# Introduction
# to
# Frequency Analysis

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SFB F003 "Optimierung und Kontrolle"
Graz, Austria

June 26, 2002

## Outline

- Basic terminology and an example
- Closed loop systems
- An Example
- Transfer function representation
- N dimensional Linear Systems
- Stability analysis closed loop system: an example
- Root Locus Plots
- Bode Plots
- Frequency analysis closed-looped systems: Nyquist plots
- An example

## Useful References


- Khoo, M.C.K., *Physiological Control Systems*, IEEE Press,New York, 2000

- F. S. Grodins, *Control Theory and Biological Systems*, Columbia University Press, New York, 1963.

- J. Zabcyk *Mathematical Control Theory: an introduction*, Birkhauser Press, N.Y. 1992

# 1 Definitions

**Definition 1.** Often the mutual influence of states forces the system to a condition called the **steady state** in which all components states are constant.

**Definition 2.** If one or more of the states of a system at a steady state are abruptly changed or **perturbed** by outside influences the system may or may not return to its steady state.

- If the system does return to steady state the steady state is called **asymptotically stable**.
- If the system does not return to steady state but remains in some bounded region around the steady state the steady state is called **stable**.
- If the system wanders further and further from the steady state the steady state is termed **unstable**.

**Definition 3. A Control system** which acts to regulate a system around a given state is called a (**regulator mechanism**).

# 2 Closed Loop Systems

**Closed loop** systems as depicted in Figure (1) monitor the output of the system and can **feed back** this knowledge to the control system to alter the control to respond to perturbations. Usually the aim is to maintain a steady state or track another signal.

## 2.1 An example

**Example 1.** The chemical respiratory control system varies the ventilation rate in response to the levels of $CO_2$ and $O_2$ in the body. The control mechanism which responds to the changing needs of the body to acquire oxygen ($O_2$) and expel carbon-dioxide ($CO_2$) acts to maintain the levels of these gases within very narrow limits.

The control system consists of three components:

- sensors which gather information;
- effectors which are nerve/muscle groups which control ventilation; and
- the control processor located in the brain which organizes information and sends commands to the effectors.

In this steady state model example , there are two compartments and a control:

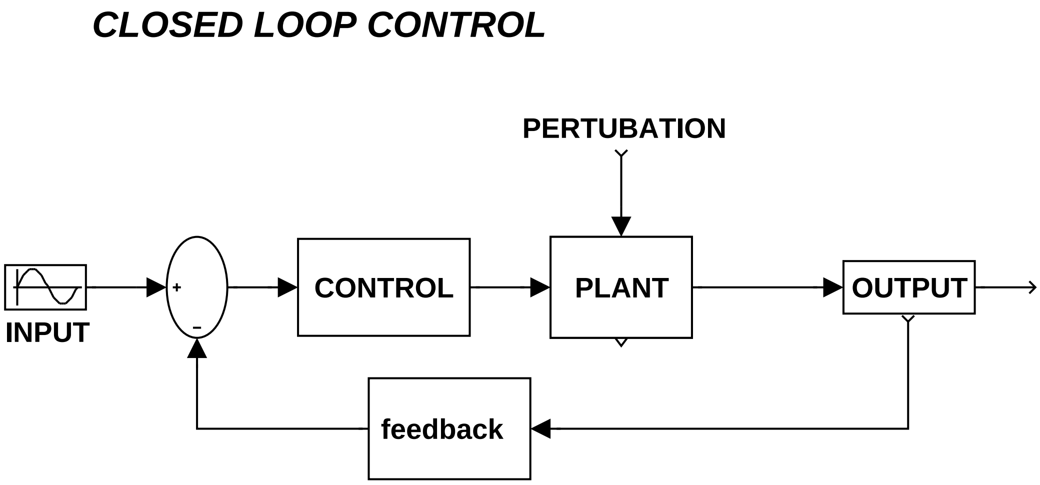


Figure 1: Closed loop control system

- a compartment which models the partial pressures of carbon dioxide ($\mathrm{Pa_{CO_2}}$) and partial pressure of oxygen and ($\mathrm{Pa_{O_2}}$) in the lungs;
- the control sensors monitor $CO_2$ and $O_2$ levels in the carotid arteries which translates this information into changes in ventilation. This control is modeled after Cunningham (1974).
- We assume that $\mathrm{Pa_{CO_2}} = \mathrm{P_{A_{CO_2}}}$.

## 2.2 model equations

We account for that part of the lungs which do not allow transfer of gases using:

$$\dot{\mathrm{V}}_{\mathrm{A}} = \dot{\mathrm{V}}_{\mathrm{I}} - \dot{\mathrm{V}}_{\mathrm{D}}$$

The mass balance equation for $CO_2$ entering and leaving the lungs is

$$\dot{\mathrm{V}}_{\mathrm{CO_2}} = k\dot{\mathrm{V}}_{\mathrm{A}}(\mathrm{F}_{\mathrm{A_{CO_2}}} - \mathrm{F}_{\mathrm{I_{CO_2}}}) \quad (1)$$

where $\dot{\mathrm{V}}_{\mathrm{CO_2}}$ is the metabolic production rate for $CO_2$. In steady state the net production of $CO_2$ must equal the net outflow from the lungs.

