Ascona Conference

"Algebraic Groups and Invariant Theory"

Springer fibers admitting singular components

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Definition of Springer fibers

- Let $V = \mathbb{C}^n$, let $u \in \mathrm{End}(V)$ nilpotent.
- A complete flag is a maximal chain of vector subspaces: $F = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = V)$, $\dim V_i = i \ \ (\forall i)$.
- $\mathcal{B} = \{\text{complete flags}\}$ is an algebraic projective variety.
- Define:

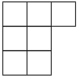
$$\mathcal{B}_u = \{F \in \mathcal{B} : u(V_i) \subset V_i \ \forall i\}.$$

  This is a projective subvariety of $\mathcal{B}$, called *Springer fiber*.
- $\mathcal{B}_u$ is connected, in general reducible.

$\mathcal{B}_u$ in geometric representation theory

- Jordan form of u represented by a Young diagram:

$$(\lambda_1 \geq \ldots \geq \lambda_r) \quad \longrightarrow \quad Y(u) =$$



(λ_i boxes in the i-th row)

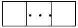
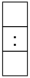
(sizes of the Jordan blocks of u)

- T.A. Springer (1976): link between $\mathcal{B}_u$ and the representation theory of $\mathbf{S}_n$:
  - structure of $\mathbf{S}_n$-module on $H^*(\mathcal{B}_u, \mathbb{Q})$, such that

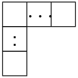
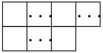
$$H^{\max}(\mathcal{B}_u, \mathbb{Q}) \cong M(\,Y(u)\,) \in \mathrm{Irr}(\mathbf{S}_n)$$

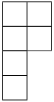
($M(Y)$: Specht module).

Particular cases

- Two simple cases:
  - If $Y(u) =$ , then u is regular and $\mathcal{B}_u = \{\text{pt}\}$.
  - If $Y(u) =$ , then $u = 0$ and $\mathcal{B}_u = \mathcal{B}$.

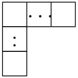
  In any other case, $\mathcal{B}_u$ is reducible.

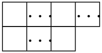
- It may happen that every component of $\mathcal{B}_u$ is nonsingular:
  - in the hook case, i.e. $Y(u) =$  (J.A. Vargas, 1979)
  - in the 2-row case, i.e. $Y(u) =$  (F. Fung, 2003).

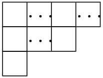
- However, for $Y(u) =$  $\mathcal{B}_u$ has a singular component (J.A. Vargas, 1979).

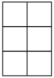
Main result

Theorem. *Every irreducible component of* $\mathcal{B}_u$ *is nonsingular exactly in four cases:*

(i) *the hook case* $Y(u) =$ 

(ii) *the 2-row case* $Y(u) =$ 

(iii) *the "2-row plus one box" case:* $Y(u) =$ 

(iv) *(exceptional case)* $Y(u) =$ .

Preliminaries: $\mathcal{B}_u$ and the combinatorics of Young

- Jordan form of u represented by a Young diagram:

$$Y(u) = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

  - (N. Spaltenstein): $\dim \mathcal{B}_u = \sum_{j=1}^{s} \frac{1}{2}\lambda_j^*(\lambda_j^* - 1)$
    where $\lambda_1^*, ..., \lambda_s^*$ are the sizes of the columns of $Y(u)$.

- *Standard tableau* = numbering of $Y(u)$ by $1, \ldots, n$, increasing along the rows and the columns.

$$\text{Example:} \quad T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

  - (N. Spaltenstein): the irreducible components of $\mathcal{B}_u$ are parameterized by the standard tableaux of shape $Y(u)$.

$\mathcal{B}_u$ *and the combinatorics of Young (continue)*

- Spaltenstein's construction of the components:
  - T standard gives rise to

$$Y_1(T) \subset \ldots \subset Y_i(T) \subset \ldots \subset Y_n(T) = Y(u),$$

where $Y_i(T) =$ shape of the subtableau $T[1, \ldots, i]$.

  - $F = (V_0, \ldots, V_n) \in \mathcal{B}_u$ gives rise to

$$Y(u_{|V_1}) \subset \ldots \subset Y(u_{|V_i}) \subset \ldots \subset Y(u_{|V_n}) = Y(u).$$

Define $\mathcal{B}_u^T = \{F \in \mathcal{B}_u : Y(u_{|V_i}) = Y_i(T) \ \ (\forall i)\}$,
we get $\mathcal{B}_u = \bigsqcup_T \mathcal{B}_u^T$.
$\mathcal{B}_u^T$ is locally closed, irreducible, $\dim \mathcal{B}_u^T = \dim \mathcal{B}_u$ $(\forall T)$.

