Well-behaved separating algebras

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Sept. 1, 2009

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- G a finite group,
- V a n-dimensional representation of G over $\mathbf{k}$.

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- G acts on $\mathbf{k}[V]$ via $(\sigma \cdot f)(v) = f(\sigma^{-1} \cdot v)$.
- Finally, $\mathbf{k}[V]^G$ is the subring of $\mathbf{k}[V]$ fixed by the action of G.

An Easy Example

- Let V be a 2-dimensional vector space over $\mathbb{C}$.
- Let $C_4 = \langle\sigma\rangle$, the cyclic group of order 4, act on V via:

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- $\sigma \cdot x = i^{-1}x = -ix$, and $\sigma \cdot y = i^{-1}y = -iy$, thus
- $\mathbf{k}[V]^G = \mathbf{k}[x^4, x^3y, x^2y^2, xy^3, y^4]$.

Separating invariants

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A subalgebra $A \subset \mathbf{k}[V]^G$ satisfying this condition is a *separating algebra*.

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Thus, as functions, x^2y^2 and xy^3 are entirely determined by x^4, x^3y, y^4, and they are not needed to separate.

- Hence, $S = \{x^4, x^3y, y^4\}$ is a separating set.
- $A = \mathbb{C}[x^4, x^3y, y^4]$ is a separating algebra.

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- The polarization of separating invariants yields separating invariants. (Domokos (2007), Draisma, Kemper, and Wehlau (2008))
- For G finite, the invariants of degree at most $|G|$ form a separating set. (Derksen and Kemper (2002))

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- If G is reductive and A is a graded separating subalgebra, then the extension $A \subset \mathbf{k}[V]^G$ is finite and $\mathbf{k}[V]^G$ is the normalization of the purely inseparable closure of A in $\mathbf{k}[V]$. (Derksen and Kemper (2002))

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- If G is reductive, $\mathbf{k}$ is a field of positive characteristic, and A is a graded subalgebra, then A is a separating subalgebra if and only if $\mathbf{k}[V]^G$ is the purely inseparable closure of A in $\mathbf{k}[V]$. (Derksen and Kemper (2008))

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- For reductive groups, separating algebras are easier to compute than the ring of invariants, and the close relationship means that we can obtain generators for the ring of invariants from a separating set. (Kemper (2003))

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- Let $\overline{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$,
- and let $\overline{V} = V \otimes_{\mathbf{k}} \overline{\mathbf{k}}$.

Definition 2. A subalgebra $A \subset \mathbf{k}[V]^G$ is a *geometric separating algebra* if and only if, for all $u, v \in \overline{V}$, if there is an $f \in \mathbf{k}[V]^G$ such that $f(u) \neq f(v)$, then there is an $h \in A$ such that $h(u) \neq h(v)$.

Geometric Separating Invariants (continued)

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- *if $V = \mathrm{Spec}(\mathbf{k}[V])$, $V/\!\!/G = \mathrm{Spec}(\mathbf{k}[V]^G)$, and $W = \mathrm{Spec}(A)$, then*

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- *if $\delta : \mathbf{k}[V] \to \mathbf{k}[V] \otimes \mathbf{k}[V]$ sends f to $f \otimes 1 - 1 \otimes f$, then*
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Theorem 4. *Suppose G is reductive. A subalgebra $A \subset \mathbf{k}[V]^G$ is a geometric separating algebra if and only if the morphism of schemes $\theta : V/\!\!/G \to W$ is radicial, that is, if the corresponding morphism of $\mathbb{F}$-points is injective for all fields $\mathbb{F}$.*

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Theorem 8 (Shephard and Todd (1954), Serre, Chevalley (1955), Clark and Ewing (1974))**.** *Suppose $|G|$ is invertible in $\mathbf{k}$, then $\mathbf{k}[V]^G$ is a polynomial ring if and only if the action of G on V is generated by reflections.*

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Corollary 9. *Suppose $|G|$ is invertible in $\mathbf{k}$. There exists a polynomial geometric separating algebra if and only if the action of G on V is generated by reflections.*

Back to the Example, once more

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Hence, G is not a reflection group, and so no geometric separating algebra is a polynomial ring. In other words, there are no geometric separating sets of size two, and thus

$$\{x^4, x^3y, y^4\}$$

is a geometric separating set of minimal size.

Consider the group G acting as follows on a 4-dimensional vector space over a field $\mathbf{k}$ of characteristic $p > 0$ containing a root z of $Z^p - Z + 1$:

$$G = \left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

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Then the ring of invariants is an hypersurface, but there is a polynomial separating set.

Remark 10.

- There are other similar examples.
- In all cases the groups are rigid groups, the isotropy subgroups are reflection groups.

Theorem 11. *Let G be a finite group. If there is a finitely generated graded geometric separating algebra which is a complete intersection, then the action of G on V is generated by bireflections.*

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Theorem 13 (Kac and Watanabe (1981), Gordeev (1982))**.** *Let G be a finite group. If the ring of G-invariants is a complete intersection ring, then the action of G on V is generated by bireflections.*

(From now on: joint work with Jonathan Elmer and Martin Kohls.)

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Theorem 14. *If V is faithful and modular, then there exists $r \geq 1$ such that, for all k, every graded geometric separating algebra in $\mathbf{k}[V^{\oplus k}]^G$ has Cohen-Macaulay defect at least $k - r - 1$. In particular, for $k > r + 1$, no graded geometric separating algebra in $\mathbf{k}[V^{\oplus k}]^G$ is Cohen-Macaulay.*

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Theorem 15. *Let G be a p-group. If there exists a graded geometric separating algebra in $\mathbf{k}[V]^G$ which is Cohen-Macaulay, then G is generated by elements acting as bireflections.*

$\mathbf{k}$ is a finite field and G is the group generated by the matrices of the form:

$$\left(\begin{array}{cccc|c} \multicolumn{4}{c|}{I_4} & \mathbf{0} \\ \hline \alpha & 0 & 0 & \delta & \\ 0 & \beta & 0 & \delta & I_m \\ 0 & 0 & \gamma & \delta & \end{array}\right),$$

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Remark 16.

- This is a reflection group (and so, in particular a bireflection group), but not a rigid group.
- The converts of Theorem 3 (for graded subalgebras) and theorem 7 do not hold.

A further interesting example

$G = C_2 \times C_2 = \langle \sigma, \tau \rangle$ is the Klein four group. Consider the 5-dimensional representation of G over a field of characteristic 2 given by

$$\sigma \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tau \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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$\mathbf{k}[V]^G$ is not Cohen-Macaulay, but there is a graded geometric separating algebra which is Cohen-Macaulay.

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Remark 17. This is, of course, a bireflection group.

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are all geometric separating algebras.

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are all geometric separating algebras.

$\mathbb{C}[x^4, x^3y, xy^3, y^4]$ is not Cohen-Macaulay.

For more details

[1] Emilie Dufresne. Separating invariants and finite reflection groups. *Adv. Math.*, 221:1979–1989, 2009. arXiv:Math.AC/0805.2605v3.

[2] Emilie Dufresne Jonathan Elmer and Martin Kohls. The Cohen-Macaulay property of separating invariants of finite groups. *Transform. Groups*, to appear. arXiv:Math.AC/0904.1069.