Metabolic rates are in $STPD$ units and ventilation volumes in $BTPS$ units. Thus we need a conversion factor:

$$\frac{V_{STPD}760}{273} = \frac{V_{BTPS}(\mathrm{P_B} - 47)}{310}$$

or

$$\begin{aligned} k &= \frac{V_{STPD}}{V_{BTPS}} \\ &= \frac{\mathrm{P_B} - 47}{863} \end{aligned}$$

Using Dalton's law for pressure volume relations:

$$\begin{aligned} \mathrm{P_{I_{CO_2}}} &= \mathrm{F_{I_{CO_2}}}(\mathrm{P_B} - 47) \\ \mathrm{P_{A_{CO_2}}} &= \mathrm{F_{A_{CO_2}}}(\mathrm{P_B} - 47) \end{aligned}$$

we substitute in (1):

$$\mathrm{P_{a_{CO_2}}} = \mathrm{P_{I_{CO_2}}} + \frac{863\dot{\mathrm{V}}_{\mathrm{CO_2}}}{\dot{\mathrm{V}}_{\mathrm{A}}} \tag{2}$$

A similar equation hold for $\mathrm{P_{a_{O_2}}}$

$$\mathrm{P_{a_{O_2}}} = \mathrm{P_{I_{O_2}}} + \frac{863\dot{\mathrm{V}}_{\mathrm{O_2}}}{\dot{\mathrm{V}}_{\mathrm{A}}} \tag{3}$$

The control equation is

$$\dot{\mathrm{V}}_{\mathrm{A}} = (1.46 + \frac{32}{\mathrm{P_{a_{O_2}}} - 38.6})(\mathrm{P_{a_{CO_2}}} - 37)$$

In the **simulink diagram** below the important organizational features of the system are diagrammed.

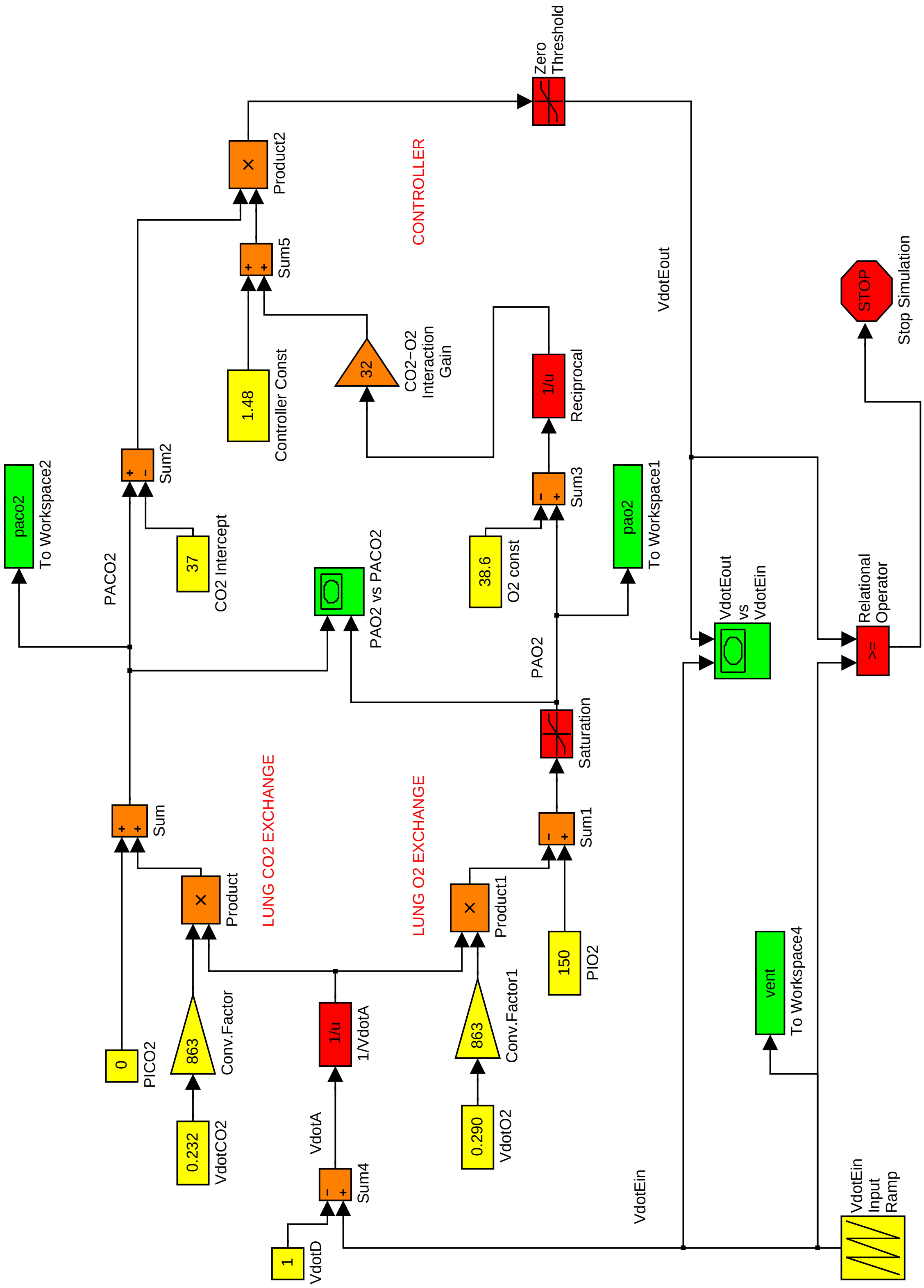


Figure 2: Comprehensive respiratory model

# 3 Solutions of a second order linear time invariant system

**Every solution** of a linear system can be written as a sum of the general solution to the homogeneous problem (where $u(t) = 0$) plus a particular solution to the non-homogeneous system (where $u(t) \neq 0$).

**Theorem 1.** *The solution of the system defined by*

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = u(t), \tag{4}$$

$$x(0) = c_0, \quad x'(0) = c_1, \quad u(t) \neq 0, \tag{5}$$

*can be written as the sum of the general solution $x_h(t)$ of the homogeneous system :*

$$a_2 \frac{d^2x_h(t)}{dt^2} + a_1 \frac{dx_h(t)}{dt} + a_0 x_h(t) = u(t), \tag{6}$$

$$x_h(0) = c_0, \quad x_h'(0) = c_1, \quad u(t) = 0, \tag{7}$$

*and the solution $x_p(t)$, of the non homogeneous problem:*

$$a_2 \frac{d^2x_p(t)}{dt^2} + a_1 \frac{dx_p(t)}{dt} + a_0 x_p(t) = u(t), \tag{8}$$

$$x_p(0) = 0, \quad x_p'(0) = 0, \quad u(t) \neq 0. \tag{9}$$

The solution $x_h(t)$ is called the **transient response** because it depends only on the uncontrolled behavior. $x_p(t)$ is referred to as the **forced response** because the control $u(t)$ changes the behavior.