$\Rightarrow$ $\mathcal{K}^T := \overline{\mathcal{B}_u^T}$ are the irreducible components of $\mathcal{B}_u$, and $\dim \mathcal{K}^T = \dim \mathcal{B}_u$ $(\forall T)$.

1st step: inductive criterion of singularity

- Let T be standard, $\mathcal{K}^T \subset \mathcal{B}_u$ the associated component. Let $T' = T[1, ..., n-1]$, it gives $\mathcal{K}^{T'} \subset \mathcal{B}_{u'}$ component.

  Theorem. *(a) $\mathcal{K}^{T'}$ is singular $\Rightarrow \mathcal{K}^T$ is singular.*
  (b) Moreover, if n lies in the last column of T, then: $\mathcal{K}^{T'}$ is singular $\Leftrightarrow \mathcal{K}^T$ is singular.

  Skip of the proof.
  - Let $\mathcal{U} = \{F = (V_0, ..., V_n) \in \mathcal{K}^T : Y(u_{|V_{n-1}}) = Y_{n-1}(T)\}$. We have $\mathcal{B}_u^T \subset \mathcal{U} \subset \mathcal{K}^T$, and $\mathcal{U}$ is open in $\mathcal{K}^T$. In case (b), $\mathcal{U}$ is closed, hence $\mathcal{U} = \mathcal{K}^T$.
  - Thus, it suffices to show: $\mathcal{K}^{T'}$ singular $\Leftrightarrow \mathcal{U}$ singular.
  - To do this, we show that $\Phi : \mathcal{U} \to \mathcal{H}$, $(V_0, ..., V_n) \mapsto V_{n-1}$ is a fibre bundle over its image, of base nonsingular, of fiber $\mathcal{K}^{T'}$. □

2nd step: construction of singular components

- Two basic singular components:

Proposition. *(a) If* $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}$*, then* $\mathcal{K}^T$ *is singular.* (Vargas)

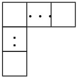
(b) If $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} \end{array}$*, then* $\mathcal{K}^T$ *is singular.*

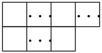
- Combining with the previous criterion, we get:

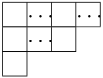
Corollary. *Whenever* $Y(u)$ *contains* $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \end{array}$ *or* $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \end{array}$ *as a subdiagram,* $\mathcal{B}_u$ *admits a singular component.*

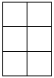
3nd step: the $(r, s, 1)$-case

- It remains to show that, in the following cases:

  (i) hook case $Y(u) =$ 

  (ii) 2-row case $Y(u) =$ 

  (iii) "2-row plus one box" case: $Y(u) =$ 

  (iv) case $Y(u) =$ ,

  every component of $\mathcal{B}_u$ is nonsingular.
- In cases (i), (ii), (iv), all the components are nonsingular.
- Thus, it remains to show:

  If $Y(u)$ has three rows of sizes $r, s, 1$, then every component of $\mathcal{B}_u$ is nonsingular.

3nd step: the $(r, s, 1)$-case (continue)

- We may assume $r = s$.
- We show that $\mathcal{K}^{T(r)}$ is nonsingular for $T(r) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & r \\ \hline r{+}1 & r{+}2 & \cdots & 2r \\ \hline 2r{+}1 \\ \cline{1-1}\end{array}$.
- Say $T \sim T'$ if $\mathcal{K}^T$, $\mathcal{K}^{T'}$ are either both singular or both nonsingular.

  We have $T' \sim T$ if T' is obtained from T by:
  - adding/deleting n in the last column,
  - the Schützenberger involution $T \mapsto Sch(T)$.

  For T of $(r, r, 1)$-type, we show $T \sim T(r')$ for some $r' \leq r$. Therefore $\mathcal{K}^T$ is nonsingular.

1st step: inductive criterion of singularity

- Let T be standard, $\mathcal{K}^T \subset \mathcal{B}_u$ component.
  Let $T' = T[1, ..., n-1]$, it gives $\mathcal{K}^{T'} \subset \mathcal{B}_{u'}$ component.

  Theorem. *(a)* $\mathcal{K}^{T'}$ *is singular* $\Rightarrow$ $\mathcal{K}^T$ *is singular.*
  (b) Moreover, if n *lies in the last column of* T*, then:*
  $\mathcal{K}^{T'}$ *is singular* $\Leftrightarrow$ $\mathcal{K}^T$ *is singular.*

- In particular, adding one box to a tableau preserves the singularity of the component.