Seminar on Optimization and Control in Physiological Systems

Introduction to delay differential equations

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1. Introduction

In the applications, the future behavior of many phenomena are assumed to be described by the solution of an ordinary differential equation. Implicit in this assumption is that the future behavior is uniquely determined by the present and independent of the past. In differential difference equations (DDE), or more generally functional differential equations (FDE), the past exerts its influence in a significant manner upon the future. Many models under scrutiny are better represented by (FDE), than by ordinary differential equations.

(DDE) and (FDE) were first encountered in the late eighteenth century by Bernoulli's, Laplace and Condorcet. However, very little was accomplished during the nineteenth century and the early part of the twentieth century. During the last sixty years and especially the last forty, the subject has been and is continuing to be investigated at a very rapid pace. The impetus has mainly been due to developments in the theory of control, mathematical biology, mathematical economics and the theory of systems which communicate through lossless channels.

In this introductory section, we indicate by means of examples (with references to their origin) the diversity of FDE.

Minorsky (1962, [1]) was one of the first investigators of modern times to study the differential-difference equation

$$\dot{x}\ (t) = F(t, x(t), x(t-r))$$

and its effect on simple feedback control systems in which the communication time cannot be neglected.

Lord Cherwell (see Wright 1961, [2]) has encountered the differential-difference equation

$$\dot{x}\ (t) = -\alpha x(t-1)(1 + x(t))$$

in his study of the distribution of primes. Variants of this equation have also been used as models in the theory of growth (see Cunningham 1954, [3]).

Volterra [4] in his study of predator-prey models studied the integro-differential equation

$$\begin{aligned} \dot{N}_1\ (t) &= [\varepsilon_1 - \gamma_1 N_2(t) - \int_{-r}^{0} F_1(-\theta) N_2(t+\theta) d\theta] N_1(t) \\ \dot{N}_2\ (t) &= [-\varepsilon_2 + \gamma_1 N_2(t) + \int_{-r}^{0} F_2(-\theta) N_1(t+\theta) d\theta] N_2(t), \end{aligned}$$

Where N_1, N_2 are the number of prey and predators, respectively.

Wangersky and Cunningham [3] have also used the equations

$$\begin{aligned}\dot{x}\ (t) &= \alpha(x(t))[\frac{m-x(t)}{m}] - bx(t)y(t)\\ \dot{y}\ (t) &= -\beta y(t) + cx(t-r)y(t-r)\end{aligned}$$

for similar models.

The equation

$$\dot{x}\ (t) = -\int_{t-r}^{t} a(t-u)g(x(u))du$$

was encountered by Ergen [5] in the theory of a circulating fuel nuclear reactor and has been studied extensively by Levin and Nohel [6]. In this model, x is the neutron density. It is also a good model in one dimensional viscoelasticity in which x is the strain and a is the relaxation function.

In the theory of control, Krasovskii [7] has studied extensively the system

$$\begin{aligned}\dot{x}\ (t) &= P(t)x(t) + B(t)u(t)\\ \dot{y}\ (t) &= Q(t)x(t)\\ \dot{u}\ (t) &= \int_{-r}^{0}[d_\theta \eta(t,\theta)]y(t+\theta) + \int_{-r}^{0}[d_\theta \eta(t,\theta)]u(t+\theta).\end{aligned}$$

In theory of lossless transmission lines, Miranker [8] and Brayton [9] have encountered the equation

$$\begin{aligned}\dot{v}\ (t) &= \alpha\ \dot{v}\ (t-r) - \beta v(t)\\ &\quad -\alpha\gamma v(t-r) + F(v(t), v(t-r))\end{aligned}$$

where α, β, γ are constants.

In his study of vibrating masses attached to an elastic bar, Rubanik [10] considered the equations

$$\begin{aligned}\ddot{x}\ (t) + \omega_1^2 x(t) &= \varepsilon f_1(x(t), \dot{x}\ (t), y(t), \dot{y}\ (t)) + \gamma_1\ \ddot{y}\ (t-r)\\ \ddot{y}\ (t) + \omega_1^2 y(t) &= \varepsilon f_2(x(t), \dot{x}\ (t), y(t), \dot{y}\ (t)) + \gamma_2\ \ddot{x}\ (t-r).\end{aligned}$$

In studying the collision problem in electrodynamics, Driver [11] encountered systems of the type

$$\dot{x}\ (t) = f_1(t, x(t), x(g(t))) + f_2(t, x(t), x(g(t)\ \dot{x}\ (g(t)),\ \ g(t) \leq t$$

El'sgol'tz and Hughes have considered the following variational problem, minimize

$$V(x) = \int_0^1 F(t, x(t), x(t-r), \dot{x}\ (t), \dot{x}\ (t-r), \ddot{x}\ (t-r))dt$$

over some class of function x. Generally, the Euler equations are of the form

$$\ddot{x}(t) = f(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t-r)).$$

In the slowing down of neutrons in a nuclear reactor the equation

$$x(t) = \int_t^{t+1} k(s)x(s)ds$$

or

$$\dot{x}(t) = k(t+1)x(t+1) - k(t)x(t)$$

seems to play an important role (see Slater and Wilf [12]).

2. A Delayed Epidemic Model

Mathematical biologist A. J. Lotka investigated, in a series of papers from 1912 on, a differential equation model of malarial epidemics due to Ross (1911) [14]. In particular (see Sharpe and Lotka (1923) [13]), he examined the effect of incubation delay. Let us first look at the model without any delay. The equations are, as given by Lotka (1923), for the human population,

$$\dot{h}(t) = \frac{bgm(t)(p-h(t))}{p - Mh(t) - rh(t)};$$

and for the mosquito population,

$$\dot{m}(t) = \frac{bfh(t)(q-m(t))}{p - Nm(t) - sm(t)}.$$

Here, p and q are the total human and mosquito populations, treated as constant quantities, which is a standard practice in simple epidemiological models. The function $h(t)$ and $m(t)$ stand for human and mosquito populations carrying the malaria organism (the infected or diseased populations), respectively. The healthy populations are $p - h(t)$ and $q - m(t)$. A fixed proportion of each of these populations is assumed to be infective, with the infective population being fh and gm, respectively. The quantities M and N are death rates, while r and s are recovery rates.