# 4 Laplace Solutions of Linear Systems

Consider again the second order system:

$$a_2 \frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = u(t)$$
$$\text{where } x(0) = 0, \quad x'(0) = 0.$$

If we take Laplace transform on each side of this equation, using the rules

described in an earlier lecture we find that:

$$a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s) = U(s),$$

or

$$X(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} U(s).$$

The quantity $X(s)$ defines the Laplace transform of the state of the system or output. The quantity $U(s)$ is the Laplace transform of the input. The quantity

$$\frac{1}{a_2 s^2 + a_1 s + a_0}$$

relates the input to the output and is referred to as the **transfer function**, usually denoted by $H(s)$. Thus:

$$X(s) = H(s)U(s). \tag{10}$$

By taking the inverse Laplace transform of the expression $H(s)U(s)$ we find x(t).

Systems can be analyzed either in the "t" domain or in the transformed 's" domain and then retransformed into the "t" domain. Furthermore, letting $h(t)$ be the inverse Laplace transform of $H(s)$, and recalling the convolution formula (see notes) we have

$$x(t) = L^{-1}\{H(s)U(s)\} \tag{11}$$

$$= \int_0^t h(t-\tau)u(\tau)\, d\tau. \tag{12}$$

We see that once $h(t)$ (or the transfer function $H(s)$) is known the system response to any input $u(t)$ can be calculated. Note also from $X(s) = H(s)U(s)$ or Figure (3) that if we could choose an input whose Laplace transform $U(s) = 1$ then the output $X(s)$ of the system for this input $U(s)$ would actually represent $H(s)$ and thus the time domain response $x(t)$ would actually represent $h(t)$. In other words, we could observe the system defining function $h(t)$ by applying a special test signal which would reveal it.

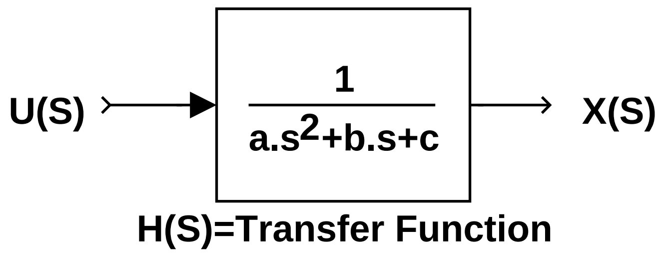


Figure 3: Open loop transfer function

# 5 Laplace Transform for n-dimensional systems

The Laplace transform method works for n-dimensional systems as well. Note that differentiation and integration are done component-wise and hence the basic process is the same as illustrated in the above examples. The main difference is in the formulation of the transfer matrix which involves finding the inverse of a matrix. Note that in terms of components

$$\mathbf{X}^*(s) = \begin{pmatrix} X_1(s) \\ ... \\ X_n(s) \end{pmatrix} = L\{\mathbf{x}(t)\} = \begin{pmatrix} L\{x_1(t)\} \\ ... \\ L\{x_n(t)\} \end{pmatrix}.$$

Recall that the general n dimensional linear system with constant coefficients takes the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Let $\mathbf{X}^*(s)$ be the Laplace transform of $\mathbf{x}(t)$ and $\mathbf{U}^*(s)$ be the Laplace transform of $\mathbf{u}(t)$. Then

$$s\mathbf{X}^*(s) - \mathbf{x}_0 = \mathbf{A}\mathbf{X}^*(s) + \mathbf{U}^*(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}^*(s) = \mathbf{x}_0 + \mathbf{U}^*(s).$$

Thus

$$\mathbf{X}^*(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{U}^*(s). \tag{13}$$

What is $(s\mathbf{I} - \mathbf{A})^{-1}$? Recall from earlier lectures that $\mathbf{X}(t) = e^{\mathbf{A}t}$ satisfies the matrix relation

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$$

and hence satisfies the matrix differential equation

$$\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t) \quad \mathbf{X}(0) = \mathbf{I}. \tag{14}$$

where $\mathbf{X}', \mathbf{A}, \mathbf{X}$, and $\mathbf{X}(0)$ are matrices. Letting $\mathbf{X}^{**}(s) = L\{e^{\mathbf{A}t}\}$, take the Laplace transform of the system (14) component by component. This gives after simpification with matrix rules

$$(s\mathbf{X}^{**}(s) - \mathbf{I}) = \mathbf{A}\mathbf{X}^{**}(s),$$

or

$$(s\mathbf{I} - \mathbf{A})^{-1} = \mathbf{X}^{**}(s) = L\{e^{\mathbf{A}t}\}.$$

When we take the inverse Laplace transform of (13) we get the same solution as with variation of constants described later using $(s\mathbf{I} - \mathbf{A})^{-1} = L\{e^{\mathbf{A}t}\}$.

**Example 2.** Solve the initial value problem

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Taking Laplace transform of both sides of the differential equation gives

$$s\mathbf{X}(s) - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \mathbf{X}(s) + \frac{1}{s-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Considering the system component-wise gives the following pair of equations to solve simultaneously:

$$\begin{aligned} (s-1)X_1(s) - 4X_2(s) &= 2 + \frac{1}{s-1}, \\ X_1(s) + (s-1)X_2(s) &= 1 + \frac{1}{s-1}. \end{aligned}$$

The solution of these equations is

$$X_1(s) = \frac{2}{s-3} + \frac{1}{s^2-1}, \quad X_2(s) = \frac{1}{s-3} + \frac{s}{(s-1)(s+1)(s-3)}.$$

Now,

$$\frac{2}{s-3} = L\{2e^{3t}\}, \text{ and } \frac{1}{s^2-1} = L\{\frac{e^t - e^{-t}}{2}\}.$$

Hence,

$$x_1(t) = 2e^{3t} + \frac{e^t - e^{-t}}{2}.$$

To invert $X_2(t)$ we use partial fractions and get

$$\frac{s}{(s-1)(s+1)(s-3)} = \frac{-1/4}{s-1} + \frac{-1/8}{s+1} + \frac{3/8}{s-3}$$

and thus

$$x_2(t) = -\frac{1}{8}e^{-t} - \frac{1}{4}e^t + \frac{11}{8}e^{3t}.$$

# 6 Systems analysis

Analytical tools for analyzing the steady state behavior of control systems includes then:

- Finding the steady states of the system where the system is fixed and does not change over time. This is done by setting all derivatives equal to zero and solving for the states which produce this condition.
- Finding the transfer function by applying test inputs or controls which approximate the impulse $\delta$ dirac function

- Studying the system response to various controls such as the **step response** or **frequency response**. In general there will be some alteration in the steady state of the system when controls are applied. This is referred to as steady state error.

