Introduction to Linear Control Theory

Lecture 3 Math notes

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Outline

- Basic terminology and examples
- Open-looped and closed loop systems
- Laplace transform method
- Transfer function representation
- Electrical Circuits and Physiological systems
- A lung mechanics model
- Simulink and Matlab

Useful References

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1 Introduction

Definition 1. A **System** is a complex organization of interacting factors, conditions or components, usually functioning to create some result. The components of the system often can exist in a wide range of **states**. These states influence each other producing a dynamic evolution of the system over time.

Definition 2. Often the mutual influence of states forces the system to a condition called the **steady state** in which all components states are constant.

Definition 3. If one or more of the states of a system at a steady state are abruptly changed or **perturbed** by outside influences the system may or may not return to its steady state.

- If the system does return to steady state the steady state is called **asymptotically stable**.
- If the system does not return to steady state but remains in some bounded region around the steady state the steady state is called **stable**.
- If the system wanders further and further from the steady state the steady state is termed **unstable**.

Definition 4. **Mathematical Control theory** is the study of the design of controls which can force a system to achieve certain goals in certain ways such as tracking a prescribed path (**servomechanism**) or regulating a system around a given state (**regulator mechanism**). The typical model for such systems consist of differential or difference equations.

Example 1. In the absence of voluntary control of breathing or neurologically induced changes in breathing, the respiratory control system varies the ventilation rate in response to the levels of CO_2 and O_2 in the body.

The control mechanism which responds to the changing needs of the body to acquire oxygen and expel CO_2 acts to maintain the levels of these gases within very narrow limits (and to a less understood degree match ventilation and blood flow). The control system consists of three components:

- sensors which gather information;
- effectors which are nerve/muscle groups which control ventilation; and
- the control processor located in the brain which organizes information and sends commands to the effectors.

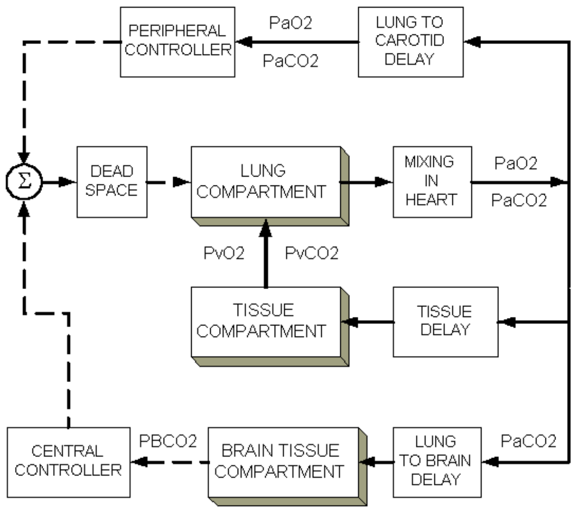

Figure 1: Block diagram of respiratory control system

2 Traditional approaches to control design

2.1 Open Loop, Closed Loop Systems and Block Diagrams

Traditional approaches to **Control Design** evolved from engineering fields, primarily electrical engineering. Indeed, early linear system theory borrowed heavily from electrical terminology and schematic diagrams as did early applications in physiology. Adaptations of models for computer simulations also used linear electrical circuit concepts and symbols such as resistance, capacitance, voltage, and current as well as analog devices to sum, differentiate, and integrate. There are clear and useful analogies between electrical systems and other dynamic systems as can be seen in the following Table 1. The primary method of analyzing **linear time-invariant** systems used Laplace transforms.

Table 1: Electrical-Physiological comparisons

Electrical	Physiological
Voltage	Fluid Pressure
Resistance	Vascular Resistance
Capacitance	Compliance
Inductance	Fluid Mass or Inertia

The traditional techniques involved **steady state analysis** and dynamic input/output analysis utilizing special input signals such as **impulse**, **step input** and **sinusoidal**.

Figures (2) and (3) show **Block Diagrams** which describe the relationships between input and output in the two main forms of Control system. Figure (2) describes **open loop control**. In this format, the system involves some physical structure (or **plant**) which is reflected in the **output** which can be either the state of the system or some measurement (or function) of the state of the system. The state of the system is altered by the application of an input which can be modified by a controller. The signal from the controller alters the state of the system and thus the output.

Closed loop systems as depicted in Figure (3) monitor the output of the system and can **feed back** this knowledge to the control system to alter the control to respond to perturbations. Usually the aim is to maintain a steady state or track another signal.

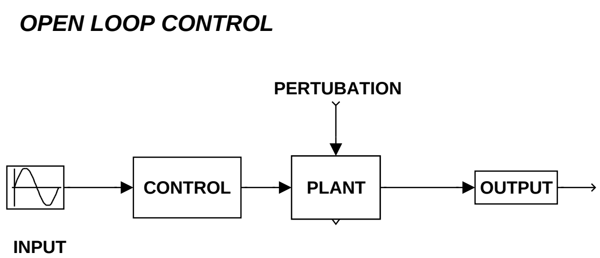

Figure 2: Open loop control system

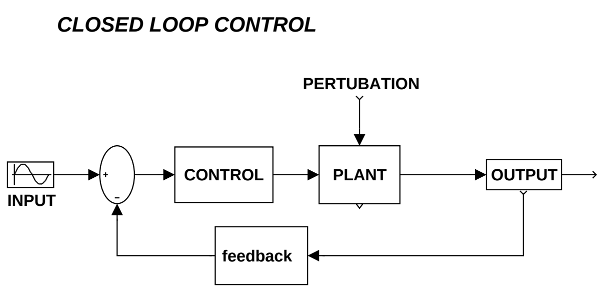

Figure 3: Closed loop control system

3 Laplace transform method

We want to give a short introduction to the methods and terminology of the technique for analyzing linear systems involving Laplace transforms.