For our presents purposes what is of most interest is the modification to include incubation delays, quoted from Ross (1911) [14] to be $u = 0.5$ month in human and $v = 0.6$ month in mosquito. We thus have

$$\begin{aligned} \dot{h}(t) &= \frac{bgm(t-u)(p-h(t-v))}{p-(M+r)h(t)} \\ \dot{m}(t) &= \frac{bfh(t-v)(q-m(t-u))}{p-(N+s)m(t)}. \end{aligned}$$

The delay is from the time of a bite to the time at which the human or mosquito is infective.

3. Delay Models in Physiology: Dynamic Diseases

There are many acute physiological diseases where the initial symptoms are manifested by an alternation or irregularity in a control system that is normally periodic, or by the onset of an oscillation in a nonoscillatory process. Such physiological diseases have been termed as dynamical diseases by Glass and Mackey (1979) , who have made a systematic study of several important and interesting physiological models with time delays. The following is one example of these models

$$\dot{x}\,(t) = \lambda - \frac{\alpha V_m x(t) x^n(t-\tau)}{\theta^n + x^n(t-\tau)}. \tag{1}$$

Here, $\lambda, \alpha, V_m, n, \tau, \theta, \beta_0$, and γ are positive constants. Equation (1) is used to study a "dynamic disease" involving respiratory disorders, where $x(t)$ denote the arterial CO_2 concentration of a mammal, λ is the CO_2 production rate, V_m denotes the maximum "ventilation"rate of CO_2, and τ is the time between oxygenation of blood in the lungs and stimulation of chemoreceptors in the brainstem. (for more detail see article by Glass and Mackey (1979))

4. Simple examples of delay differential equation

As we see the above, there are many different types of equations that occur in the applications, some which depend only upon the past state, some which depend upon the past state as well as the rate change of the past state and some which depend upon the future. The solutions behave differently for each of these types of equations. To recognize some of the difficulties, let us discuss in an intuitive manner some very simple examples.

♡ Example 1:

Consider first the linear **retarded equation**

$$x'(t) = -x(t-r), \;\; r > 0. \tag{2}$$

What is the minimum amount of data that is necessary for (2) to define a function for $t \geq 0$? A moment of reflection indicates that we must specify a function on the interval $[-r, 0]$. If φ is a given continuous function defined on $[-r, 0]$, then there is only one function $x(t)$ defined on $[-r,\infty)$ which coincides with φ on $[-r,0]$ and satisfies (2) for $t>0$. In fact, if x is such a function, then is must satisfy

$$x(t)=\varphi(0)-\int_0^t x(s-r)ds,\ t\geq 0 \tag{3}$$

and in particular,

$$x(t)=\varphi(0)-\int_0^t \varphi(s-r)ds,\ 0\leq t\leq r.$$

This latter equation uniquely defines x on $[0,r]$. Once x is known on $[0,r]$, then (2) uniquely defines x on $[r,2r]$, etc.

The following observation about (2) are important:

→ ($\mathbb{I}$) For any continuous function φ defined on $[-r,0]$, there is a unique solution x of (2) on $[-r,\infty)$. Designate this solution by $x(\varphi)$.

→ ($\mathbb{II}$) The solution $x(\varphi)$ has a continuous derivative for $t>0$, but not at $t=0$ unless $\varphi(\theta)$ has a left hand a derivative at $\theta=0$ and $\varphi^{'}(0)=-\varphi(-r)$. The solution $x(\varphi)$ is smoother than the initial data.

→ ($\mathbb{III}$) For a given φ on $[-r,0]$, the solution $x(\varphi)(t)$ of (2) need not be defined for $t\leq -r$. In fact, if $x(\varphi)(t)$ is defined for $t\leq -r$, say $x(\varphi)(t)$ is defined for $t\geq -r-\varepsilon$, $\varepsilon>0$, then $\varphi(\theta)$ must have a continuous first derivative for $\theta\in(-\varepsilon,0]$. If a solution $x(\varphi)$ does exist for $t\leq -r$, then $x(\varphi)(t)$ for $t\leq -r$ has in general fewer derivatives than φ.

♡ Example 2:

As a second example, consider the **advanced equation**

$$\frac{dy(\tau)}{d\tau}=y(\tau+r),\ r>0. \tag{4}$$

If we let $\tau=-t$, $x(t)=y(-t)$, then x satisfies (2). Therefore, the natural problem for (2) is for $\tau\leq 0$. On the other hand, if this equation describes a physical system, then it must be integrated for $\tau\geq 0$. As in ($\mathbb{III}$) above, any such solution must satisfy some special conditions and, in general, has fewer derivatives than the initial data.

♡ Example 3:

As another example, consider the neutral equation

$$\dot{x}\,(t)-c\,\dot{x}\,(t-r)-dx(t-r)=0,\ r>0,\ c\neq 0. \tag{5}$$

In this situation, it is a little more difficult to begin the discussion since many different possibilities are available for the concept of a solution. In any case, if (5) is to define a function for $t\geq 0$, then we must specify a function on $[-r,0]$. If we suppose that φ is a function on $[-r,0]$ which has a continuous first derivative, then one can certainly find a function which satisfies (5) for $t > 0$ and even has a continuous first derivative except at the points $t = kr,\ k = 0, 1, 2, ...$

In fact

$$\dot{x}\ (t) = c\,\dot{x}\ (t - r) + dx(t - r)$$

can be integrated successively in steps of length r. If $\dot{\varphi}\ (0) \neq c\ \dot{\varphi}\ (-r) + d\varphi(-r)$, then $\dot{x}$ is discontinuous at $t = 0$. Consequently, $\dot{x}$ will be discontinuous at $t = kr,\ k = 1, 2, ...$ Since $c \neq 0$, we can also write

$$\dot{x}\ (t - r) = \frac{1}{c}\left[\dot{x}\ (t) - dx(t - r)\right]$$

and, therefore, define $x(t)$ for $t \leq -r$. The following observations are now immediate:

→ (IV) For any function φ defined on $[-r, 0]$ with $\dot{\varphi}\ (\theta)$ continuous, there is a unique solution $x(\varphi)$ of (5) on $(-\infty, +\infty)$ which has a continuous first derivative for $t \neq kr,\ k = \pm 1, \pm 2, ...$

→ (V) The solution $x(\varphi)$ has essentially the same smoothness properties as the initial data. Compare this with hyperbolic partial differential equations. One can also interpret (5) in integrated form as

$$(6) \qquad x(t) - cx(t - r) = \varphi(0) - c\varphi(-r) + d\int_0^t x(s - r)ds,\ t \geq 0.$$

A solution can now be defined for a continuous initial function. For $c = 0$, this now includes the retarded equation (2).