# 7 Solutions of the linear system

The set of equations:

$$\begin{aligned} \mathbf{x}'(\mathbf{t}) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) & (15) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) & (16) \end{aligned}$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$, defines the solution implicitly. In this case (in general, the only case) we can find an explicit representation for the solution. To begin, consider the uncontrolled system:

$$\mathbf{x}'(\mathbf{t}) = \mathbf{A}\,\mathbf{x}(t). \quad (17)$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$, and $\mathbf{u}(t) = 0$. We know from earlier lectures that the solution is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0.$$

Consider next the case:

$$\mathbf{x}'(\mathbf{t}) = \mathbf{B}\,\mathbf{u}(t), \quad (18)$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$. A simple integration (for vectors, entry by entry) yields the solution

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{B}\mathbf{u}(\tau)d\tau.$$

Now consider the system (15) above and let $\mathbf{x}(t)$ be the solution of this system. Further let

$$\mathbf{z}(t) = e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t). \qquad \mathbf{z}(t_0) = \mathbf{x}(t_0). \quad (19)$$

Differentiating $\mathbf{z}(t)$ with respect to $t$ (using the product rule) gives:

$$\begin{aligned} \mathbf{z}'(\mathbf{t}) &= -\mathbf{A}e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t) \quad + e^{-\mathbf{A}(t-t_0)}\mathbf{x}'(\mathbf{t}) \\ &= -\mathbf{A}e^{-\mathbf{A}(t-t_0)}\mathbf{x}(t) + e^{-\mathbf{A}(t-t_0)}(\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)) \\ &= e^{-\mathbf{A}(t-t_0)}\mathbf{B}\mathbf{u}(t). \end{aligned}$$

Hence

$$\mathbf{z}(t) = \mathbf{z_0} + \int_{t_0}^{t} e^{-\mathbf{A}(\tau-t_0)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

So that relating $x$ and $z$ by Equation (19) above we have:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}(t-t_0)}\mathbf{z}(t)\\ &= e^{\mathbf{A}(t-t_0)}\mathbf{z}(t_0) + e^{\mathbf{A}(t-t_0)}\int_{t_0}^{t} e^{-\mathbf{A}(\tau-t_0)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ &= e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\end{aligned}$$

and the observation output variable $\mathbf{y}(t)$ is given by

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t).$$

# 8 System modes

The eigenvalues which are found via the characteristic equation determines a solution called a **fundamental mode** Suppose that $\mathbf{A}$ has the $n$ distinct eigenvalues $\lambda_1, ..., \lambda_n$ and corresponding eigenvectors $\mathbf{e_1}, ..., \mathbf{e_n}$. Then these vectors form a basis for $R^n$ and are linearly independent. Thus any initial vector $\mathbf{x}(0) = \sum_{i=1}^{n} \mu_i \, \mathbf{e_i}$, Let us furthermore define the $n \times n$ matrix $\mathbf{T}$ consisting of the eigenvectors of $\mathbf{A}$ as columns, $\mathbf{T} = (e_1, ..., e_n)$. Then

**Theorem 2.** *For the above defined vectors and matrices we have:*

*i)* $\mathbf{T}$ *diagonalizes* $\mathbf{A}$, *that is,*

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = diag(\lambda_1, ..., \lambda_n). \tag{20}$$

*ii) The transition matrix has the form*

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} = \mathbf{T}\, diag\,(e^{\lambda_1 t} ... e^{\lambda_n t})\, \mathbf{T}^{-1}. \tag{21}$$

*iii) The solution of (17) can be written in the form*

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0)\\ &= \sum_{i=1}^{n} \mu_i e^{\mathbf{A}t}\mathbf{e_i}\\ &= \sum_{i=1}^{n} \mu_i \, e^{\lambda_i t} \, \mathbf{e_i},\end{aligned}$$

*where the scalars* $\mu_i$ *depend on the initial condition* $\mathbf{x}(0)$ *via* $(\mu_1, ..., \mu_n)^T = \mathbf{T}^{-1}\mathbf{x}(0)$.

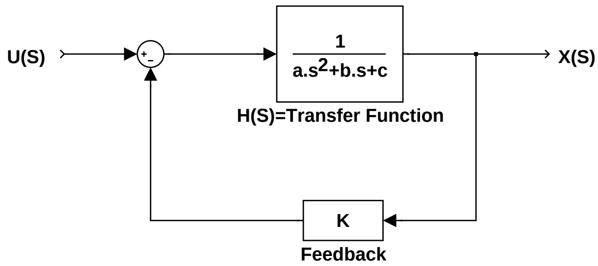


Figure 4: Closed loop transfer function

# 9 Closed loop transfer functions for LTI system

The ideas involved in analyzing a closed loop system such as represented in Figure (4) can be illustrated using the following example.

**Example 3.** In this example, which is presented in Khoo 2000, a simple linearized model is used to study the relation between the input of air pressure $P_{ao}$ at the airway opening and the pressure $P_A$ at the alveoli. Using simple mechanical laws of pressure drop, air flow resistance, and lung compliance (similar to electrical diagrams), the following second order differential equation is derived:

$$P_{ao}(t) = LC\frac{d^2P_A(t)}{dt^2} + RC\frac{dP_A(t)}{dt} + P_A. \tag{22}$$

Here $P_A(0) = 0$, ${P_A}'(0) = 0$, $u(t) = P_{ao}$, and $LC$, and $RC$ are constant. The transfer function of this system takes the form

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + 1} = H(s), \tag{23}$$

where $H(s)$ is the transfer function of the open loop system as represented in Equation (10).