This method is most successful when applied to linear systems with constant coefficients (linear-time-invariant or LTI), though it is sometimes useful for time dependent coefficients. The method can be used to analyze systems with single input and single output (**SISO**) as in Figure 2 but can be used to analyze multiple inputs multiple outputs (**MIMO**) by successive applications of the SISO method.

The system itself can be defined either by a set of linear differential equations in the **time domain** which are then transformed by the Laplace transform or by blocks directly defined by **transfer functions** representing electrical components in the "s" domain. Algebraic manipulations are employed to find the transformed unknown solution (with initial conditions incorporated).

The approach to analyzing a system employs the study of the system steady state response to various inputs including the **impulse input** and **step input** as well as the studying system response to trigonometric inputs (**frequency response**).

One important advantage of this approach to analyzing linear systems is that the linearity and the algebraic feature of the Laplace transform allow for the modularizing of the total system into subsystems and for the combining of subsystems into larger super systems in a convenient way.

3.1 Laplace transform

The following definitions and examples illustrate the Laplace transform method for solving linear systems. For a more thorough treatment see [7] and [1].

Definition 5. Let $f(t)$ be a given real-valued function defined on the Interval $0 \leq t < \infty$. The **Laplace transform** of $f(t)$, denoted by $L\{f\} = F$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt$$

We assume now, that s is a real number (in general it can be complex). If $F(s)$ exists at some s_0 it can be shown, that $F(s)$ also exists for all real numbers $s > s_0$.

Example 2. Let $f(t) = 1$ for all $t \geq 0$. Then for $s > 0$

$$F(s) = \int_0^\infty e^{-st} f(t) = \lim_{T \to \infty} \int_o^T e^{-st} dt = \lim_{T \to \infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \frac{1}{s}$$

Hence, $L\{1\} = 1/s$ exists for all $s > 0$.

Example 3. Let $f(t) = e^{at}$ for some real number a. Then for $s > a$

$$F(s) = \int_0^\infty e^{-st}e^{at} = \lim_{T\to\infty}\int_o^T e^{-(s-a)t}dt = \frac{1}{s-a}$$

Hence, $L\{e^{at}\} = (s-a)^{-1}$ exists for all $s > a$.

Example 4. Analogous you can get $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Definition 6. A **function** defined on $[0, \infty)$ is said to be **of exponential order** if there are real constants $M \geq 0$ and a such that $|f(t)| \leq Me^{at}$ for all $t \geq 0$.

Theorem 1. *If f is of exponential order and at least piecewise continuous over $[0, T]$, $T > 0$ then the Laplace transform $L\{f\} = F(s)$ exists for all $s > a$. Moreover $|F(s)| \leq M(s-a)^{-1}$ for all $s > a$.*

Example 5. The unit step function is defined by

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

We see clearly that $u_c(t)$ is of exponential order, thus we can use Laplace transform

$$\begin{aligned} L\{u_c\} &= \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^c e^{-st}\cdot 0\,dt + \int_c^\infty e^{-st}\cdot 1\,dt \\ &= \lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_c^T = \frac{e^{-sc}}{s} \qquad \text{for any } s > 0 \end{aligned}$$

Important calculation rules for the **Laplace operator** (with c_1, c_2 real, f_1, f_2 of exponential order):

1. **Linearity:**
$$L\{c_1f_1 + c_2f_2\} = c_1L\{f_1\} + c_2L\{f_2\}$$

2. **Convolution:**
$$L\{\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\} = L\{f_1(t)\} * L\{f_2(t)\}$$

3. **Integration:**
$$L\{\int_0^t f(\tau)\,d\tau\} = \frac{1}{s}L\{f\}$$

4. **Differentiation:** There exists a real b depending on f, that for $s > b$
$$\begin{aligned} L\{\frac{d^n}{dt^n}f(t)\} &= s^n F(s) - s^{n-1}f_0 - \ldots - s f_0^{(n-2)} - f_0^{(n-1)} \\ \text{with } f_0^{(v)} &= \lim_{t\to 0+} \frac{d^v f(t)}{dt^v} \end{aligned}$$

5. **Shifting:**
$$L\{f(t-b)\} = e^{-bs}L\{f(t)\}$$

6. **Similarity:**
$$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \text{ with } a > 0$$

7. **Damping:**
$$L\{e^{-\alpha t}f(t)\} = F(s+\alpha)$$

8. **Multiplicity:**
$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

9. **Division:**
$$L\{\frac{1}{t}f(t)\} = \int_s^\infty F(q)\,dq$$

To justify the inversion of the Laplace transform we have the following result:

Theorem 2. *If $x_1(t)$ and $x_2(t)$ are two functions of exponential order, both continuous and their Laplace transforms equal on an interval $s_0 < s < \infty$, then $x_1(t) = x_2(t)$ for all $t \geq 0$. This is not true for only piecewise-continuous functions! If they are piecewise-continuous, then $x_1(t) = x_2(t)$ on $0 \leq t < \infty$* except *on a set $\{t_n\}$ of isolated points.*