♡ Example 4:

As a final example, consider the equation of **mixed type**

$$\dot{x}\ (t) + ax(t - r) + bx(t + r) = 0,\ r > 0,\ a \neq 0,\ b \neq 0.$$

For this equation it is not at all clear what information is needed for (6) to define a function for $t \geq 0$ since the derivative of x depends upon past as well as future values. This equation seems to dictate that boundary conditions should be specified in order to obtain a solution in the same way as one does for elliptic partial differential equations.

Just looking at the examples above from the point of view of the information needed to obtain solutions of the equations and the resulting smoothness properties of the solutions, we have seen there are distinct types in a manner suggestive of the types in partial differential equations. To gain more insight into the differences in these types, let us look at their corresponding characteristic equation. As for linear ordinary differential equations with constant coefficients, the characteristic equation is obtained by trying to find a λ such that $e^{\lambda t}$ is a solution of the differential equation.

Characteristic equation:

For equation (2) :

$$\dot{x}\ (t) = -x(t-r),\ r > 0.$$

the characteristic equation is

$$(7) \qquad \lambda + e^{-\lambda r} = 0 \Leftrightarrow \lambda e^{\lambda r} = -1 = e^{(2k+1)\pi i},\ k = 0, \pm 1, \pm 2, ...$$

It is clear that λ satisfies (7) if and only if

$$\lambda r + \ln |\lambda| = (2k+1)\pi i,\ k = 0, \pm 1, \pm 2, ...$$

or

$$\mathrm{Re}(\lambda r + \ln |\lambda|) = 0 \Rightarrow r\mathrm{Re}\lambda = -\mathrm{Re} \ln |\lambda| .$$

Therefore,

$$\begin{aligned} \mathrm{Re}\lambda &\to -\infty \text{ as } |\lambda| \to \infty \text{ if } r > 0 \text{ (retarded)} \\ \mathrm{Re}\lambda &\to +\infty \text{ as } |\lambda| \to \infty \text{ if } r < 0 \text{ (advanced).} \end{aligned}$$

Since (7) is an entire function of λ, this implies there are only a **finite number of roots to the right of any line** $\mathrm{Re}z = \gamma$ if $r > 0$ (**retarded**) and there are only a **finite number of roots to the left of any line** $\mathrm{Re}z = \gamma$ **if** $r < 0$ (**advanced**). Also, as $r \to 0^+$, $\mathrm{Re}\lambda \to -\infty$ unless $|\lambda| \to 1$ and as $r \to 0^-$, $\mathrm{Re}\lambda \to +\infty$ unless $|\lambda| \to 1$. It is natural to expect that the asymptotic behavior of the solutions will be depicted by the supremum of real parts of the λ satisfying the characteristic equation. If this is so, then for $r \to 0^+$ then equation degenerates nicely (as far asymptotic properties at $t = \infty$ are concerned) to the ordinary differential equation

$$\dot{x}\ (t) = -x(t).$$

5. A General initial value problem

Suppose $r \geq 0$ is given real number, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is a real or complex n-dimensional linear vector space with norm $|\varphi|$. (norm sup for example), $C([a,b], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-r,0]$ we let $C([-r,0], \mathbb{R}^n)$ and designate the norm of an element φ in C by $\|\varphi\| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$. Even though single bars are used for norms in different space, no confusion should arise. If $\sigma \in \mathbb{R}$, $A \geq 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let

$x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$. If $\dot{x}\ (t) = dx(t)/dt$ and $f : \mathbb{R} \times C \to \mathbb{R}^n$ is a given function, we say that the relation

$$\dot{x}\ (t) = f(t, x_t) \tag{8}$$

is a functional differential equation of retarded type or simply a functional differential equation. A function x is said to be a solution of (8) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$ and $x(t)$ satisfies (8) for $t \in (\sigma, \sigma + A)$. In such a case, we say x is a solution of (8) on $[\sigma - r, \sigma + A)$. For a given $\sigma \in \mathbb{R}$ and a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say $x = x(\sigma, \varphi)$ is a solution of (8) with initial value φ at σ or simply a solution of (8) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (8) on $[\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \varphi) = \varphi$. We say system (8) is linear if

$$f(t, \varphi) = L(t, \varphi) + h(t),$$

where $L(t, \varphi)$ is linear in φ; linear, homogeneous if $h \equiv 0$ and linear nonhomogeneous if $h \not\equiv 0$. We say system (8) is autonomous if $f(t, \varphi) = g(\varphi)$ where g does not depend on t.

Lemma 1. *If $\sigma \in \mathbb{R}$, $\varphi \in C$ are given and $f(t, \varphi)$ is continuous, then finding a solution of (8) through (t, φ) is equivalent to solving the integral equation*

$$x(t) = \varphi(0) + \int_0^t f(s, x_s)ds,\ t \ge \sigma,\ x_\sigma = \varphi.$$

Existence and uniqueness of the solution

In this section, we give a basic existence theorem for the initial value problem of (8) assuming that f is continuous. We need

Lemma 2. *If $x \in C([\sigma - r, \sigma + \alpha], \mathbb{R}^n)$, then x_t is a continuous function of t for t in $[\sigma, \sigma + \alpha]$.*

Proof: *Since x is continuous on $[\sigma - r, \sigma + \alpha]$, it is uniformly continuous and thus for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x(t) - x(\tau)| < \varepsilon \text{ if } |t - \tau| \le \delta.$$

or

$$|x(t) - x(\tau)| \overset{uniformly}{\to 0} \quad \text{if } t \to \tau.$$

Consequently, for t, τ in $[\sigma, \sigma + \alpha]$, $|t - \tau| < \delta$, we have

$$|x(t+\theta) - x(\tau + \theta)| < \varepsilon \text{ for all } \theta \text{ in } [-r, 0].$$

This proves the lemma.