Assume now that the air flow to the lungs needs to be regulated to avoid damage. We assume a mechanism that measures the alveolar pressure $P_A$ and produces a negative feedback to the input control $P_{ao}$, that is the new control $u(t) = P_{ao}(t) - KP_A(t)$. The transfer function for this system (after rearranging terms) takes the form

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s), \tag{24}$$

where $G(s)$ is the transfer function of the closed loop system.

The relationship

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s) \tag{25}$$

allows for the study of transient (or impulse response), steady-state response to step inputs or sinusoidal inputs. Further, it can be used to study stability and the dependence of the response to changes in the parameters $LC$ and $RC$. We see that

- the impulse response $g(t) = L^{-1}\{\frac{1}{LCs^2+RCs+(1+K)}\}$;
- the response to a step input or other input can be found by the convolution formula;
- the roots (called **poles**) of the expression $LCs^2 + RCs + (1 + K)$ are actually the same as the eigenvalues of the characteristic equation described earlier. These roots determine the stability qualities of the solution.

A method for graphically representing the change in the poles as the parameters change is called the **root-locus method**. This representation draws the curves generated by the poles as plotted in the complex plane.

- The lung mechanics model has a second degree polynomial in the denominator, so there will be two roots for each choice of parameter values.
- The movement of these roots as $K$ is varied from $K = 0$ to $K = \infty$ is represented in the figure below.
- This example illustrates that "closing the loop" allows the designer to manipulate the characteristics of the the poles which determine the stability of the system.
- By varying $K$, it is possible to move the poles to the left half plane where they will have negative real part. This is called **pole placement**.

# 10 Frequency domain analysis

We mention that the expression in (26) for $G(s)$ can be used to study the response to a sinusoidal input as well. One need only replace $s = i\omega$ into (30) and one can derive the amplitude and phase shift in the output for a sine wave input of given frequency and unity amplitude. We have:

$$\frac{P_A(s)}{P_{ao}(s)} = \frac{1}{LCs^2 + RCs + (1 + K)} = G(s), \tag{26}$$

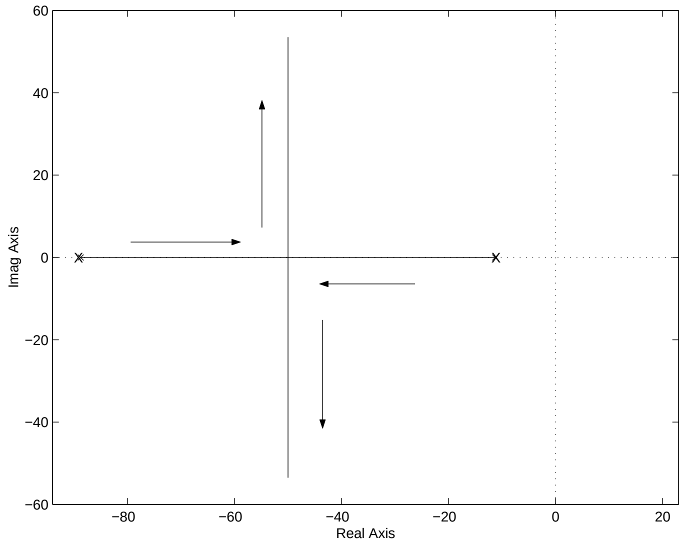


Figure 5: Closed loop root locus plot

and with the substitution $s = i\omega$ we derive

$$H_0(\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + (1+K)} \tag{27}$$

$$= \|H_0(\omega)\| e^{j\phi(\omega)} \tag{28}$$

A motivation for this technique is as follows:

- The time domain relation is: $P_{ao}(t) = LC\frac{d^2P_A(t)}{dt^2} + RC\frac{dP_A(t)}{dt} + P_A + KP_A$.
- assuming a complex input $P_{ao}(t) = X_0e^{j\omega t}$ this forces an output of the same form $P_A(t) = Ze^{j\omega t}$.
- substituting these forms of input and output in the equation above implies $Z = H_0(\omega)X_0$
- Here $H_0(\omega) = \frac{1}{LC(j\omega)^2+RCj\omega+(1+K)}$
- Substituting for $Z$ in the expression for $P_A(t)$ we have that $P_A(t) = \|H_0\|X_0e^{j(\omega)t+\phi(\omega))}$

In Figure (6) we see the relation between input (solid line) and output (dashed line) when the input frequencies are 1, 3, and 8 Hz (cycles/sec) for the lung model with proportional feedback. In this figure, the change in amplitude and phase are represented as functions of the input frequency response in radians. That is the gain is:

$$\|H_0(\omega)\| = \frac{1}{\sqrt{LC(j\omega)^2 + RCj\omega + (1+K)}}, \tag{29}$$

and phase shift:

$$\phi(\omega) = -tan^{-1}(\frac{RC\omega}{-LC(\omega)^2 + (1+K)}). \tag{30}$$

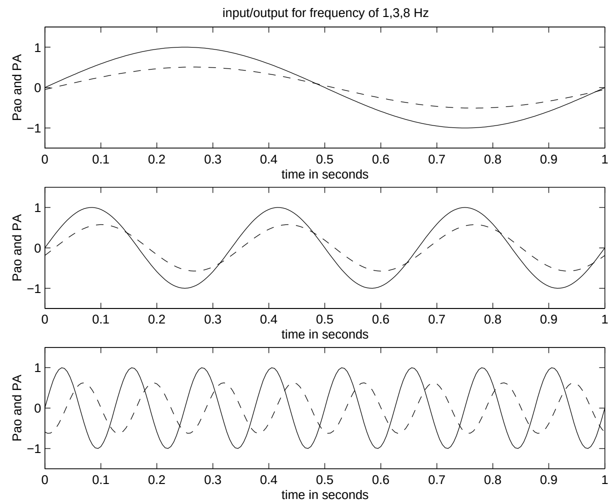


Figure 6: Closed loop frequency response

The bode plot which provides a summary of all frequency inputs is given as: The Bode plot gives information about the frequency response.