$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s}$	1	$\frac{1}{s+\alpha}$	$e^{-\alpha t}$
$\frac{1}{s^2}$	t	$\frac{1}{(s+\alpha)(s+\beta)}$	$\frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})$
$\frac{1}{s^2+\alpha^2}$	$\frac{1}{\alpha}\sin(\alpha t)$	$\frac{s}{s^2+\alpha^2}$	$\cos(\alpha t)$
$\frac{1}{(s+\beta)^2+\alpha^2}$	$\frac{1}{\alpha}e^{-\beta t}\sin(\alpha t)$	$\frac{s}{(s+\beta)^2+\alpha^2}$	$e^{-\beta t}\left(\cos(\alpha t) - \frac{\beta}{\alpha}\sin(\alpha t)\right)$
$\frac{1}{s^n}$	$\frac{1}{(n-1)!}t^{n-1}$	$\frac{1}{(s+\alpha)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-\alpha t}$

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\ln\frac{s-\alpha}{s-\beta}$	$\frac{1}{t}(e^{\beta t}-e^{\alpha t})$	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\sqrt{\frac{\sqrt{s^2+\alpha^2}+s}{s^2+\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$	$\sqrt{\frac{\sqrt{s^2-\alpha^2}+s}{s^2-\alpha^2}}$	$\sqrt{\frac{2}{\pi t}}\cos(\alpha t)$
$\arctan\frac{\alpha}{s}$	$\frac{\sin(\alpha t)}{t}$		

$[t]$ =the biggest natural number n with $n \leq t$.

$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s(e^{\alpha s}-1)}$	$\left[\frac{t}{\alpha}\right]$	$\frac{1}{s(1-e^{-\alpha s})}$	$\left[\frac{t}{\alpha}\right]+1$

$F(s)$	$f(t)$
$\frac{e^{-\alpha s}-e^{-\beta s}}{s}$	$\begin{cases} 0 & \text{für } 0<t<\alpha \\ 1 & \text{für } \alpha<t<\beta \\ 0 & \text{für } \beta<t \end{cases}$
$\frac{\left(e^{-\alpha s}-e^{-\beta s}\right)^2}{s^2}$	$\begin{cases} 0 & \text{für } 0<t<2\alpha \\ t-2\alpha & \text{für } 2\alpha<t<\alpha+\beta \\ 2\beta-t & \text{für } \alpha+\beta<t<2\beta \\ 0 & \text{für } 2\beta<t \end{cases}$
$\frac{e^{-\alpha s}}{s+\beta}$	$\begin{cases} 0 & \text{für } 0<t<\alpha \\ e^{-\beta(t-\alpha)} & \text{für } \alpha<t \end{cases}$
$\frac{1}{s(1+e^{-\alpha s})}$	$\begin{cases} 1 & \text{für } 2n\alpha<t<(2n+1)\alpha \\ 0 & \text{für } (2n+1)\alpha<t<(2n+2)\alpha \end{cases}$ $n=0,1,2,\ldots$

3.2 Transfer function

We let $x(t)$ denote the state of the system which can be thought of also as the **output** of the system, while $u(t)$ denotes the **input** either as a direct input to the system or a control signal. $u(t)$ is a function of t and affects the equations of the state. In this case $x(t)$ and $u(t)$ are functions but the discussion carries over for vectors as well with division replaced by matrix inversion. Let

$$a\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+cx(t)=u(t),$$
$$x(0)=0, \quad x'(0)=0.$$

If we take Laplace transform on each side of this equation, using the rules described in the first lecture we find that:

$$as^2X(s)+bsX(s)+cX(s)=U(s),$$

or

$$X(s) = \frac{1}{as^2 + bs + c} U(s).$$

The quantity $X(s)$ defines the Laplace transform of the state of the system or output. The quantity $U(s)$ is the Laplace transform of the input. The quantity

$$\frac{1}{as^2 + bs + c}$$

relates the input to the output and is referred to as the **transfer function**, usually denoted by $H(s)$. Thus:

$$X(s) = H(s)U(s). \tag{1}$$

By taking the inverse Laplace transform of the expression $H(s)U(s)$ we find x(t). Systems can be analyzed either in the "t" domain or in the transformed 's" domain and then retransformed into the "t" domain. Furthermore, letting $h(t)$ be the inverse Laplace transform of $H(s)$, and recalling the convolution formula

$$L\{[f_1 * f_2](t)\} = L\{\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\} = L\{f_1(t)\} L\{f_2(t)\}$$

we have

$$x(t) = L^{-1}\{H(s)U(s)\} \tag{2}$$
$$= \int_0^t h(t-\tau) u(\tau)\, d\tau. \tag{3}$$

We see that once $h(t)$ (or the transfer function $H(s)$) is known the system response to any input $u(t)$ can be calculated. Note also from (1) and Figure (4) that if we could choose an input whose Laplace transform $U(s) = 1$ then the output $X(s)$ of the system for this input $U(s)$ would actually represent $H(s)$ and thus the time domain response $x(t)$ would actually represent $h(t)$. In other words, we could observe the system defining function $h(t)$ by applying a special test signal which would reveal it.

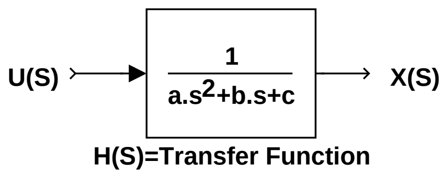

Figure 4: Open loop transfer function

To see how feedback works in this setting we let $u(t) = r(t) - Kx(t)$, $x(0) = 0$, $x'(0) = 0$ and $0 < K < 1$. In this case, the control $r(t)$ is influenced by the state $x(t)$ and this influence is to reduce the control $r(t)$ when $x(t)$ increases thereby acting to feed back the growth of $x(t)$. The fed back information is a multiple of x and is called **proportional control**. Other feed back controls are possible which involve information about $'x$ (derivative control) or the integral of x (integral control).