♡ Existence

Theorem 1. *Suppose Ω is an open set in $\mathbb{R} \times C$ and $f : \Omega \to \mathbb{R}^n$ is continuous. If $(\sigma, \varphi) \in \Omega$, then there is a solution of (8) passing through (σ, φ).*

♡ uniqueness

Theorem 2. *Suppose Ω is an open set in $\mathbb{R} \times C$ and $f : \Omega \to \mathbb{R}^n$ is continuous and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in Ω. If $(\sigma, \varphi) \in \Omega$, then there is a unique solution of (8) with initial value φ at σ.*

6. Remarks on the map defined by solutions

In this section, we give some specific examples of functional differential equation in order to contrast the behavior with ordinary differential equations.
The examples will also serve to formularize the reader with the idea of looking at the solution of (8) in the space C rather than $\mathbb{R}^n$.

Throughout this section, suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous and for any $(\sigma, \varphi) \in \mathbb{R} \times C$, there is a unique solution $x = x(\sigma, \varphi)$ of (8) passing through (σ, φ). $x(\sigma, \varphi)(t)$ is continuous in (t, σ, φ) in its domain of definition.

♡ first example

Two distinct solutions of (8) considered in $\mathbb{R} \times \mathbb{R}^n$ **may intersect an infinite number of times**. In fact, consider the **scalar** equation

$$\dot{x}\ (t) = -x(t - \frac{\pi}{2}) \tag{9}$$

has a unique solution through each $(\sigma, \varphi) \in \mathbb{R} \times C$, but it also has the solutions $x(t) = \sin t$ and $x(t) = \cos t$. The sets $\{(t, \sin t), t \in \mathbb{R}\}$, $\{(t, \cos t), t \in \mathbb{R}\}$ in $\mathbb{R} \times \mathbb{R}$ intersect an infinity number of times on any interval $[\sigma, \infty)$ and yet are not identical on any interval.

The above example shows that it is probably impossible to develop a geometric theory for (8) by defining trajectories in $\mathbb{R} \times \mathbb{R}^n$ as $\{(t, x(\sigma, \varphi)(t)), t \geq \sigma\}$.

On the other hand, it seems reasonable to have the definition of a trajectory of a solution so that it will depict the evolution of the state of the system. Furthermore, the state of the system should be that part of the system which uniquely determines the future behavior. From our basic existence and uniqueness theorem the state at time t therefore, should be $x_t(\sigma, \varphi)$ and the trajectory through (σ, φ) should be the set $\cup_{t \geq 0}(t, x_t(\sigma, \varphi))$ in $\mathbb{R} \times C$. For the geometric theory of functional differential equation, the map is going to be $x_t(\sigma, .)$. Therefore, for $t \geq \sigma$, define the operator $T(t, \sigma) : C \to C$ by the relation

$$T(t, \sigma)\varphi = x_t(\sigma, \varphi).$$

The operator $T(t, \sigma)$ is continuous. From the hypothesis of uniqueness of solutions of (8), for given φ, $\psi \in C$, if there is a $\tau \geq \sigma$ such that $T(\tau, \sigma)\varphi = T(\tau, \sigma)\psi$, then $T(t, \sigma)\varphi = T(t, \sigma)\psi$ for $t \geq \tau$.

For autonomous equations, it is more natural to consider the orbits of solutions rather than the trajectories; that is, the path traced out by the solution in the phose space X rather than the graph of the solutions in $\mathbb{R} \times C$. If the phase space for Equation (9) is chosen as $\mathbb{R}$ and the orbits as $\cup_{t \geq 0} x(0, \phi)(t)$, then the orbits for the solutions $x(t) = \sin t$ and $x(t) = \cos t$ coincide and are equal to the interval $[-1, 1]$. That the orbits coincide is expected because $\sin(t + \pi/2) = \cos t$, Equation (9) is autonomous and therefore, a solution shifted in phose is still a solution. The difficulty encountered by choosing the phase space $\mathbb{R}$ is that the orbit of one solution may completely contain the orbit of another solution and not be related in any way to a phase shift. The orbit of the solution $x = 0$ is contained in the orbit of $\cos t$.

On the other hand, if the phase space is chosen as $C = C([-\pi/2, 0], \mathbb{R})$, then the orbit of the solution sin$t$ of Equation (9) is the set,

$$\Gamma = \{\psi : \psi(\theta) = \sin(t + \theta), -\pi/2 \leq \theta \leq 0, \text{ for } t \in [0, \infty)\},$$

of points in C. The set Γ as before is also the orbit of the solution $\cos t$. Furthermore, because of uniqueness of solutions and one-to-oneness of the mapping $T(t, \sigma)\varphi$, any solution x of Equation (9) for which there is a τ with $x_t \in \Gamma$ must be a phase shifts on sin t. Therefore, Γ is determined by phase shifts of a solution. Finally, Γ is a closed curve in C which is intuitively satisfying since $\sin t$ is periodic.

This simple example suggests the geometric theory for Equation (9) will probably be richer if the map $T(t, \sigma)$ is used. However, in some situations, it is very advantageous to know that $T(t, \sigma)\phi$ is determined by taking a restriction over an interval of a function in $\mathbb{R}^n$.

