebras $S_{1,4}$

From (5) and (6) for $\Gamma = \{1, 4\}$, by putting $s = 2$ and $m_1 = 1, m_2 = 4$ and by introducing the designations $z_1 = b, z_2 = e$, we find the initial form of the generating function

$$\varphi_{1,4}^{(0)}(u) = (1 - u^{-2})\psi_1^{(0)}(u)\psi_4^{(0)}(u),$$

where

$$\psi_1^{(0)}(u) = \frac{1}{(1 - u^2 b)(1 - b)^2(1 - u^{-2} b)},$$

$$\psi_4^{(0)}(u) = \frac{1}{(1 - u^5 e)(1 - u^3 e)^2(1 - ue)^2(1 - u^{-1} e)^2(1 - u^{-3} e)^2(1 - u^{-5} e)}$$

Following [1], we remark the generalized Hilbert series $H(S_{1,4}, u, b, e)$ is a solution of Cayley equation

$$H(S_{1,4}, u, b, e) - u^{-2} H(S_{1,4}, u^{-1}, b, e) = \varphi_{1,4}^{(0)}(u).$$

The generalized Hilbert series for the graded algebra $S_{1,4}$

Lemma

The generalized Hilbert series for the graded algebra $S_{1,4}$ of polynomial solutions of system (1)-(2) with $\Gamma = \{1, 4\}$ is a rational function of $u,\ b,\ e$ and has the form

$$H(S_{1,4}, u, b, e) = \frac{N_{1,4}(u)}{D_{1,4}(u)}, \tag{7}$$

where

$$D_{1,4}(u) = (1-b)(1-b^2)(1-bu^2)(1-be^2)^2(1-b^3e^2)^2(1-b^5e^2)\times \\ \times(1-e^2)(1-e^4)^2(1-e^6)^2(1-e^8)^2(1-eu)^2(1-eu^3)^2(1-eu^5), \tag{8}$$

$$N_{1,4}(u) = \sum_{k=0}^{13} R_k(b, e)u^k, \tag{9}$$

where $R(b, e)$ is a polynomial of b and e and $gradR(b, e) = 58$.

The common Hilbert series for the graded algebra $S_{1,4}$

Corollary

The common Hilbert series for the graded algebra $S_{1,4}$ of polynomial solutions for system (1)-(2) *has the form*

$$H_{S_{1,4}}(u) = \frac{n_{1,4}(u)}{d_{1,4}(u)},$$

where

$$\begin{gathered}
n_{1,4}(u) = 1 + u + u^2 + 5u^3 + 17u^4 + 39u^5 + 100u^6 + 218u^7 + \\
+467u^8 + 865u^9 + 1586u^{10} + 2685u^{11} + 4467u^{12} + 6889u^{13} + \\
+10423u^{14} + 14934u^{15} + 20921u^{16} + 27849u^{17} + 36293u^{18} + \\
+45278u^{19} + 55254u^{20} + 64697u^{21} + 74134u^{22} + 81782u^{23} + \\
+88328u^{24} + 91866u^{25} + 93539u^{26} + 91866u^{27} + \ldots + u^{51} + u^{52}, \\
d_{1,4}(u) = (1-u^2)(1-u^3)(1-u^4)^3(1-u^5)^2(1-u^6)^3(1-u^7)(1-u^8)^2.
\end{gathered}$$

The Krull dimension for algebra $\varrho(S_{1,4})$

Lemma

The Krull dimension for algebra $S_{1,4}$ is equal to 13, i.e. the maximum number of algebraically independent homogeneous polynomial solutions for system (1)-(2) *in algebra $S_{1,4}$ is equal to 13.*

Hilbert series for algebras $S_{1,4}$

Definition

We say that the linearity-independent polynomial solution of the system (1) of the type $(d) = (\delta, d_1, d_2, \ldots, d_s)$ is the irreducible polynomial of this system if it cannot be presented as a polynomial of lowest degrees δ, d_i $(i = \overline{1, s})$ of this system.

According to Definition of Hilbert series in (3) we have that $dim_{\mathbb{R}} S_{1,4}^{(\delta, d_1, d_2)}$ coincides with the coefficients of $u^{\delta} b^{d_1} e^{d_2}$ in the expansion $H(S_{1,4}, u, b, e)$ from (7)-(9) in u, b and e.

