

**Introduction**  
**to**  
**Input Analysis: Impulse, Step, and**  
**Frequency Analysis**

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## Outline

- Lung Models: solutions and transfer functions
  - First and second order models with various parameters
  - Proportional, Derivative and Integral Feedback
  - Impulse and Step Inputs
  
- Root Locus Plots

## Closed loop transfer functions for LTI system

The ideas involved in analyzing a closed loop system such as represented in Figure (1) can be illustrated using the following example.

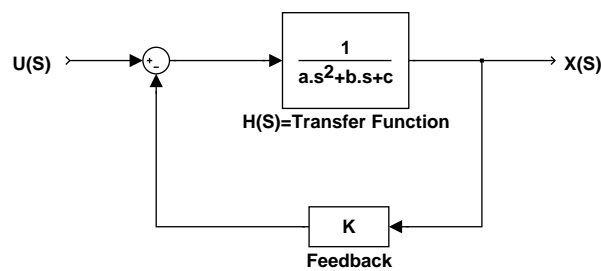


Figure 1: Closed loop transfer function

**Example 1.** In this example, a simple linearized model is used to study the relation between the input of air pressure  $P_{ao}$  at the airway opening and the pressure  $P_A$  at the alveoli. Using simple mechanical laws of pressure drop, air flow resistance, and lung compliance (similar to electrical diagrams), the following second order differential equation is derived:

$$P_{ao}(t) = LC \frac{d^2 P_A(t)}{dt^2} + RC \frac{dP_A(t)}{dt} + P_A. \quad (1)$$

Here  $P_A(0) = 0$ ,  $P_A'(0) = 0$ ,  $u(t) = P_{ao}$ , and  $LC$ , and  $RC$  are constant. The transfer function of this system

takes the form

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + 1} = H(s), \quad (2)$$

where  $H(s)$  is the transfer function of the open loop system as represented in Equation (1).

Assume now that the air flow to the lungs needs to be regulated to avoid damage. We assume a mechanism that measures the alveolar pressure  $P_A$  and produces a negative feedback to the input control  $P_{ao}$ , that is the new control  $u(t) = P_{ao}(t) - KP_A(t)$ . The transfer function for this system (after rearranging terms) takes the form

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s), \quad (3)$$

where  $G(s)$  is the transfer function of the closed loop system.

Note that  $P_{ao}(s) = 1$  for impulse and  $P_{ao}(s) = \frac{1}{s}$  for step function.

## Simplified First Order Model

Consider the simplified model where  $L = 0$ .

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{RCs + (1 + k)} = G(s), \quad (4)$$

Let  $\lambda = 1 + k$  and  $\tau = RC$ . We get transfer function

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{\tau} \left( \frac{1}{s + \frac{\lambda}{\tau}} \right) \quad (5)$$

and for impulse input we get for both open and closed loops:

$$P_A(t) = \frac{1}{\tau} \exp\left(-\frac{\lambda}{\tau}t\right), \quad (6)$$

The time for the system to reach the zero equilibrium is influenced by  $k$ . For step input we get

$$P_A(s) = \frac{1}{\tau} \left( \frac{1}{s \left( s + \frac{\lambda}{\tau} \right)} \right), \quad (7)$$

$$P_A(t) = \frac{1}{\lambda} \left( 1 - \exp \left( -\frac{\lambda}{\tau} t \right) \right), \quad (8)$$

So we can see that with the step input open loop case we generate a long term response of  $P_A(t) = 1$  but in the closed loop case there is some steady state error in the long run behavior.

Comparing open and closed loop control for PROPORTIONAL CONTROL:

- Impulse response
  - Transient response is faster for closed loop.
  - However, oscillatory frequency increases for closed loop although amplitude diminishes.
- Step response
  - transient response is also faster for closed loop.
  - frequency of oscillations increases and amplitude drops.
  - Steady state error is introduced.

## Second Order Model

Consider the second order model:

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + \lambda} = H(s), \quad (9)$$

Where we again let  $\lambda = 1 + k$  so that by setting  $\lambda = 1$  we have  $k = 0$  and open loop and for larger  $\lambda$  we have closed loop ( $k > 0$ ). Dividing by  $LC$  we have

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{\lambda}{LC}} = H(s), \quad (10)$$

The eigenvalues or poles are given as

$$\alpha_{1,2} = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{\lambda}{LC}} \quad (11)$$

We simulate with  $L = 0.01$ ,  $C = 0.1$ , and  $R$  various values. If  $R = 0$  we end up with the time domain solution:

$$P_A(t) = \frac{1}{\sqrt{\lambda LC}} \sin\left(\sqrt{\frac{\lambda}{LC}}t\right) \quad (12)$$

So we have oscillations. What happens if  $R \neq 0$ ?