♡ second example

There are functional differential equations for which there is a t_0 with $x(t) = 0$ for all $t \geq t_0$. Consider the equation

$$\begin{cases} \dot{x}\ (t) &= -\alpha(t)x(t-1), \ t \geq 0 \\ x_0(s) &= \varphi(s), \ -1 \leq s \leq 0 \end{cases}$$

where

$$\alpha(t)=\begin{cases} 2\sin^2 \pi t & , \quad t\in[2n,2n+1] \\ 0 & , \quad t\in(2n-1,2n) \end{cases}$$

for each integer, we have

$$\boxed{\forall t\geq 4,\ x_t=0.}$$

we want to proof that $x_t(-1)=0$, for any $t\geq 4$, then we want to see

$$x(t-1)=0,\ t\geq 4 \Leftrightarrow x(s)=0,\ s+1=t\geq 4 \Leftrightarrow x(s)=0,\ s\geq 3.$$

We have for :

♦$s\in[0,1]$,

$$\dot{x}\ (s)=-2\sin^2(\pi s)\varphi(s-1)\Rightarrow x(s)=x(0)-\int_0^s 2\sin^2(\pi\tau)\varphi(\tau-1)d\tau$$

♦$s\in[1,2]$,

$$\dot{x}\ (s)=0\Rightarrow x(s)=x(1)=x(2)$$

♦$s\in[2,3]$,

$$\begin{aligned} \dot{x}\ (s) &= -2\sin^2(\pi s)\varphi(s-1) \\ \dot{x}\ (s) &= -2\sin^2(\pi s)x(1) \\ x(s) &= x(2)-\int_2^s 2\sin^2(\pi\tau)x(1)d\tau \\ x(s) &= x(1)(1-\int_2^s 2\sin^2(\pi\tau)d\tau). \end{aligned}$$

♦$s\in[3,4]$,

$$\dot{x}\ (s)=0\Rightarrow x(s)=x(3)=x(4)$$

For that we want to proof

$$\int_2^3 \sin^2(\pi\tau)d\tau=\frac{1}{2}.$$

In fact

$$\begin{aligned} \int_2^3 \sin^2(\pi\tau)d\tau &= -\int_2^3 \frac{1}{\pi}\sin(\pi\tau)\cos^{'}(\pi\tau)d\tau \\ &= -\frac{1}{\pi}([\sin(\pi\tau)\cos(\pi\tau)]_2^3-\int_2^3 \pi\cos(\pi\tau)\cos(\pi\tau)d\tau) \\ &= 0+\int_2^3 \cos^2(\pi\tau)d\tau \\ &= 1-\int_2^3 \sin^2(\pi\tau)d\tau \end{aligned}$$

then
$$\int_2^3 \sin^2(\pi\tau)d\tau = \frac{1}{2}.$$
Finally
$$x(s) = x(3) = x(4) = x(1)(1 - \int_2^3 \sin^2(\pi\tau)d\tau) = x(1)(1 - \frac{2}{2}) = 0.$$

7. Definitions of Stability

Let $C_H = \{\varphi \in C : |\varphi| < H\}$, $\mathbb{R}^+ = [0, \infty)$. In this section, we consider the system (8) with $f(t, 0) \equiv 0$, $t \in \mathbb{R}^+$, $f : \mathbb{R}^+ \times C_H$ is continuous and satisfies enough additional hypotheses to ensure that the solution $x(\sigma, \varphi)(t)$ through (σ, φ) is continuous in (σ, φ, t) in the domain of definition of the function.

Definition 1. (**a**) *The solution $x = 0$ of (8) is called* ***stable*** *at t_0 if $t_0 \geq 0$ and*
(*i*) *there is a $b = b(t_0) > 0$ such that φ in C_b implies the solution $x(t_0, \varphi)$ of (8) exists for $t \geq t_0$ and $x_t(t_0, \varphi)$ is in C_H for $t \geq t_0$*;
(*ii*) *For every $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that φ in C_δ implies the solution $x(t_0, \varphi)$ of (8) satisfies $x_t(t_0, \varphi)$ in C_ε for all $t \geq t_0$.*
(**b**) *The solution $x = 0$ of (8) is called asymptotically stable at t_0 if it is stable and there is an $H_0 = H_0(t_0)$ such that φ in C_{H_0} implies the solution $x(t_0, \varphi)$ of satisfies*
$$\lim_{t\to\infty} |x_t(t_0, \varphi)| = 0.$$
(**c**) *The solution $x = 0$ of (8) is unstable at t_0 if it is not stable at t_0.*

In ordinary differential equations, a system which enjoys either one of the above types of stability at t_0 enjoys the same type of stability at t_1 for any $t_1 \geq t_0$. The basic reason for this fact is that the mapping induced by the solutions of ordinary differential equations for which solutions are uniquely defined by their initial values takes a sphere of initial values into a set which contains a sphere. Also, continuity with respect to initial values implies the above remark is also true for any $t_1 \leq t_0$ provided only that solutions of the equation exist on $[t_1, t_0]$.

For functional-differential equations, the latter property holds for exactly the same reason; namely, if the solution $x = 0$ of (8) is stable at t_0 in the sense of definition 1 (**a**) or 1 (**b**), then it is stable at $t_1 \leq t_0$ in the same sense provided that the solutions exist on $[t_1, t_0]$.

However, stability of the solution $x = 0$ of (8) at t_0 does not necessarily imply stability of $x = 0$ at $t_1 \geq t_0$. In fact, consider equation
$$\dot{x}\,(t) = b(t)x(t - 3\pi/2).$$

For $t_0 = 0$, the solution is given by

$$x(0,\varphi)(t) = \begin{cases} \varphi(0), & 0 \le t \le 3\pi/2 \\ (-\sin t)\varphi(0), & t \ge 3\pi/2, \end{cases}$$

and so the solution $x = 0$ of is clearly stable for $t_0 = 0$. On the other hand, for any $t_1 \ge 3\pi$, the solution $x(t,\varphi)$ of must satisfy the equation

$$\dot{x}\ (t) = x(t - 3\pi/2). \tag{10}$$

For any constant a and any λ for which $\lambda = \exp(-3\pi\lambda/2)$, the function $x(t) = a\exp(\lambda t)$ satisfies (10). Since there is a $\lambda_0 > 0$ satisfying this equation, the solution $x = 0$ is unstable for any $t_1 > 3\pi$.

8. Sufficient conditions for stability of general systems

In this section, we give sufficient conditions for stability of the solution $x = 0$ of and illustrate the results with examples. If $V : \mathbb{R}^+ \times C_H \to \mathbb{R}$ is continuous we let

$$\dot{V}\ (t,\varphi) = \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t+h, x_{t+h}(t,\varphi)) - V(t,\varphi)]$$

where $x_{t+h}(t,\varphi)$ is the solution of (8) through (t,φ). $\dot{V}\ (t,\varphi)$ is the upper right hand derivative of $V(t,\varphi)$ along the solutions of (8).

