Exercise 3.6

Existence. In $\mathbb{R}^3$, a plane is the set all points which pairwise define vectors that can be written as linear combinations of two linearly independent vectors. Here we assume that students may know that all linear combinations of two linearly independent vectors define a plane in the space.

From exercise 2.6 we get that a 2-dimensional vector space has two linearly independent vectors, while three vectors must be linearly dependent. We denote by $\vec{a}$ and $\vec{b}$ two linearly independent vectors of V_2. Given point P, there exists exactly one point Q_1 such that $\vec{a} = \overrightarrow{PQ_1}$ and exactly one point Q_2 such that $\vec{b} = \overrightarrow{PQ_2}$.

It follows that the set of points X such that $\overrightarrow{PX}$ is linear combination of $\vec{a} = \overrightarrow{PQ_1}$ and of $\vec{b} = \overrightarrow{PQ_2}$ defines a plane. Thus $E = \left\{X \in \mathcal{P} | \overrightarrow{PX} \in V_2\right\}$ is indeed a plane.

Uniqueness. We show that E is unique. Suppose that $\bar{E}$ is another plane such that $P \in \bar{E}$ and $\left\{\overrightarrow{XY} | X, Y \in \bar{E}\right\} = V_2$. Then we show that $E = \bar{E}$. We have:

$$X \in E \Rightarrow \overrightarrow{PX} \in V_2 = \left\{\overrightarrow{XY} | X, Y \in \bar{E}\right\} \Rightarrow X \in \bar{E},$$

$$X \in \bar{E} \wedge P \in \bar{E} \Rightarrow \overrightarrow{PX} \in V_2 \Rightarrow X \in E,$$

so that $E = \bar{E}$ and E is unique.