# 11 Stability analysis

For the general linear time invariant system with proportional feedback the equations for the transfer function follow directly as in Example 3. For a given input $u(t)$, output $x(t)$, feedback proportion $K$ and open loop transfer function $H(s)$ we have by direct application of the Laplace transform:

$$\frac{X(s)}{U(s)} = \frac{H(s)}{1 + KH(s)} = G(s)$$

where $G(s)$ is the transfer function of the closed loop system. Compare this formulation with the one given equation (30). As in every case, we find the poles by solving

$$1 + KH(s) = 0.$$

That is we find the roots of the expression in the denominator of the transfer function. The techniques given in Example 3 can be applied to these expression to study the steady state and dynamic dependence of the general system on the parameters.

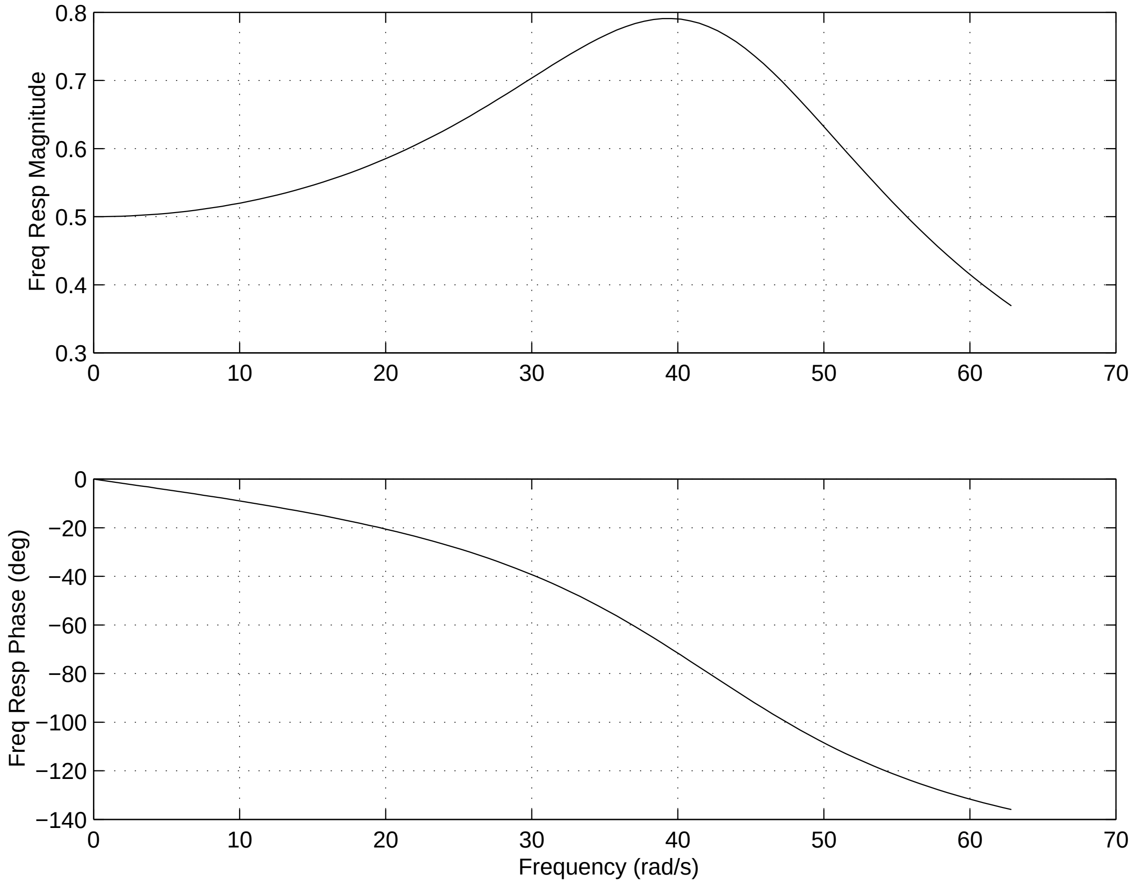


Figure 7: Bode plot frequency response

# 12 Nyquist criterion

## 12.1 How to derive Nyquist plot

### 12.1.1 By Bode plot

If you have the Bode plot, it's rather easy to derive the Nyquist plot. The Bode plot shows for each frequency $\omega$ the magnitude ratio and the phase shift of the input-output relation. Watch out, in the Bode plot you have a log-log plot for the magnitude!

To get the Nyquist plot, you plot for every frequency a dot in polar axis. The radius equals the magnitude ratio and the angle equals the phase shift. Thus you get for every frequency one point.

### 12.1.2 By Transfer function

The Transfer function is a function from $\mathbf{C} \to \mathbf{C}$. If you plug in $s = i\omega$ in the Transfer function, then you have a function from $\mathbf{R} \to \mathbf{C}$. The plot of this is the Nyquist plot.

## 12.2 The Nyquist Criterion

Let the open loop system be

$$Y(s) = G(s)X(s).$$

Assuming you know the number of poles and zeros of the open loop system in the right hand plane, you are able to determine the stability of a closed loop system with proportional feedback $k \cdot Y(s)$. You have to plot the Nyquist plot for $k \cdot G(s)$ to count the encirclements.

- N = number got by: Let each counterclockwise encirclement of the point P(-1/0) in the complex plane be counted as +1 and each clockwise encirclement as -1.
- P = number of poles in right half plane of the open loop system
- Z = number of poles in right half plane of the closed loop system

$$Z = P - N$$

The closed loop is stable, if there are no poles in the right half plane.

### 12.2.1 Where does this come from? Why at the point P(-1/0)?

As mentioned above we have $G(s)$ is the transfer function of the open loop:

$$Y(s) = G(s)X(s) \quad \text{or} \quad \frac{Y(s)}{X(s)} = G(s),$$

With proportional feedback we get

$$Y(s) = G(s)\,[X(s) + k \cdot Y(s)]$$

which yields

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + k \cdot G(s)} \equiv T(s)$$

We know poles and zeros of $G(s)$. How are they related to poles and zeros of $T(s)$?