Theorem 3. *Suppose f takes closed bounded sets of $\mathbb{R}^+ \times C_H$ into closed bounded sets of $\mathbb{R}^n$. Suppose $u(s)$, $v(s)$, $w(s)$ are continuous functions for s in $[0,H)$, $u(s)$, $v(s)$ positive and nondecreasing for $s \neq 0$, $u(0) = v(0) = 0$, $w(s)$ nonnegative, and nondecreasing. If there is a continuous function $V : \mathbb{R}^+ \times C_H \to \mathbb{R}$ such that*

$$\begin{aligned} u(|\varphi(0)|) &\le V(t,\varphi) \le v(|\varphi|) \\ \dot{V}\ (t,\varphi) &\le -w(|\varphi(0)|) \end{aligned}$$

then the solution $x = 0$ of (8) is uniformly stable. If, in addition, $w(s) > 0$ for $s > 0$, $w(s)$ nondecreasing, then the solution $x = 0$ of (8) is uniformly asymptotically stable.

Example

Consider the scalar equation

$$\dot{x}\ (t) = -ax(t) - b(t)x(t-r)$$

where $a > 0$, $b(t)$ is continuous and bounded for all $t \ge 0$. If x is scalar, take $|x|$ as the absolute value of x. If

$$V(\varphi) = \frac{1}{2a}\varphi^2(0) + \mu\int_{-r}^{0}\varphi^2(\theta)d\theta,$$

where μ is to be determined. We can apply Theorem 3 and if b is a constant, then the exact region of stability is indicated in the Figure 1

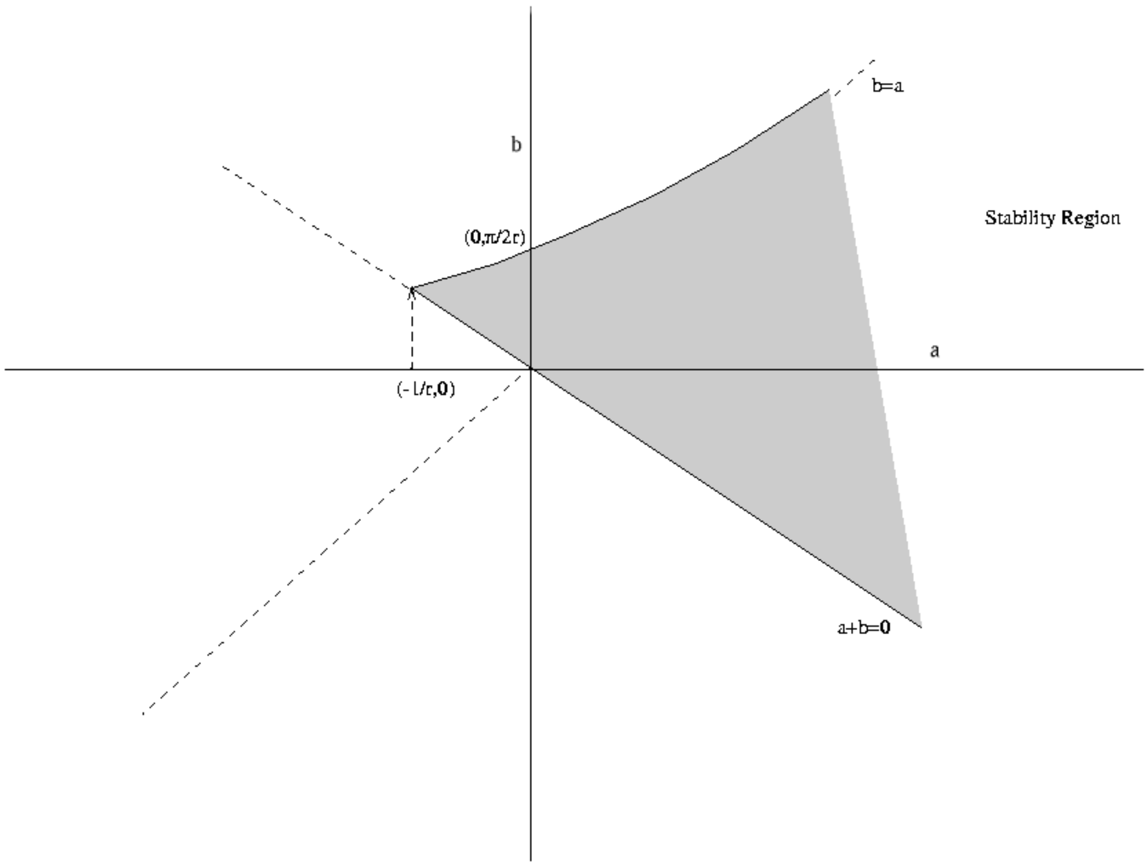

FIGURE 1. Delay

The region $|b| < a$ is the maximum region for which stability is assured for all values of r, $0 \leq r < \infty$. On the other hand, as $r \to 0$ the true region of stability for approaches the half-plane $b + a > 0$.

9. Sufficient conditions for instability

In this section, we give a sufficient condition for the instability of the solution $x = 0$ of (8) and give some examples to illustrate result.

Theorem 4. *Suppose $V(\varphi)$ is a continuous bounded scalar function on C_H. If there exist a γ, $0 < \gamma < H$ and an open set U in C such that*
(i) $V(\varphi) > 0$ on U, $V(\varphi) = 0$ on the boundary of U,
(ii) 0 belongs to the closure of $U \cap C_\gamma$,

(iii) $V(\varphi) \leq u(|\varphi(0)|)$ *on* $U \cap C_\gamma$,
(iv) $\dot{V}^*(\varphi) \geq w(|\varphi(0)|)$ *on* $[0,\infty) \times U \cap C_\gamma$,

$$\dot{V}^*(\varphi) = \lim_{h \to 0^+} \frac{1}{h}[V(x_{t+h}(t,\varphi)) - V(\varphi)],$$

where $u(s)$, $w(s)$ *are continuous, increasing and positive for* $s > 0$, *then the solution* $x = 0$ *of (8) is unstable. More specifically, each solution* $x_t(t_0, \varphi)$ *of (8) with initial function* φ *in* $U \cap C_\gamma$ *at* t_0 *must reach the bounded of* C_γ *in a finite time.*