Applying Laplace transforms gives

$$H(s) = \frac{1}{as^2 + bs + c + K}.$$

3.3 Impulse response

Once the transfer function $H(s)$ is known, $h(t)$ can be found and the output for any input can be derived by applying the convolution formula (2) and integrating the control $u(t-\tau)$ with $h(t)$. In this set up, note that **if** there were a function whose Laplace transform was X(s) = 1 then the application of this input would produce a response which actually described the underlying transfer function which could be used to analyze systems whose internal structure was unknown. See Figure (4).

Strangely, there is no function which has Laplace transform $X(s) = 1$. The following family of functions in Equation (4) are concentrated around $x = 0$ and have Laplace transform as close to 1 as we want and can be thought of as approximating the ideal of a function whose Laplace transform **is** one. Let

$$f_0(t) = \frac{u_0(t) - u_a(t)}{a} \tag{4}$$

where $u_b(t)$ indicates a step function which jumps from 0 to 1 at $x = b$. That is:

$$u_b(t) = \begin{cases} 0 & \text{if } t < b \\ 1 & \text{if } t \geq b. \end{cases}$$

The idealized function is referred to as the **Dirac delta function** and is denote by $\delta(x)$. Note that these approximating functions become more and more concentrated around $x = 0$ but always with area under the curve equal to one. Hence, the applied input "impulse" gets more and more concentrated but with equal total "force". Figure (5) illustrates some of these functions with the impulse beginning at $x = 0$. The application of very short impulses, then, will give an approximate picture of the inverse Laplace of the transfer

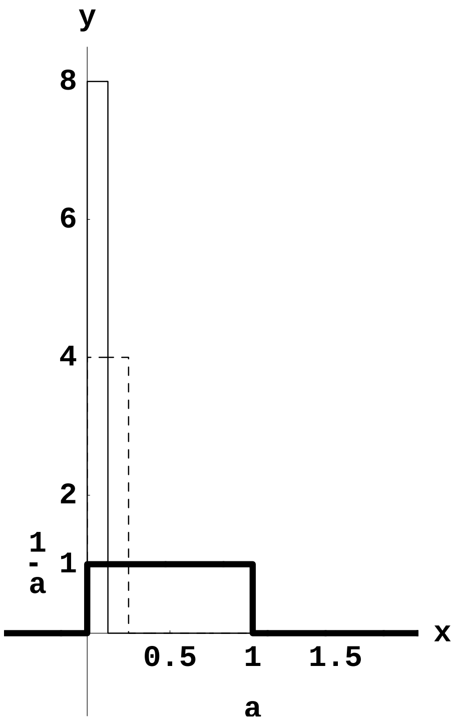

Figure 5: Impulse function approximations

function $H(s)$ of the system. Note also, that the step response $u(t) = 1$ is found by

$$\int_0^t h(w)dw.$$

3.4 Model analysis

To study the nature and behavior of a given system there are a number of analytical and simulation techniques which include:

- Finding the steady states of the system where the system is fixed and does not change over time. This is done by setting all derivatives equal to zero and solving for the states which produce this condition;
- Finding the transfer function by applying test inputs or controls which approximate the impulse δ dirac function;
- Studying the system response to various controls such as the **step response** and **impulse response**. This is referred to as **transient response analysis**;

- Studying the system response to sinusoidal inputs which is referred to as **Frequency domain analysis**;
- Studying the stability of the system.

We will also see that in general there will be some alteration in the steady state of the system when a closed loop proportional feedback control is applied. This is referred to as steady state error. We will develop several models and develop techniques using Simulink to carry out these investigations.

4 Electrical Circuits and Physiological systems

To develop the systems to analyze we introduce some design terminology which links electrical and physiological systems. As we mentioned before, there are clear and useful analogies between electrical systems and other dynamic systems such as we see in Table 2. The following symbols given in

Table 2: Electrical-Physiological comparisons

Electrical	Physiological
Voltage	Fluid Pressure
Resistance	Vascular Resistance
Capacitance	Compliance
Inductance	Fluid Mass or Inertia

Figures 6, 7, 8, and 9 are used for the electrical quantities above.

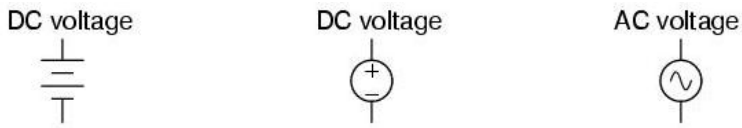

Figure 6: Electrical symbols for voltage or pressure

We also will apply Kirchhoff's Laws to set up flow relations.

Definition 7. Kirchhoff's First Law states that the algebraic sum of the across-variable values (voltages) around any **closed** loop must be zero.

Definition 8. Kirchhoff's Second Law states that the algebraic sum of all through-variable values (currents) into any given node must be zero.

Figure 7: Electrical symbols for inductance or inertia

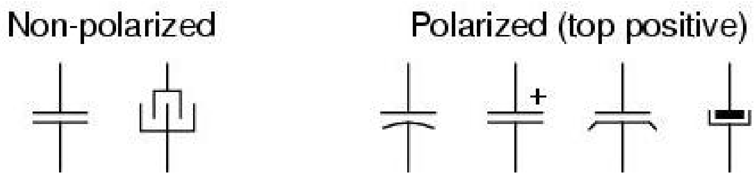

Figure 8: Electrical symbols for capacitance or compliance

We also need to note the following rules for combining resistances and capacitances which follow from Kirchhoff's Laws.