Let $G(s)$ be given by

$$G(s) = \frac{b(s)}{a(s)}$$

then

$$1 + k \cdot G(s) = 1 + k \cdot \frac{b(s)}{a(s)} = \frac{a(s) + k \cdot b(s)}{a(s)}$$

thus

$$T(s) = \frac{\frac{b(s)}{a(s)}}{\frac{a(s)+k \cdot b(s)}{a(s)}} = \frac{b(s)}{a(s) + k \cdot b(s)}$$

Good. What can we tell from this?

- The number of poles of $1 + k \cdot G(s)$ in the right half plane is equal to the number of poles of $G(s)$ in the right half plane.
- The number of poles of $T(s)$ in the right half plane equals the number of zeros of $1 + k \cdot G(s)$ in the right half plane.

We know (explanation follows) that the number derived by counting the encirclements $N$ around the origin is related to the number of poles $P$ and zeros $Z$ in the right half plane - for a certain graph. $N = P - Z$

**1+kG(s)** If we consider the graph of $1 + k \cdot G(s)$, which is the graph of $k \cdot G(s)$ shifted by one (i.e. origin is P(-1/0)), we can derive the number of encirclements $N$ by counting (assuming P(-1/0) the origin).

The number of poles $P$ in the right half plane is the same as of $G(s)$ and thus is known.

Thus we can calculate $Z$, the number of zeros of $1 + k \cdot G(s)$, which is the number of poles of $T(s)$ in the right half plane.

Another possibility is to map the Nyquist plot of $G(s)$ and then count the encirclements of the point P(-$\frac{1}{k}$/0).

### 12.2.2 Where do the encirclements come from?

Given is the analytic function $g(s)$, with

$$g(s) = \frac{\prod_{i=1}^{\alpha}(s - z_i)^{m_i}}{\prod_{i=1}^{\beta}(s - p_i)^{n_i}},$$

where $z_i$ are the zeros and $p_i$ the poles. Now we define

$$h(s) \equiv \frac{g'(s)}{g(s)} = \frac{d}{ds}[\log\, g(s)],$$

Now we may take the logarithm of $g(s)$...

$$\log\, g(s) = \sum_{i=1}^{\alpha} m_i \log(s - z_i) - \sum_{i=1}^{\beta} n_i \log(s - p_i).$$

...and plug it in the above equation

$$h(s) = \frac{d}{ds}[\log\, g(s)] = \sum_{i=1}^{\alpha} \frac{m_i}{s - z_i} - \sum_{i=1}^{\beta} \frac{n_i}{s - p_i}.$$

It follows that the poles of $h(s)$ have to be the poles and the zeros of $g(s)$.

We can now evaluate the contour integral of $h(s)$ around any closed path that contains the poles and zeros of $g(s)$.

With the cauchy integral theorem

$$\frac{1}{2\pi i} \oint_\gamma \frac{m_i}{s - z_i} ds = m_i \quad \text{if } z_i \text{ inside } \gamma,\ \gamma \text{ clockwise}$$

we get

$$\frac{1}{2\pi i} \oint_\gamma h(s) ds = \sum_{i=1}^{\alpha} m_i - \sum_{i=1}^{\beta} n_i = Z - P,$$

where Z is the number of zeros of $g(s)$ and P the number of poles inside the contour.

As we want to know the number of poles on the right hand side, the contour must enclose all of them. Most common is to use a D-contour, which goes along the imaginary axes and makes a half circle in the right half plane. If the radius is large enough, we enclose all poles and zeros of the right half plane.

For more details please look for yourself. It's not easy to find good references...

On the other hand we can start all over and exploit the logarithm

$$\log g(s) = \log |g(s)| + i\angle(g(s)),$$

$\angle(g(s))$ is the phase angle of g(s). Plugging in yields to

$$\frac{1}{2\pi i}\oint_\gamma h(s)ds = \frac{1}{2\pi i}\oint_\gamma \frac{d}{ds}[\log g(s)]ds = \frac{1}{2\pi i}[\log g(s)]|_{s_1}^{s_2}$$

with expansion of logarithm

$$\frac{1}{2\pi i}\oint_\gamma h(s)ds = \frac{1}{2\pi i}\{\log |g(s)| + i\angle(g(s))\}|_{s_1}^{s_2}$$

Since it is a closed contour $\log |g(s_1)| = \log |g(s_2)|$. Thus

$$\frac{1}{2\pi i}\oint_\gamma h(s)ds = \frac{1}{2\pi}\{\angle(g(s_2)) - \angle(g(s_1))\} = N$$

N is the number of encirclements of the origin by $g(s)$.

(The last statement leads into problems, because the argument function doesn't yield to unique solutions. As part of a proper solution, we would have to split up the integral into parts, so that the angles are correct.)

Overall we get

$$\frac{1}{2\pi i}\oint_\gamma h(s)ds = N = Z - P$$

## 13 Nyquist criterion: an example

We discussed a steady state model for the respiratory control before. Now we look at a different model, which doesn't take into account the $O_2$, but distinguishes the two different controllers Gpp and Gcc. Before we had

$$P_{ACO_2} = P_{ICO_2} + \frac{863\dot{V}_{CO_2}}{\dot{V}_A}$$

now we have the dynamic equation

$$V_{lung}\frac{dP_{ACO_2}}{dt} = \left(\dot{V}_E - \dot{V}_D\right)(P_{ICO_2} - P_{ACO_2}) + 863Q\,(C_{vCO_2} - C_{aCO_2})$$

Suppose that small perturbations are imposed on $\dot{V}_E$ $(\Delta\dot{V}_E)$ which lead to changes in $P_{ACO_2}$ and $C_{aCO_2}$. (We ignore the effect of arterial blood gas fluctuations on mixed veneous $CO_2$ concentration and assume dead space ventilation remains constant.) We may neglect terms of $\Delta P_{ACO_2}\Delta\dot{V}_E$ and get thus

$$V_{lung}\frac{d(\Delta P_{ACO_2})}{dt} = -\left(\dot{V}_E - \dot{V}_D\right)\Delta P_{ACO_2} + (P_{ICO_2} - P_{ACO_2})\,\Delta\dot{V}_E - 863Q\Delta C_{aCO_2}$$