Example

Consider the equation

$$\dot{x}(t) = -ax(t) - b(t)x(t-r)$$

where $a + b < 0$ and r is any positive constant. We wish to prove by use of Lyapunov functions that the solution $x = 0$ of this equation in unstable. The exact region of stability for this equation is shown in Figure 1.
The region $a+b < 0$ is the interior of the intersections of the instability regions as a function of r.
If F is any given function and

$$V(x_t) = \frac{x^2(t)}{2} - \frac{1}{2}\int_{t-r}^{t} F(t-u)\,[x(u) - x(t)]^2\,du$$

10. Oscillations and delay

With the origin of the introduction of delays into certain models one finds the need to take account oscillatory phenomena.
The simple model of the demography is can be written

$$\frac{dN}{dt} = aN(t)$$

where the parameter $a = r - d$, and it is supposed that r and d depend on the total of the population, we have the equation

$$\frac{dN}{dt} = [r(N(t)) - d(N(t))]\,N(t) \tag{11}$$

It is natural to suppose that $r(0) > d(0)$ and that $r(N)$ is decreasing and $d(N)$ is increasing, with $r(\infty) < d(\infty)$. The all solutions of (11) are monotonous increasing if $N(0) < N^*$, decreasing if $N(0) > N^*$ where N^* is the unique root of the equation

$$r(N(t)) = d(N(t))$$

The functions $r(N(t))$ and $d(N(t))$ represent the answers which the population brings to the level reproduction and mortality at any moment. These answers are made starting from the evaluation of the size of the population at any moment. If it is supposed now that the rate of reproduction is not controlled instantaneously but with a delay which is of about a period of gestation, one leads to the model following

$$\frac{dN}{dt} = [r(N(t-\tau)) - d(N(t))]\,N(t) \tag{12}$$

N^* is still the unique stationary solution not equal to zero of the equation. The Linearization of (12) around of N^* gives the following equation

$$\frac{dy}{dt} = r'(N^*)N^*y(t-\tau) - d'(N^*)N^*y(t)$$

Let

$$\rho = -r'(N^*)N^*,\ \delta = d'(N^*)N^*$$

we have $\rho \geq 0$ and $\delta \geq 0$. One can in fact of bringing in case where $\delta = 0$, by making the change of variable

$$z(t) = \exp(\delta t)y(t)$$

one has then

$$z'(t) = -\rho e^{\delta\tau} z(t-\tau)$$

Thus, we suppose, without loss of general information, that $\delta = 0$, and we suppose

$$\rho > 0$$

One thus considers the equation

$$\frac{dy}{dt} = -\rho y(t-\tau) \tag{13}$$

The characteristic equation of (13) is

$$\lambda = -\rho e^{-\lambda\tau}$$

Posing $z = -\lambda\tau$ and $\zeta = \rho\tau > 0$, the equation can be written

$$ze^{-z} = \zeta \tag{14}$$

The equation (14) has two strictly positive real roots $z_1 < 1 < z_2$ if $\zeta \in \left]0, \frac{1}{e}\right[$, $z = 1$ or root doubles if $\zeta = \frac{1}{e}$ and no real root if $\zeta > \frac{1}{e}$. In the case when $\zeta > \frac{1}{e}$ all the solutions of (13) are oscillating. It is said that the equation (13) is oscillating. It is also checked that in this case all the solutions of the nonlinear equation (12) are oscillating around solution N^*.

The theory of the oscillations studies the conditions under which an equation is oscillating. This theory was developed with strength in years

80*: it was generalized with all kinds of equations belonging to the category of the functional differential equations.*

11. Stability function of delay

The effect of the delay on stability was examined by many authors. In many examples, a system which was stable remainder in the presence of one or more delays, this until the delay reaches a value threshold, beyond which the system becomes unstable. In certain examples, stability can be found by increasing the delay and there can be a finished or infinite number of intervals where the system is alternatively stable then unstable: it is the phenomenon of the switch of stability.

11.1. **Tepic result.** One considers the equation

$$\frac{dx}{dt} = ax(t) + bx(t - \tau) \tag{15}$$

The characteristic equation is :

$$\lambda - a - b\exp(-\lambda\tau) = 0$$

By making the change of parameter: $z = \lambda\tau$, $p = a\tau$ and $q = b\tau$, the equation becomes

$$pe^z + q - ze^z = 0 \tag{16}$$

There is the following result:

Theorem 5. *'A condition necessary and sufficient so that all the roots of the equation* (16) *are the real part strictly negative is that*
(i) $p < 1$ *and*
(ii) $p < -q < (\theta^2 + p^2)^{1/2}$*, where* θ *is the only root of the equation* $\theta = p\tan\theta$*,* $0 < \theta < \pi$ *where* $\theta = \pi/2$ *if* $p = 0$*.*

General result: if the equation (15) is stable for $\tau = 0$, then or well it is stable for all $\tau \geq 0$, or there is a value τ^* such that the equation is stable for $\tau < \tau^*$ and unstable for all $\tau > \tau^*$.

12. The Logistic Equation

Let $x(t)$ denote the population size at time; let b and d denote the birth rate and death rate, respectively, on the time interval $[t, t + \Delta t]$, where $\Delta t > 0$. Then

$$x(t + \Delta t) - x(t) = bx(t) - dx(t)\Delta t. \tag{17}$$

Dividing (17) by Δt and letting Δt approach zero, we obtain

$$\frac{dx}{dt} = bx - dx = rx, \tag{18}$$

where $r = b - d$ is the intrinsic growth rate of the population. The solution of equation (18) with an initial population $x(0) = x_0$ is given by

$$x(t) = x_0 e^{rt}. \tag{19}$$

The function (19) represents the traditional exponential growth if $r > 0$ or decay if $r < 0$ of a population. Such a population growth. Verhulst (1836) proposed the following logistic equation

$$\frac{dx}{dt} = rx(1 - \frac{x}{K}) \tag{20}$$

where $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of population.. Solving (20) we obtain ($x(0) = x_0$)

$$x(t) = \frac{x_0 K}{x_0 - (x_0 - K)e^{-rt}}. \tag{21}$$

The asymptotic behaviour of solution (21) is described in fig (2).