- Given R_1 and R_2 are in series then the total resistance R is given by

$$R = R_1 + R_2 \tag{5}$$

- If R_1 and R_2 are in parallel we have

$$R = (\frac{1}{R_1} + \frac{1}{R_2})^{-1}. \tag{6}$$

- If capacitances are in series we have

$$C = (\frac{1}{C_1} + \frac{1}{C_2})^{-1}. \tag{7}$$

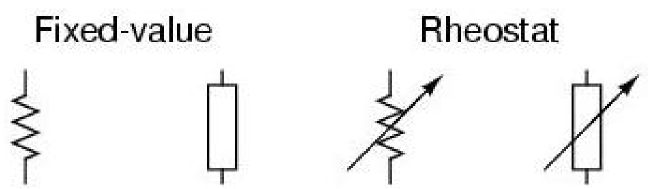

Figure 9: Electrical symbols for resistance

- If capacitances are in parallel we have

$$C = C_1 + C_2 \tag{8}$$

Finally, we note some important electrical relations:

$$C = \frac{\Delta V}{\Delta P} \tag{9}$$

$$R = \frac{\Delta P}{Q} \tag{10}$$

$$V = \int Q dt \tag{11}$$

5 Lung mechanics model

We now establish the first model which will utilize the ideas described above. This will be a model of air flow through the mechanical structure of the lungs. The simplest form of this model will be an open loop control model which can be used to study the effects of various respiratory parameters on lung function. Medical applications exist such as methods for artificial respiration. The set up for this model as it is given represents an artificial ventilator generating forces at the airway opening producing positive pressures relative to ambient pressure on inspiration (otherwise there are some changes needed in the representation).

5.1 Lung mechanics physical model

The lung mechanics model considers the relation between air flow volumes Q and various pressures in the lungs. The model considers air flow resistance R, compliances C of air flow compartments, and pressures P. An electrical analogy schematic diagram is given in Figure 10 and Table 3 gives the meaning of the parameter symbols.

As the air flows into the air passages, it encounters different resistances in the central and peripheral airways. Furthermore, spatial volumes are effected by the compliant nature of these structures. The model also includes the effect of shunting of a portion of the air away from the alveoli compartment as a result of disstention of the conducting airways and gas compression.

The relation among quantities in the mechanical air flow model are represented in Figure 10 which utilizes the analogous electrical symbols to represent Q, R, C, and P.

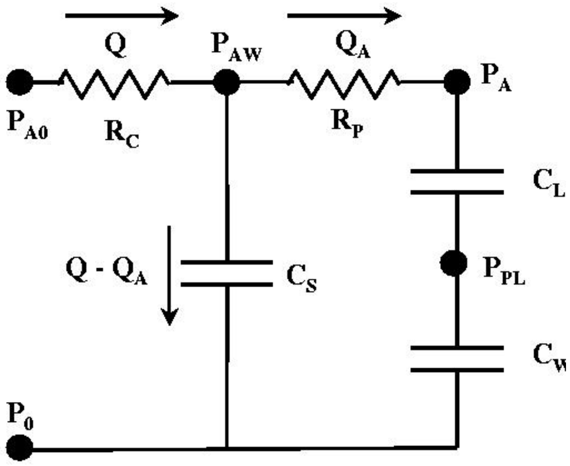

Figure 10: lung mechanics schematic design

Table 3: Physiological parameters

Parameter	Symbol
C_W	chest wall compliance
C_L	lung compliance
C_S	shunt compliance
R_P	peripheral airway resistance
R_C	central airway resistance
P_{ao}	pressure at airway opening
P_{pl}	pressure in pleural space
P_{aw}	pressure in central airway
P_A	pressure in alveoli
P_0	ambient pressure
Q_A	airflow in alveoli
Q	total airflow

Beginning with the airflow P_{ao} at the mouth (the input will be modeled as a sinusoidal flow) we follow the path of the airflow through the system. Note that the ambient pressure P_0 is set to zero. Compliances C multiplied by pressure changes give volume change of stored air (see Eq. 9).

1. As the air flow travels first through the central conducting tubes and then the periheral conducting tubes, pressure drops occur resulting in pressures P_{aw} leaving the central airway and finally P_A in the alveoli.

2. Different resistances for the central and peripheral airway are utilized.

3. The flow Q_A represents the air flow rate through the peripheral airways to the alveoli. Part of the total airflow $Q - Q_A$ does not reach the alveoli but is diverted by compliance C_S representing lost volume due to expanding of airways and the phenomenon of gas compression.

4. The compliance C_L and C_W together form the effective air storage space available for alveoli air. Being in series the total effective compliance is smaller that either individually (see Eq. 7). The compliance C_L represents the expansion of the lung space (alveoli) and the quantity C_W represents chest wall expansion which increase by a like volume. The reason for the series arrangement requires some explanation. C_W represents the expansion of the chest wall due to the pressure P_{pl} in the pleural space between lung and chest wall. When the breathing is by ventilator (so that we have positive pressures at each point), the expansion of the lung space must also push out the chest wall. Thus there must be a positive pressure expanding the lungs which then further pushes on the interpleural space which then pushes the chest wall. The compliance of the pleural space is in series because the combined reaction to the overall pressure change is reduced somewhat like a series of springs which don't completely transfer the force. This acts to reduce the overall compliance as can be seen from Eq. 9.

5. As the sinusoidal flow of air at the mouth is impressed on the system we have an alternating or oscillatory flow pattern in the system.