## Derivative and Integral Control

Consider the case where  $L = 0$ ,  $\lambda = 1$ ,  $\tau = RC$ . We get

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{\tau s + 1} = H(s), \quad (13)$$

For derivative feedback we have the relation  $P_{ao} - kP'_A$ . We have then

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{(\tau + k)s + 1} \quad (14)$$

The time domain solution for the steady state response is:

$$P_A(t) = 1 - \exp^{-\left(\frac{t}{\tau+k}\right)} \quad (15)$$

Notice that for derivative control there is no steady state error but the response is slower. A similar story holds for integral control.



## Root Locus Plots

The relationship

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s) \quad (16)$$

allows for the study of transient (or impulse response), steady-state response to step inputs or sinusoidal inputs. Further, it can be used to study stability and the dependence of the response to changes in the parameters  $LC$  and  $RC$ . We see that

- the impulse response  $g(t) = L^{-1}\left\{\frac{1}{LCs^2+RCs+(1+K)}\right\}$ ;
- the response to a step input or other input can be found by the convolution formula;
- the roots (called **poles**) of the expression  $LCs^2 + RCs + (1 + K)$  are actually the same as the eigenvalues of the characteristic equation described earlier. These roots determine the stability qualities of the solution.

A method for graphically representing the change in the poles as the parameters change is called the **root-locus method**. This representation draws the curves generated by the poles as plotted in the complex plane.

- The lung mechanics model has a second degree polynomial in the denominator, so there will be two roots for each choice of parameter values.
- The movement of these roots as  $K$  is varied from  $K = 0$  to  $K = \infty$  is represented in the figure below.
- This example illustrates that "closing the loop" allows the designer to manipulate the characteristics of the the poles which determine the stability of the system.
- By varying  $K$ , it is possible to move the poles to the left half plane where they will have negative real part. This is called **pole placement**.

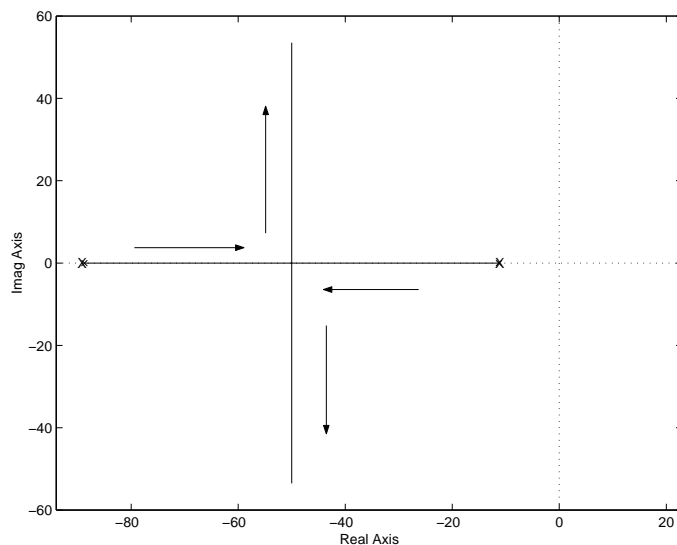


Figure 2: Closed loop root locus plot

## Frequency domain analysis

We mention that the expression in (17) for  $G(s)$  can be used to study the response to a sinusoidal input as well. One need only replace  $s = i\omega$  into (21) and one can derive the amplitude and phase shift in the output for a sine wave input of given frequency and unity amplitude. We have:

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s), \quad (17)$$

and with the substitution  $s = i\omega$  we derive

$$H_0(\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + (1 + K)} \quad (18)$$

$$= \|H_0(\omega)\|e^{j\phi(\omega)} \quad (19)$$

A motivation for this technique is as follows:

- The time domain relation is:  $P_{ao}(t) = LC\frac{d^2P_A(t)}{dt^2} + RC\frac{dP_A(t)}{dt} + P_A + KP_A$ .
- assuming a complex input  $P_{ao}(t) = X_0e^{j\omega t}$  this forces an output of the same form  $P_A(t) = Ze^{j\omega t}$ .
- substituting these forms of input and output in the equation above implies  $Z = H_0(\omega)X_0$
- Here  $H_0(\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + (1+K)}$
- Substituting for  $Z$  in the expression for  $P_A(t)$  we have that  $P_A(t) = \|H_0\|X_0e^{j(\omega)t + \phi(\omega)}$

In Figure (3) we see the relation between input (solid line) and output (dashed line) when the input frequencies are 1, 3, and 8 Hz (cycles/sec) for the lung model with proportional feedback. In this figure, the change in amplitude and phase are represented as functions of the input frequency response in radians. That is the gain is:

$$\|H_0(\omega)\| = \frac{1}{\sqrt{LC(j\omega)^2 + RCj\omega + (1 + K)}}, \quad (20)$$

and phase shift:

$$\phi(\omega) = -\tan^{-1}\left(\frac{RC\omega}{-LC(\omega)^2 + (1 + K)}\right). \quad (21)$$

