Introduction
to
Laplace Transform Method

for

Quant 3 Lecture 3 Math Notes

Jerry Batzel

Institute for Mathematics and Scientific Computing
University of Graz, Austria

1 Laplace transform method

We want to give a short introduction to the methods and terminology of the technique for analyzing linear systems of differential equationsinvolving Laplace transforms.

This method is most successful when applied to linear systems with constant coefficients (linear-time-invariant or LTI), though it is sometimes useful for time dependent coefficients.

The linaer system describing a model defined by a set of linear differential equations in the **time domain** can be transformed by the Laplace transform into **transfer functions** in the "s" domain. Algebraic manipulations are employed to find the transformed unknown solution (with initial conditions incorporated).

The approach is used analyze a system for steady state response to various inputs including the **impulse input** and **step input** as well as the studying system response to trigonometric inputs (**frequency response**).

One important advantage of this approach to analyzing linear systems is that the linearity and the algebraic feature of the Laplace transform allow for the modularizing of the total system into subsystems and for the combining of subsystems into larger super systems in a convenient way.

1.1 Laplace transform

The following definitions and examples illustrate the Laplace transform method for solving linear systems.

Definition 1. Let $f(t)$ be a given real-valued function defined on the Interval $0 \leq t < \infty$. The **Laplace transform** of $f(t)$, denoted by $L\{f\} = F$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t) dt$$

We assume now, that s is a real number (in general it can be complex). If $F(s)$ exists at some s_0 it can be shown, that $F(s)$ also exists for all real numbers $s > s_0$.

Example 1. Let $f(t) = 1$ for all $t \geq 0$. Then for $s > 0$

$$F(s) = \int_0^\infty e^{-st} f(t) = \lim_{T\to\infty} \int_o^T e^{-st} dt = \lim_{T\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^T = \frac{1}{s}$$

Hence, $L\{1\} = 1/s$ exists for all $s > 0$.

Example 2. Let $f(t) = e^{at}$ for some real number a. Then for $s > a$

$$F(s) = \int_0^\infty e^{-st}e^{at} = \lim_{T\to\infty}\int_o^T e^{-(s-a)t}dt = \frac{1}{s-a}$$

Hence, $L\{e^{at}\} = (s-a)^{-1}$ exists for all $s > a$.

Example 3. Analogous you can get $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Definition 2. A **function** defined on $[0,\infty)$ is said to be **of exponential order** if there are real constants $M \geq 0$ and a such that $|f(t)| \leq Me^{at}$ for all $t \geq 0$.

Theorem 1. *If f is of exponential order and at least piecewise continuous over $[0,T]$, $T > 0$ then the Laplace transform $L\{f\} = F(s)$ exists for all $s > a$. Moreover $|F(s)| \leq M(s-a)^{-1}$ for all $s > a$.*

Example 4. The unit step function is defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

We see clearly that $u_c(t)$ is of exponential order, thus we can use Laplace transform

$$\begin{aligned} L\{u_c\} &= \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^c e^{-st}\cdot 0\,dt + \int_c^\infty e^{-st}\cdot 1\,dt \\ &= \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_c^T = \frac{e^{-sc}}{s} \qquad \text{for any } s > 0 \end{aligned}$$

Important calculation rules for the **Laplace operator** (with c_1, c_2 real, f_1, f_2 of exponential order):

1. **Linearity:**
$$L\{c_1f_1 + c_2f_2\} = c_1L\{f_1\} + c_2L\{f_2\}$$

2. **Convolution:**
$$L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\} * L\{f_2(t)\}$$

3. **Integration:**
$$L\{\int_0^t f(\tau)\,d\tau\} = \frac{1}{s}L\{f\}$$

4. **Differentiation:** There exists a real b depending on f, that for $s > b$

$$\begin{aligned} L\{\frac{d^n}{dt^n} f(t)\} &= s^n F(s) - s^{n-1} f_0 - \ldots - s f_0^{(n-2)} - f_0^{(n-1)} \\ \text{with } f_0^{(v)} &= \lim_{t \to 0+} \frac{d^v f(t)}{dt^v} \end{aligned}$$

5. **Shifting:**

$$L\{f(t-b)\} = e^{-bs} L\{f(t)\}$$

6. **Similarity:**

$$L\{f(at)\} = \frac{1}{a} F(\frac{s}{a}) \text{ with } a > 0$$

7. **Damping:**

$$L\{e^{-\alpha t} f(t)\} = F(s+\alpha)$$

8. **Multiplicity:**

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

9. **Division:**

$$L\{\frac{1}{t} f(t)\} = \int_s^\infty F(q)\, dq$$

To justify the inversion of the Laplace transform we have the following result:

Theorem 2. *If $f_1(t)$ and $f_2(t)$ are two functions of exponential order, both continuous and their Laplace transforms equal on an interval $s_0 < s < \infty$, then $f_1(t) = f_2(t)$ for all $t \geq 0$. This is not true for only piecewise-continuous functions! If they are piecewise-continuous, then $f_1(t) = f_2(t)$ on $0 \leq t < \infty$* except *on a set $\{t_n\}$ of isolated points.*