6. The model can be used to study the effect of a variety of parameter combinations on the functioning of the system.

5.2 Lung mechanics mathematical model

Note first that given a 2nd order system where $y(t)$ is the output and $x(t)$ is the input:

$$a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + y(t) = b_2\frac{d^2x(t)}{dt^2} + b_1\frac{dx(t)}{dt} + b_0x(t) \tag{12}$$

we have as transfer function:

$$\frac{Y(s)}{X(s)} = \frac{b_2s^2 + b_1s + b_0}{a_2s^2 + a_1s + 1}.$$

the form of which we will develop for the following lung model.

$$\frac{d^2P_{ao}(t)}{dt^2} + \frac{1}{R_PC_T}\frac{dP_{ao}(t)}{dt} = R_C\frac{d^2Q(t)}{dt^2} + (\frac{1}{C_S} + \frac{R_C}{R_PC_T})\frac{dQ(t)}{dt} + \frac{1}{R_PC_S}(\frac{1}{C_L} + \frac{1}{C_W})Q. \quad (13)$$

Here,

$$C_T = (\frac{1}{C_L} + \frac{1}{C_W})\frac{1}{C_S})^{-1}.$$

The derivation of the lung model equation is made via Kirchhoff's Laws applied to the electrical circuit representation given in Figure 10. This figure was developed to represent the components involved in the mechanical flow of air to the lungs via the analogous electrical-physiologucal relations given in Table 1 or Table 2.

To translate the electrical circuit model Fig. 10 to a mathematical format we begin with the node P_{aw} and apply Kirchhoff's Second Law to conclude that if the airflow to the alveoli is given by Q_A then the shunted air flow is $Q - Q_A$. Furthermore, when we apply Kirchhoff's Second Law to the circuit containing C_S, R_P, C_L, and C_W we find that

$$R_PQ_Q + (\frac{1}{C_L} + \frac{1}{C_W})\int Q_Adt = \frac{1}{C_S}\int(Q - Q_A)dt. \quad (14)$$

Now we apply Kirchhoff's First law to the circuit containing R_C and C_S to obtain

$$P_{ao} = R_CQ + \frac{1}{C_S}\int(Q - Q_A)dt. \quad (15)$$

Differentiating Eq. 14 and Eq. 15 we arrive after substitution and simplification at Eq. 13.

At this stage we could set up the transfer function using Laplace transforms and simplifying as exemplified in Eq. 12. We get:

$$\frac{Q(s)}{P_{ao}(s)} = \frac{s^2 + 420s}{s^2 + 620s + 4000}.$$

Here $C_L = 0.2$ $LcmH_2O^{-1}$, $C_W = 0.2$ $LcmH_2O^{-1}$, $C_S = 0.005$ $LcmH_2O^{-1}$, $R_C = 1.0$ cmH_2OsL^{-1}, and $R_P = 0.5$ cmH_2OsL^{-1}. Also we have for initial conditions $P_{ao}(0) = 0$ and $Q(0) = 0$. The following block diagram (Fig. 11) illustrates this transfer relation with some additional input and output symbols which will be discussed in the next section. Note that the above formulation allows us to test a variety of inputs to the system **but not** to vary the parameters without rederiving the transfer function. We will consider a Simulink model for the above system which reflects more directly the components of the system given in Fig. (). Solving that system numerically will allow us to vary the parameters values to study various situations.

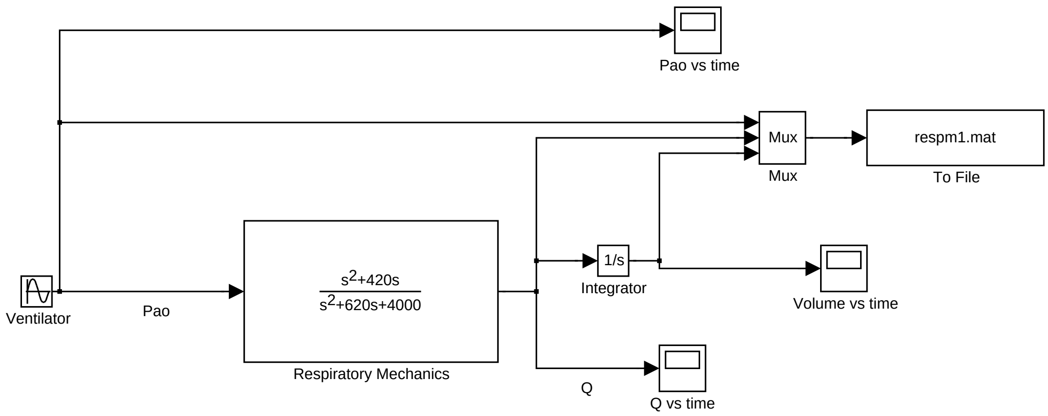

Figure 11: lung mechanics transfer function

6 Simulink and Matlab

Matlab is a numerical package well suited for solving linear systems, differential equations, and engineering design problems. It has a programming language which is easier to debug thus allowing a wide range of programs and numerical schemes to be developed. Simulink is a specialized package with graphical user interface for design and study of engineering control systems especially for modularly designed hierarchical systems.. The basic option list which will be found when Simulink is called up from Matlab is given in figure **??**. Sources include all kinds of inputs such as signal generators, step, pulse, and ramp functions. Sinks are really nothing more than output devices. Other categories include continuous operations including integrator, transfer functions etc. These and other categories will be explored throughout the course.