Hilbert series for algebras $S_{1,4}$

From [6] it is known that the equality

$$\overline{N}(\delta, d_1, d_2) = N(\delta, d_1, d_2) - N'(\delta, d_1, d_2) + N''(\delta, d_1, d_2) \qquad (10)$$

takes place, where $N(\delta, d_1, d_2) = dim_{\mathbb{R}} S_{1,4}^{(\delta, d_1, d_2)}$, $N'(\delta, d_1, d_2)$ is the number of all polynomials of this type, which can be with the help of operations of multiplication from irreducible polynomials of lowest orders δ and lowest degrees d_1, d_2, $N''(\delta, d_1, d_2)$ is the number of linearly-dependent between composite $N'(\delta, d_1, d_2)$ polynomials (i.e. $N''(\delta, d_1, d_2)$ is the number of linearly-independent polynomial relations (syzygies) of given type), and $\overline{N}(\delta, d_1, d_2)$ is the number of polynomial solutions of this type under investigation which are included in generators of algebra $S_{1,4}$.

The representative form of the generating function for algebra of polynomials $S_{1,4}$

Definition

Following [5], the representative form of generating functions (RFGF) for algebra of polynomials $S_{1,4}$ of system (1)-(2) will be defined as such function $\overline{\varphi}_{1,4}(u)$ which is received from generalized Hilbert series (7)-(9) for this algebra by the multiplication of the numerator and denominator by some polynomial $M(u, b, e)$, where in the obtained result each factor of the new denominator has already the form $1 - u^{\delta} b^{d_1} e^{d_2}$, in which the monomial $u^{\delta} b^{d_1} e^{d_2}$ corresponds to an irreducible polynomial of the type (δ, d_1, d_2).

The representative form of the generating function for algebra of polynomials $S_{1,4}$

The representative form of the generating functions for algebra of polynomials $S_{1,4}$ of the system (1)-(2) takes the form

$$\overline{\varphi}_{1,4}(u) = \frac{\overline{N}_{1,4}(u)}{\overline{D}_{1,4}(u)}, \tag{11}$$

where $\overline{N}_{1,4}(u) = N_{1,4}(u)M(u,b,e), \overline{D}_{1,4}(u) = D_{1,4}(u)M(u,b,e)$, and $M(u,b,e) = (1+ue)^2(1+u^3e)|(1+e^2)$.

Screening rule for polynomials

Following [5], we define screening rule for polynomials when the RFGF of the system (1)-(2) is known. Let us designate by $\eta(\delta, d_1, d_2)$ the positive factor of $u^\delta b^{d_1} e^{d_2}$ in the numerator $\overline{N}_{1,4}(u)$ of RFGF which corresponds to the number of linearly independent polynomials of the type (δ, d_1, d_2), not containing as factors the polynomials, whose types are represented by every factor of the denominator of RFGF.
Let $\eta'(\delta, d_1, d_2)$ be the number of all polynomials of the type (δ, d_1, d_2), obtained with the help of multiplication from irreducible polynomials of the lowest degrees, whose types are represented only in the numerator $\overline{N}_{1,4}(u)$. Then, $\eta(\delta, d_1, d_2) > \eta'(\delta, d_1, d_2)$, the number of irreducible polynomials of the type (δ, d_1, d_2) is not less than the difference $\eta(\delta, d_1, d_2) - \eta'(\delta, d_1, d_2)$, and for $\eta(\delta, d_1, d_2) \leq \eta'(\delta, d_1, d_2)$ is not less than the difference $\eta(\delta, d_1, d_2) - \eta'(\delta, d_1, d_2)$ of syzygies.

The main theorem

With the help of Franklin's theorem and screening rule we obtain

Theorem

The lower bound of the number of generators for the algebra $S_{1,4}$ is not less than 311 irreducible polynomials distributed in 58 types as follows:

$$(0,1,0),(0,2,0),6(0,0,4),7(0,0,6),15(0,0,8),14(0,0,10),$$
$$3(0,1,2),6(0,1,4),15(0,1,6),16(0,1,8),(0,2,2),8(0,2,4),$$
$$15(0,2,6),3(0,3,2),10(0,3,4),7(0,3,6),(0,4,2),5(0,4,4),$$
$$(0,5,2),3(0,5,4),2(1,0,3),11(1,0,5),20(1,0,7),2(1,0,9),$$
$$(1,1,1),8(1,1,3),20(1,1,5),2(1,2,1),9(1,2,3),4(1,2,5),$$
$$(1,3,1),3(1,3,3),3(1,4,3),(2,1,0),3(2,0,2),6(2,0,4),$$
$$4(2,1,2),9(2,1,4),3(2,2,2),2(2,3,2),(3,0,1),6(3,0,3),$$
$$9(3,0,5),2(3,1,1),6(3,1,3),(3,2,1),(4,0,2),6(4,0,4),$$
$$3(4,1,2),(5,0,1),3(5,0,3),(5,1,1),2(6,0,2),2(6,0,4),$$
$$(6,1,2),(7,0,3),(9,0,3).$$

Results: Summary

- The lower bounds of types of polynomial solutions for the system (1)-(2) can be arranged in a parallelepiped with dimensions (9,5,10).
- The first two Betti numbers from Poincare series were determined ($\sigma_1 \geq 311,\ \sigma_2 \geq 298$).