Thus given a laplace transform $F(s)$ of a function $f(t)$ we can find the "inverse laplace transform" of $F(s)$ to recover the original $f(t)$. The following tables give the laplace transforms $F(s)$ for typical functions $f(t)$ where $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s}$	1	$\frac{1}{s+\alpha}$	$e^{-\alpha t}$
$\frac{1}{s^2}$	t	$\frac{1}{(s+\alpha)(s+\beta)}$	$\frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})$
$\frac{1}{s^2+\alpha^2}$	$\frac{1}{\alpha}\sin(\alpha t)$	$\frac{s}{s^2+\alpha^2}$	$\cos(\alpha t)$
$\frac{1}{(s+\beta)^2+\alpha^2}$	$\frac{1}{\alpha} e^{-\beta t}\sin(\alpha t)$	$\frac{s}{(s+\beta)^2+\alpha^2}$	$e^{-\beta t}\left(\cos(\alpha t) - \frac{\beta}{\alpha}\sin(\alpha t)\right)$
$\frac{1}{s^n}$	$\frac{1}{(n-1)!} t^{n-1}$	$\frac{1}{(s+\alpha)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-\alpha t}$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\ln\frac{s-\alpha}{s-\beta}$	$\frac{1}{t}(e^{\beta t}-e^{\alpha t})$	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\sqrt{\frac{\sqrt{s^2+\alpha^2}+s}{s^2+\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$	$\sqrt{\frac{\sqrt{s^2-\alpha^2}+s}{s^2-\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$
$\arctan\frac{\alpha}{s}$	$\frac{\sin(\alpha t)}{t}$		

$[t]$ =the biggest natural number n with $n \le t$.

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s(e^{\alpha s}-1)}$	$\left[\frac{t}{\alpha}\right]$	$\frac{1}{s(1-e^{-\alpha s})}$	$\left[\frac{t}{\alpha}\right]+1$

$F(s)$	$f(t)$
$\frac{e^{-\alpha s}-e^{-\beta s}}{s}$	$\begin{cases} 0 & \text{für} \quad 0<t<\alpha \\ 1 & \text{für} \quad \alpha<t<\beta \\ 0 & \text{für} \quad \beta<t \end{cases}$
$\frac{\left(e^{-\alpha s}-e^{-\beta s}\right)^2}{s^2}$	$\begin{cases} 0 & \text{für} \quad 0<t<2\alpha \\ t-2\alpha & \text{für} \quad 2\alpha<t<\alpha+\beta \\ 2\beta-t & \text{für} \quad \alpha+\beta<t<2\beta \\ 0 & \text{für} \quad 2\beta<t \end{cases}$
$\frac{e^{-\alpha s}}{s+\beta}$	$\begin{cases} 0 & \text{für} \quad 0<t<\alpha \\ e^{-\beta(t-\alpha)} & \text{für} \quad \alpha<t \end{cases}$
$\frac{1}{s(1+e^{-\alpha s})}$	$\begin{cases} 1 & \text{für} \quad 2n\alpha<t<(2n+1)\alpha \\ 0 & \text{für} \quad (2n+1)\alpha<t<(2n+2)\alpha \end{cases}$ $n=0,1,2,\ldots$

1.2 Transfer function

We let $x(t)$ denote the state of the system which can be thought of also as the **output** of the system, while $u(t)$ denotes the **input** either as a direct input to the system or a control signal. $u(t)$ is a function of t and affects the equations of the state. In this case $x(t)$ and $u(t)$ are functions but the discussion carries over for vectors as well with division replaced by matrix inversion. Let

$$a\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+cx(t)=u(t),$$
$$x(0)=0, \quad x'(0)=0.$$

If we take Laplace transform on each side of this equation, using the rules above we find that:

$$as^2X(s)+bsX(s)+cX(s)=U(s),$$

or

$$X(s) = \frac{1}{as^2 + bs + c} U(s).$$

The quantity $X(s)$ defines the Laplace transform of the state of the system or output. The quantity $U(s)$ is the Laplace transform of the input. The quantity

$$\frac{1}{as^2 + bs + c}$$

relates the input to the output and is referred to as the **transfer function**, usually denoted by $H(s)$. Thus:

$$X(s) \quad = \quad H(s)U(s). \tag{1}$$

By taking the inverse Laplace transform of the expression $H(s)U(s)$ we find x(t). Systems can be analyzed either in the "t" domain or in the transformed 's" domain and then retransformed into the "t" domain. Furthermore, letting $h(t)$ be the inverse Laplace transform of $H(s)$, and recalling the convolution formula

$$L\{[f_1 * f_2](t)\} = L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\}L\{f_2(t)\}$$

we have

$$\begin{aligned} x(t) &= L^{-1}\{H(s)U(s)\} && (2) \\ &= \int_0^t h(t-\tau)u(\tau)\,d\tau. && (3) \end{aligned}$$

We see that once $h(t)$ (or the transfer function $H(s)$) is known the system response to any input $u(t)$ can be calculated.