Some elementary symbols for Simulink program diagrams are given in figure 13. These and other basic structures are coupled together in "calculation flow charts" which are patterned after electrical circuit and other system schematic diagrams or block diagrams. The Simulink diagrams for the lung mechanic model are given in Figures 11 and 14. We will now discuss how

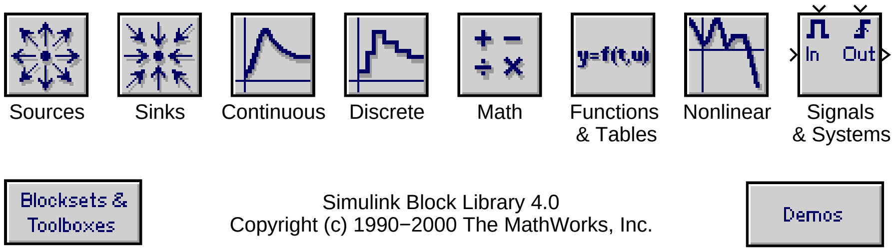

Figure 12: Basic Simulink categories

these schematics are developed. Figure 11 represents transfer function relationships and also includes some output symbols. This model is constructed with the following steps:

1. A new Simulink design window is called up by clicking and dragging on the

2. From the **continuous** window a transfer function symbol is dragged into the design window.

3. Having done the mathematics to derive the Laplace transform one needs only specify the coefficients of the transfer function to set it up

4. For the first simulations we choose a sinusoidal input from the **source** window which we drag into the design window and connect from arrow to receptor.

5. The outputs for P_{ao}, Q, and V are represented by the **scope** figures.

6. With this set up we only need click on simulate to start the action. Time span, numerical step size, and other simulation options can be set and will be discussed in the lab.

The model represented in Fig. 11 while easy to construct requires the solution of the Laplace transformation and does not allow new variations in parameters. The design in Figure 14 follows more closely the modular approach set forth in the lung mechanics schematic diagram and hence allows for a wide range of variation in parameter settings to be studied. The development of this model will now be discussed.

To translate an electrical or physiological system to a Simulink flow diagram one needs to keep in mind the following points.

- In a certain sense, Simulink diagrams are "calculation flow charts" which describe the interrelation between quantities in the system.

- The lung mechanics model in Figure 14 shows computational relations between pressures P, resistances R, capacitances C, and flows Q.

- All computational relations at **nodes** must be defined via the parts of the model.

For example, let's begin to trace out the calculation starting with P_{ao}.

1. The air flow Q through resistance R_C depends on pressure difference $P_{ao} - P_{aw}$ which is calculated using the **sum** box with inputs P_{ao} and $-P_{aw}$. The triangle $\frac{1}{R_C}$ says to multipy the constant $\frac{1}{R_C}$ with $P_{ao} - P_{aw}$ which thus calculates Q as in Eq. 10. At the next branch point we have another **sum** box with inputs Q and the shunted flow $Q_S = Q - Q_A$ which is subtracted leaving Q_A.

2. Q_A is multiplied by R_P calculating the pressure $P_{aw} - P_A$ using Eq. 10. The second branch using Q_A is used to calculate the pressures at the compliances. First integrating Q which is air flow rate we get the air volume V_A moving through the alveoli and then using Eq. 9 we multiply V_A with $\frac{1}{C_L}$ generating $P_A - P_{pl}$ and with $\frac{1}{C_W}$ generating $P_{pl} - P_0$.

3. Adding the three pressures in the last item gives $P_{aw} - P_0$ which is actually P_{aw} since we set $P_0 = 0$. We link this output back to the original sum box as input P_{aw} which then calculates the difference $P_{ao} - P_{aw}$ completing the calculation loop.

4. To complete the calculation we need the input Q_S which we find by multiplying P_{aw} with C_S as denoted by the triangle. This generates a volume V_S which when differentiated gives the required Q_S.

5. Note that subtlties are introduced into the discussion when we point out that the Eqs. 9, 10, and 11 bounce back and forth between **changes** in volumes and pressures and the absolute quantities.

In this way we set up the calculation dependencies of the various loops instead of generating the transfer function. We can thus numerically solve this system with various parameter values.

Finally, outputs from this model are given in Figures **??**, 16, and 17.