If we approximate the blood $CO_2$ dissociation curve with a straight line ($\Delta C_{aCO_2} = K_{CO_2}\Delta P_{ACO_2}$) and impose assumption of alveolar-arterial $P_{CO_2}$ equilibration ($P_{ACO_2} = P_{aCO_2}$) we obtain:

$$V_{lung}\frac{d(\Delta P_{aCO_2})}{dt} + \left(\dot{V}_E - \dot{V}_D + 863QK_{CO_2}\right)\Delta P_{ACO_2} \quad = \quad (P_{ICO_2} - P_{aCO_2})\,\Delta\dot{V}_E$$

Taking the Laplace transform and rearranging the terms of the transfer function: $H_{lung}(s)$ to

$$H_{lung}(s) \equiv \frac{\Delta P_{aCO_2}}{\Delta\dot{V}_E} = \frac{-G_{lung}}{\tau_{lung}s+1} \qquad \text{equals } H(t) \quad = \quad -\frac{G_{lung}}{\tau_{lung}}e^{-\frac{1}{\tau_{lung}}t}$$

We add some delay as the $CO_2$ moves to the central and peripheral controller.

$$\begin{aligned}\Delta P_{pCO_2}(t) &= \Delta P_{aCO_2}(t-T_p)\\ \Delta P_{cCO_2}(t) &= \Delta P_{aCO_2}(t-T_c)\end{aligned}$$

And the chemoreflex responses follow the equations

$$\begin{aligned}\tau_p\frac{d\dot{V}_p}{dt} + dotV_p &= G_p[P_{pCO_2} - I_p]\\ \tau_c\frac{d\dot{V}_c}{dt} + dotV_c &= G_c[P_{cCO_2} - I_c]\end{aligned}$$

with $\dot{V}_c + \dot{V}_p = \dot{V}_E$ Using some tricks we get to

$$\Delta\dot{V}_p(s) = \frac{G_p}{\tau_p s+1}\Delta P_{pCO_2} \tag{31}$$

$$\Delta\dot{V}_c(s) = \frac{G_c}{\tau_c s+1}\Delta P_{cCO_2} \tag{32}$$

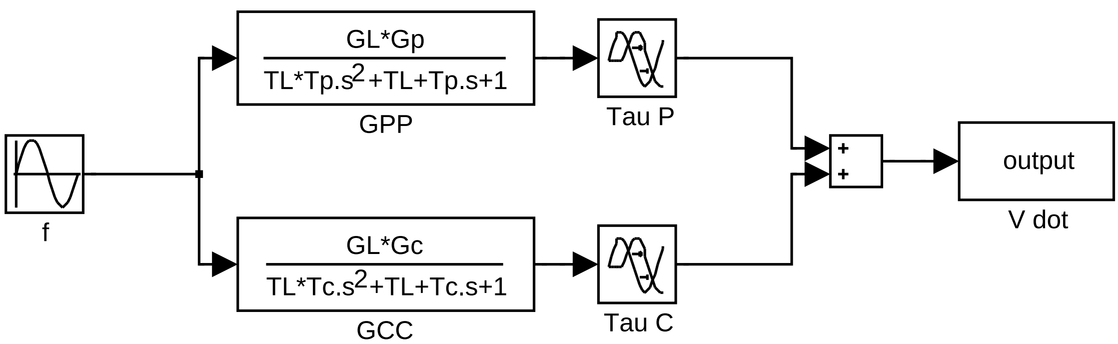


Figure 8: Normal-CHF respiratory model

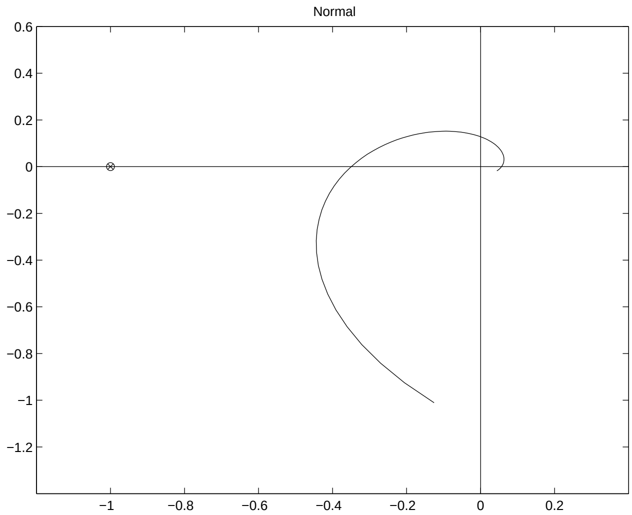


Figure 9: Nyquist plot normal respiratory model

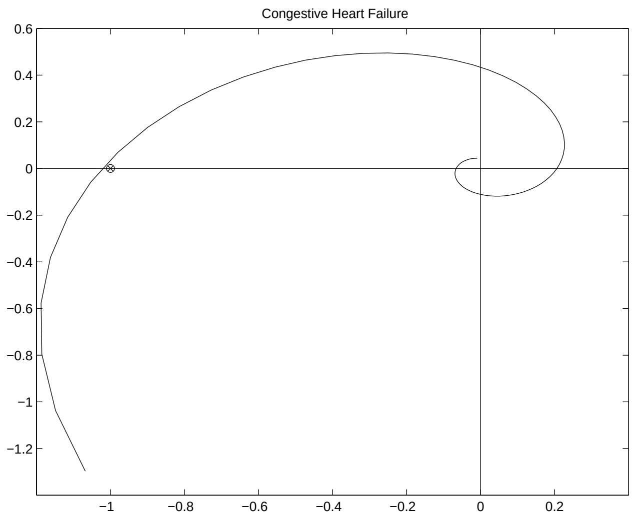


Figure 10: Nyquist plot CHF respiratory model