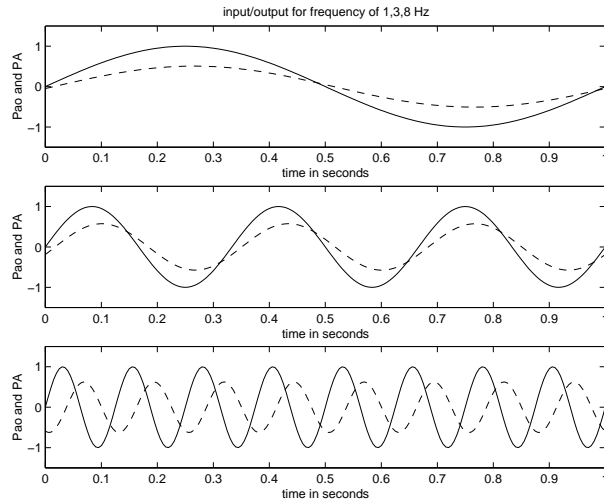


Figure 3: Closed loop frequency response

The bode plot which provides a summary of all frequency inputs is given as: The Bode plot gives informa-

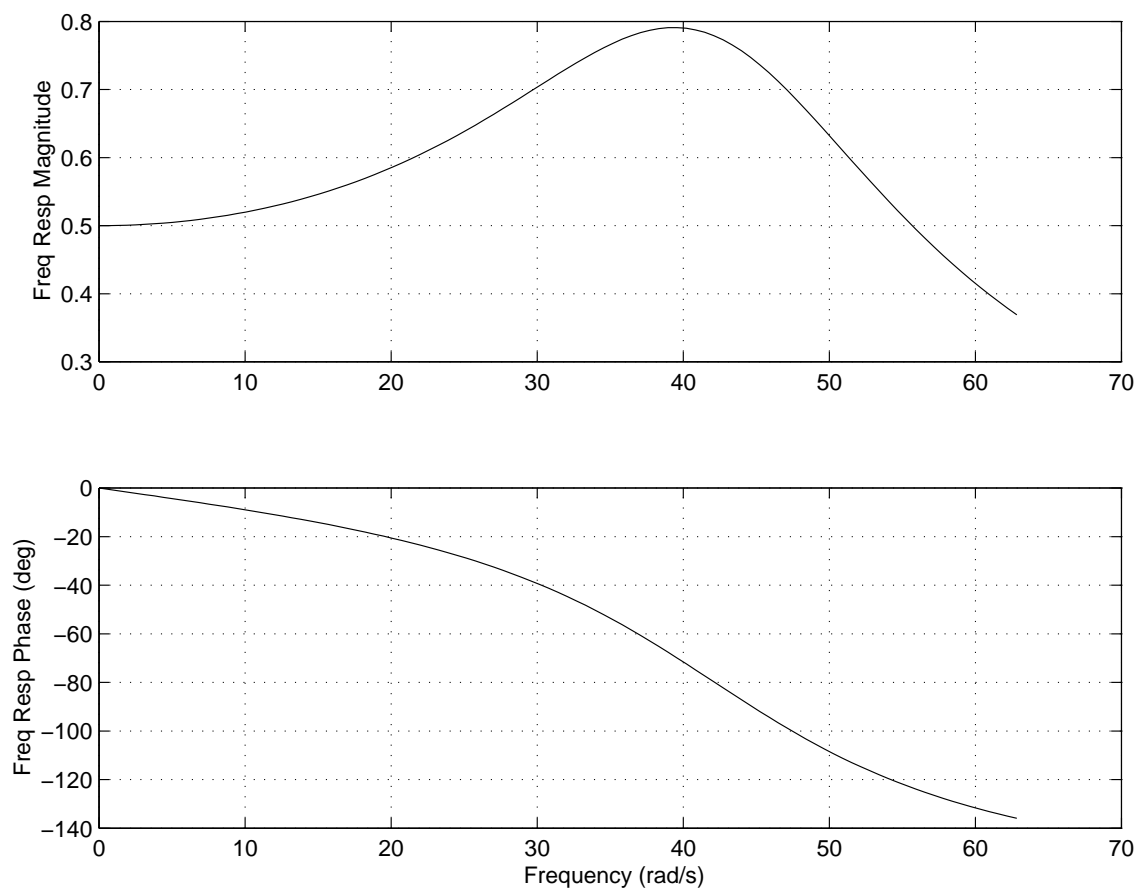


Figure 4: Bode plot frequency response

tion about the frequency response.