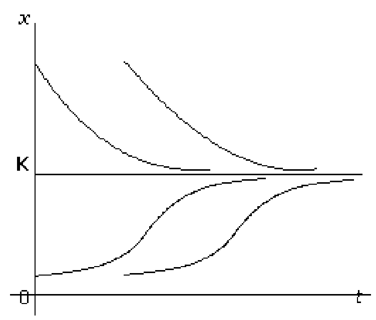

FIGURE 2. Solution

If $x_0 < K$, the population grows, approaching K asymptotically as $t \to \infty$. If $x_0 > K$, the population decreases, again approaching K asymptotically as $t \to \infty$. If $x_0 = K$, the population remains in time at $x = K$. The above analysis can be summarized into following theorem

Theorem 6. *The positive equilibrium $x = K$ of the logistic equation (20) is globally stable; that is, $\lim_{t\to\infty} x(t) = K$ for solution $x(t)$ of with any initial value $x(0) = x_0$.*

13. Hopf bifurcation

13.1. **Hutchinson's Equation.** In the classical logistic model it is assumed that the growth rate of a population at any time t depends on the relative number of individuals at that time. Hutchinson (1948) proposed the following more realistic logistic equation

$$\frac{dx(t)}{dt} = rx(t)(1 - \frac{x(t-\tau)}{K}) \tag{22}$$

where r and K have the same meaning as in the classical logistic equation (22), $\tau > 0$ is constant. Equation (22) is often referred to as the delayed logistic equation.

The initial value of equation (22) is given by

$$x(\theta) = \phi(\theta) > 0, \theta \in [-\tau, 0]$$

where ϕ is continuous on $[-\tau, 0]$. an equilibrium $x = x^*$ of (22) is stable if for any given $\varepsilon > 0$ there is a $\delta > 0$ such that $|\phi(t) - x^*| \leq \delta$ on $[-\tau, 0]$ satisfy $|x(t) - x^*| \leq \varepsilon$ for all $t \geq 0$. If in addition there is a $\delta_0 > 0$ such that $|\phi(t) - x^*| \leq \delta_0$ on $[-\tau, 0]$ implies $\lim_{t\to\infty} x(t) = x^*$, then x^* is called asymptotically stable.

Notice that equation (22) has equilibria $x = 0$ and $x = K$. Small perturbations from $x = 0$ satisfy the linear equation $\frac{dx(t)}{dt} = rx(t)$, which shows that $x = 0$ is unstable with exponential growth. We thus only need to consider the stability of the positive equilibrium $x = K$. Let $X = x - K$. Then,

$$\frac{dX}{dt} = -rX(t-\tau) - \frac{r}{K}X(t)X(t-\tau)$$

Thus, the linearized equation is

$$\frac{dX}{dt} = -rX(t-\tau). \tag{23}$$

We look for solution of the form $X(t) = ce^{\lambda t}$, where c is a constant and the eigenvalues λ are solutions of the characteristic equation

$$\lambda + re^{-\lambda r} = 0, \tag{24}$$

which is a transcendental equation. By the linearization theory $x = K$ is a asymptotically stale if all eigenvalues of (24) have negative real parts. In fact, we have the following conclusions.

Theorem 7. *(i) If $0 \leq r\tau < \frac{\pi}{2}$, then the positive equilibrium $x = K$ of equation (22) is asymptotically stable.*
(ii) If $r\tau > \frac{\pi}{2}$, then $x = K$ is unstable.
(iii) When $r\tau = \frac{\pi}{2}$, a Hopf bifurcation occurs at $x = K$; that is, periodic

solutions bifurcate from $x = K$. The periodic solutions exist for $r\tau > \frac{\pi}{2}$ and are stable.

The above theorem can be illustrated by Fig where the solid curves represent stability while the dashed lines indicated instability.

By (iii), the solution of the Hutchinson's equation (22) can exhibit stable limit cycle periodic solution for a large range of values of $r\tau$, the product of the birth rate r and the delay τ. If T is the period the $x(t+T) = x(t)$ for all t. Roughly speaking, the stability of a periodic solution means that if a perturbation is imposed the solution returns to the original periodic solution as $t \to \infty$ with possibly a phase shift. The period of the solution at the critical delay value is $\frac{2\pi}{\nu_0}$ (Hassard, Kazarinoff and Wan (1981)), thus, it is 4π.

13.2. **Van der Pol's equation.** The equations for the RLC electric circuit illustrated below

can be written as

$$
\begin{aligned}
i_C &= C\frac{dv_C}{dt},\ v_L = L\frac{di_l}{dt},\ v_R = \phi(i_R) \\
i_R &= i_L = -i_C,\ v_R + v_L = v_C,
\end{aligned}
$$

where the i's are the currents in the branches indicated by the subscripts and where $v_R = \phi(i_R)$ is generalized Ohm's law, characteristic of the "resistor" R, which is actually an active device. If we set $i_L = x$, $v_C = (L/C)^{1/2}y$ and $t = (LC)^{1/2}\tau$, then the equations take the form

$$\begin{aligned} x^{'} &= -y - f(x) \\ y^{'} &= x \end{aligned}$$

where $f(x) = (L/C)^{1/2}\phi(x)$ and $x^{'} = \frac{dx}{dt}$ denote differentiation which respect to the scaled time variable τ; Further, if the resistance is described by the function

$$f(x) = -\mu x + x^3,$$

then the system is a form of van der Pol's equation. The parameter μ controls the amount of "gain" of the device R.

For all values of μ, $(x, y) = (0, 0)$ is a stationary point.

Now

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^3 \\ 0 \end{pmatrix}$$

so the linear stability of this stationary solution id determined by the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(\mu \pm \sqrt{\mu^2 - 4}).$$

For $\mu \leq -2$ the eigenvalues are real and negative; for $-2 < \mu < 0$ they form a complex conjugate pair with negative real part; for $0 < \mu < 2$ they form a complex conjugate pair with positive real part; and for $\mu \geq 2$ they are real and positive.

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