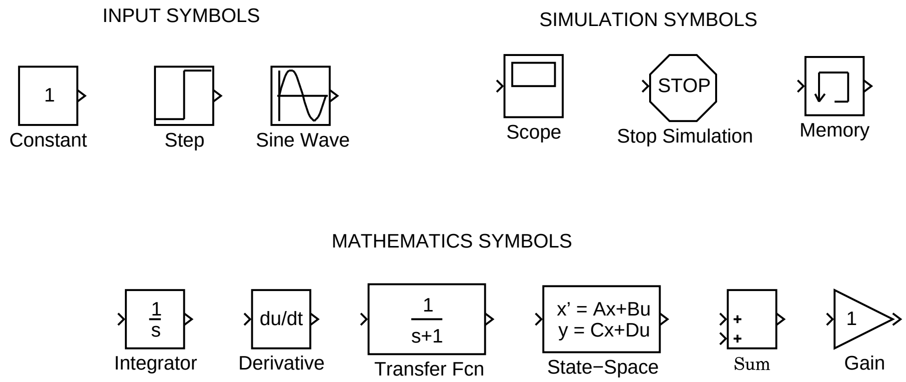

Figure 13: Basic Simulink mathematics symbols

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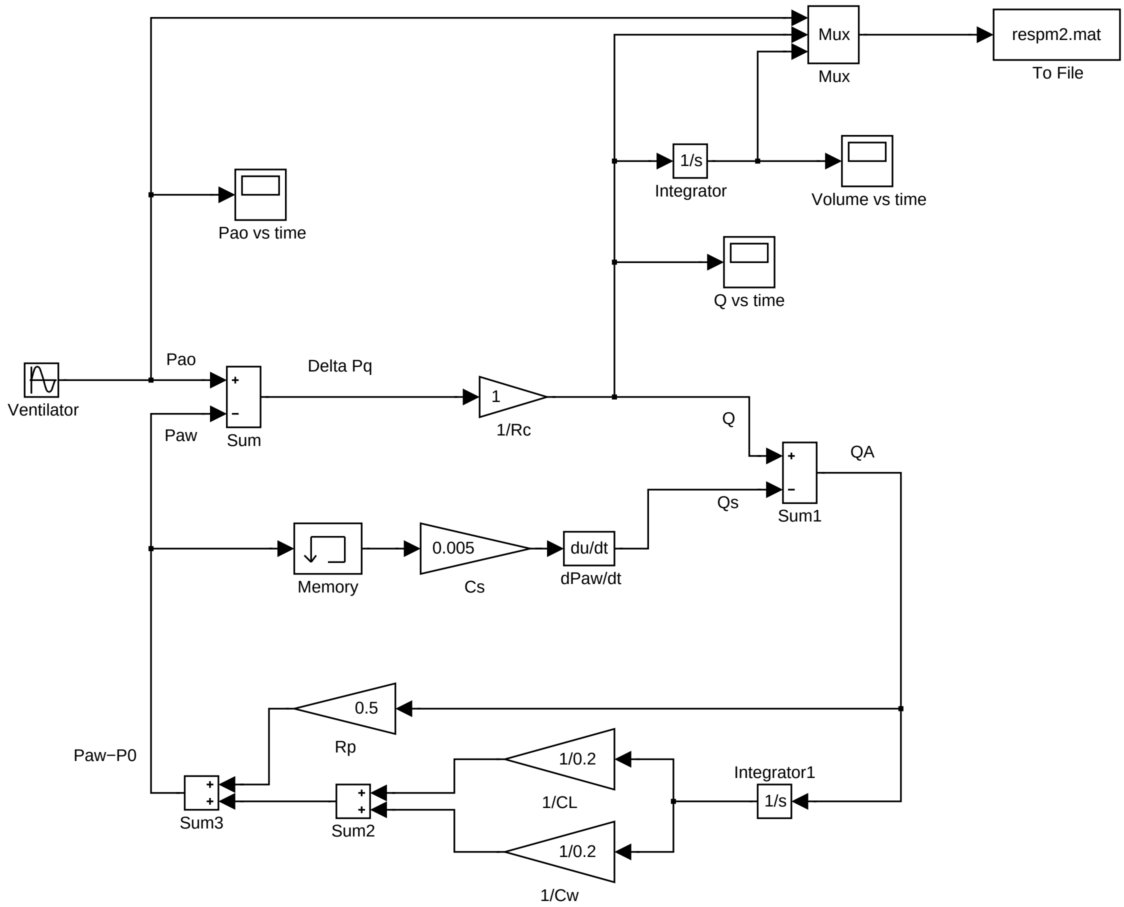

Figure 14: lung mechanics Simulink design

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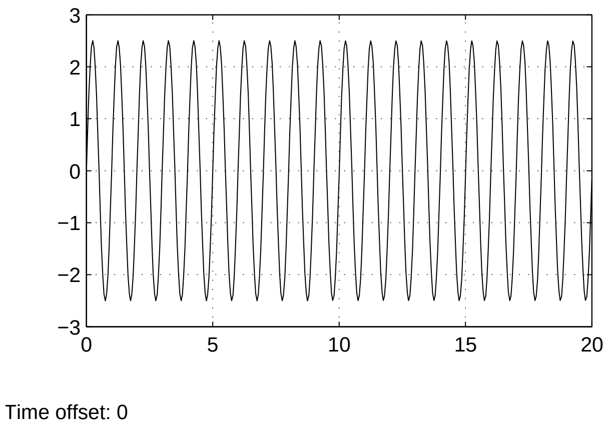

Figure 15: Pao vs time

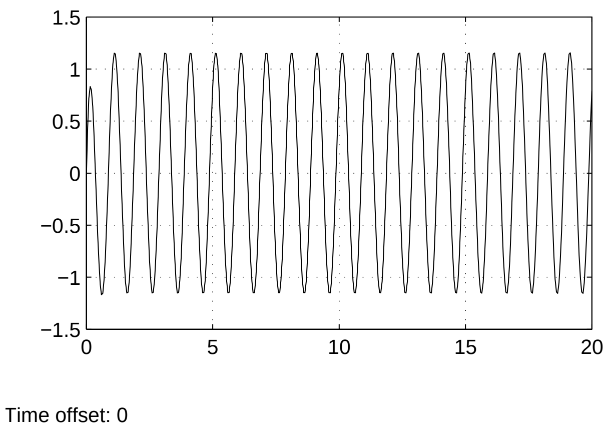

Figure 16: Q vs time

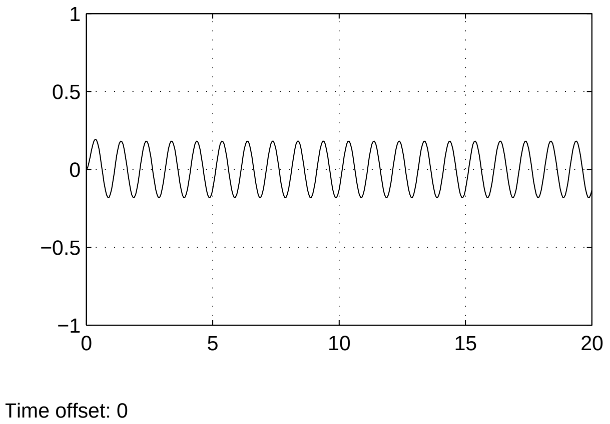

Figure 17: Volume vs time